Probability 21-325 Homework 11 (due 10th April) TT

1. Show that if for every $\delta > 0$ we have $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \delta) < \infty$, then $X_n \xrightarrow[n\to\infty]{a.s.} X$.

2. Show that if there is a sequence of positive numbers δ_n convergent to 0 such that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \delta_n) < \infty$, then $X_n \xrightarrow[n\to\infty]{a.s.} X$.

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{P}(|X_i| < 1) = 1$. Show that $X_1 X_2 \cdot \ldots \cdot X_n$ converges to 0 a.s. and in L_1.

4. Let $X_1, X_2, \ldots$ be i.i.d. random variables with density g which is positive. Show that for every continuous function f such that $\int_{\mathbb{R}} |f| < \infty$, we have $\frac{1}{n}\sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)} \xrightarrow[n\to\infty]{a.s.} \int_{\mathbb{R}} f$. (This provides a method of numerical integration.)

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = -1)$ with $\frac{1}{2} < p < 1$. Let $S_n = X_1 + \ldots + X_n$ (a random walk with a drift to the right). Show that $S_n \xrightarrow[n\to\infty]{a.s.} \infty$.

6. Find $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^1 \ldots \int_0^1 \sqrt{x_1^2 + \ldots + x_n^2}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$ (or show the limit does not exist).

7. Let f be a continuous function on $[0, 1]$. Find $\lim_{n\to\infty} \int_0^1 \ldots \int_0^1 f(\sqrt[n]{x_1 \cdot \ldots \cdot x_n})\mathrm{d}x_1 \ldots \mathrm{d}x_n$ (or show it does not exist).

8. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{E}X_i^- < \infty$ and $\mathbb{E}X_i^+ = +\infty$. Show that $\frac{X_1 + \ldots + X_n}{n}$ tends to ∞ a.s.

9. Show that for every $0 < p < 1$, Minkowski's inequality in L_p fails, that is for every $0 < p < 1$, there are random variables X and Y such that $\|X + Y\|_p > \|X\|_p + \|Y\|_p$. Show that for $0 < p < 1$ and every random variables X and Y, we have

$$\|X + Y\|_p^p \le \|X\|_p^p + \|Y\|_p^p.$$

10. Show that for every $x \in \mathbb{R}$, we have $\frac{e^x + e^{-x}}{2} \le e^{x^2/2}$.

11.* *Khinchin's inequality*: for every $p > 0$, there are positive constants A_p, B_p which depend only on p such that for every n and every real numbers $a_1, \ldots, a_n$, we have

$$A_p \left(\sum_{i=1}^{2} a_i^2 \right)^{1/2} \le \left(\mathbb{E} \left| \sum_{i=1}^{n} a_i \varepsilon_i \right|^p \right)^{1/p} \le B_p \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2},$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. symmetric random signs.