Probability 21-325 **Homework 14 (due 1st May)** **TT**

1. Let S be the number of ones when throwing a fair die 18000 times. Find a good approximation to $\mathbb{P}(2950 < S < 3050)$. How can you bound the error you make?

2. Let G be a standard Gaussian random vector in $\mathbb{R}^n$. Let $\|G\| = \sqrt{G_1^2 + \ldots + G_n^2}$ be its magnitude. Let $a_n = \mathbb{P}(\sqrt{n} - 1 \leq \|G\| \leq \sqrt{n} + 1)$. Find $a = \lim_{n\to\infty} a_n$ and show that $|a_n - a| \leq \frac{15}{\sqrt{n}}$ for all $n \geq 1$.

3. Show that $e^{-n} \sum_{k=1}^{n} \frac{n^k}{k!} \xrightarrow[n\to\infty]{} \frac{1}{2}$.

Hint: Poiss(n) random variable is a sum of n i.i.d. Poiss(1) random variables.

4. Suppose that a random variable X with variance one has the following property: $\frac{X+X'}{\sqrt{2}}$ has the same distribution as X, where X' is an independent copy of X. Show that $X \sim N(0,1)$.

5. A roulette wheel has slots numbered 1–36 (18 red and 18 black) and two slots numbered 0 and 00 that are painted green. You can bet \$1 that the ball will land in a red (or black) slot and win \$1 if it does. What is the expected value of your winnings after 361 spins of the wheel and what is approximately the probability that it will be positive?

6. A biased coin showing heads with probability p is thrown 2500 times. What is approximately the probability of getting no heads when a) $p = \frac{1}{2500}$, b) $p = \frac{1}{5}$? How about the probability of getting 500 heads?

7. Consider a simple random walk on $\{0, 1, \ldots, N\}$ with absorbing barriers at 0 and N. Find the probability u_k that the walk is absorbed at N if it begins at a point k, $0 \leq k \leq N$. Why is this called the Gambler's Ruin problem?

8. Show that for an asymmetric simple random walk on the integers, the number of revisits of the walk to its starting point is a geometric random variable.

9. Let $(S_n^{(1)})_{n\geq 0}, \ldots, (S_n^{(1)})_{n\geq 0}$ be independent symmetric random walks on the integers, each starting at 0. Consider the random walk $S_n = (S_n^{(1)}, \ldots, S_n^{(1)})$ on the lattice $\mathbb{Z}^d$. In which dimensions d is this walk recurrent and in which transient?

10. We flip a biased coin showing heads with probability $0 < p < 1$ a random number of times N which is a Poisson random variable with parameter λ, independent of the coin tosses. Let X and Y be the number of times heads and tails show up. Find the distribution of X and Y. Prove that X and Y are independent. What is the conditional distribution of N given $X = k$?

11.* Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{2n\log n}} \leq 1\right) = 1.$$