Probability 21-325 Homework 13 (due 24th April) TT

1. Find the characteristic functions of random variables with distribution Ber(p), Bin(n,p), Poiss(λ), Unif($[-1,1]$).

2. Compute the characteristic function of a random variable with density $\frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ (this distribution is often called symmetric exponential, or double-sided exponential or Laplace). Using the inversion formula for the density, find the characteristic function of a Cauchy random variable with density $\frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

3. Let $X_1, X_2, \ldots$ be i.i.d. standard Cauchy random variables. Show that for any reals $a_1, \ldots, a_n$, the sum $a_1X_1 + \ldots + a_nX_n$ has the same distribution as $(|a_1| + \ldots + |a_n|)X_1$.

4. Prove Scheffé's lemma: If $X_1, X_2, \ldots$ is a sequence of continuous random variables with densities $f_1, f_2, \ldots$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$ for some probability density f, then $\int_{\mathbb{R}} |f - f_n| \xrightarrow[n\to\infty]{} 0$. Conclude that then $X_n \xrightarrow{d} X$ for a random variable X with density f (in other words, pointwise convergence of densities implies convergence in distribution). Considering $f_n(x) = (1 + \cos(2\pi nx))\mathbf{1}_{[0,1]}(x)$, show that the converse statement does not hold.

5. Let $U_1, \ldots, U_{2n+1}$ be i.i.d. random variables uniform on $[0,1]$. Order them in a nondecreasing way and call the $n+1$ term (the middle one) M_n. Show that M_n has density $(2n+1)\binom{2n}{n}x^n(1-x)^n\mathbf{1}_{[0,1]}(x)$. Find $\mathbb{E}M_n$ and $\mathrm{Var}(M_n)$. Show that $\sqrt{8n}\left(M_n - \frac{1}{2}\right)$ converges in distribution to a standard Gaussian random variable.

6. Prove that if a sequence (X_n) of random variables converges in probability to a random variable X, then $X_n \xrightarrow{d} X$.

Hint: There is a direct proof in the textbook by Grimmett and Welsh. Alternatively, argue by contradiction using Theorems 5.5 and 3.6.

7. Show that for a random variable X the following are equivalent

(a) X is symmetric, that is X and $-X$ have the same distribution

(b) X and εX have the same distribution, where ε is an independent random sign

(c) X and $\varepsilon|X|$ have the same distribution, where ε is an independent random sign

(d) the characteristic function of X is real valued.

8.* Prove that a sequence (X_n) of random variables converges in probability to a constant random variable $X = c$ if and only if $X_n \xrightarrow{d} c$.

Hint: Upper bound the indicator function $\mathbf{1}_{\{|x-c|>\varepsilon\}}$ *by* $g(x) = \frac{|x-c|}{\varepsilon}\mathbf{1}_{\{|x-c|\le\varepsilon\}} + \mathbf{1}_{\{|x-c|>\varepsilon\}}$ *which is continuous.*

9.* For sequences of random variables (X_n) and (Y_n), we have $X_n \xrightarrow{d} c$ and $Y_n \xrightarrow{d} Y$ for a constant $c \in \mathbb{R}$ and a random variable Y. Show that then $X_n + Y_n \xrightarrow{d} c + Y$.

Hint: Use the tightness of (Y_n)*, convergence in probability of* (X_n) *and the uniform continuity of a continuous function on a bounded interval.*

10.* Find an example when $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ do not imply that $X_n + Y_n \xrightarrow{d} X + Y$. Show that if all the random variables are independent, the statement is true.

11.* Suppose that for sequences of random variables (X_n) and (Y_n), we have $X_n \xrightarrow{d} c$ and $Y_n \xrightarrow{d} Y$ for a constant $c \in \mathbb{R}$ and a random variable Y. Show that then $X_n Y_n \xrightarrow{d} cY$.

Hint: First show the statement with $c = 0$ *using* $X_n \xrightarrow{\mathbb{P}} 0$*, the tightness of* (Y_n) *and* $\mathbb{P}(|X_n Y_n| > \varepsilon) \le \mathbb{P}\left(|X_n| > \frac{\varepsilon}{M}\right) + \mathbb{P}(|Y_n| > M)$*. For the general case, use* $X_n Y_n = (X_n - c)Y_n + cY_n$.

12.* Let $X_1, X_2, \ldots$ be i.i.d. standard Gaussian random variables. Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $\frac{M_n}{\sqrt{2\log n}} \xrightarrow{\mathbb{P}} 1$.

Hint: Use results of HW12 Q8. Q7 and Q10 from this HW may be of use, too.

13.* If for a random variable X, $\phi''_X(0)$ exists, then $\mathbb{E}|X|^2 < \infty$.

14.* Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ uniformly distributed on the sphere $\{x \in \mathbb{R}^n, x_1^2 + \ldots + x_n^2 = n\}$. Show that X_1 converges in distribution to a standard Gaussian random variable.