Probability 21-325 **Homework 10 (due 3rd April)** **TT**

1. Show that if $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$ and $X_n \xrightarrow[n\to\infty]{\mathbb{P}} Y$, then $\mathbb{P}(X = Y) = 1$ (in other words, the limit in probability is unique).

2. Let $X_1, X_2, \ldots$ be i.i.d. integrable random variables. Prove that $\frac{1}{n}\max_{k\le n}|X_k|$ converges to 0 in probability.

3. Show that if $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$ and $Y_n \xrightarrow[n\to\infty]{\mathbb{P}} Y$, then $X_n + Y_n \xrightarrow[n\to\infty]{\mathbb{P}} X + Y$.

4. Show that if $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$ and $Y_n \xrightarrow[n\to\infty]{\mathbb{P}} Y$, then $X_nY_n \xrightarrow[n\to\infty]{\mathbb{P}} XY$.

5. Prove that a sequence of random variables X_n converges a.s. if and only if for every $\varepsilon > 0$, $\lim_{N\to\infty}\mathbb{P}\left(\bigcap_{n,m\ge N}|X_n - X_m| < \varepsilon\right) = 1$ (the Cauchy condition).

6. Does a sequence of independent random signs $\varepsilon_1, \varepsilon_2, \ldots$ converge a.s.?

7. Let $X_1, X_2, \ldots$ be independent random variables, $X_n \sim \text{Poiss}(1/n)$. Does the sequence X_n converge a.s., in L_2, in probability?

8. Let X be a random variable such that $\mathbb{E}e^{\delta|X|} < \infty$ for some $\delta > 0$. Show that $\mathbb{E}|X|^p < \infty$ for every $p > 0$.

9. Let X be a random variable such that $\mathbb{E}e^{tX} < \infty$ for every $t \in \mathbb{R}$. Show that the function $t \mapsto \log\mathbb{E}e^{tX}$ is convex on $\mathbb{R}$.

10. Let X be a random variable such that $\mathbb{E}|X|^p < \infty$ for every $p > 0$. Show that the function $p \mapsto \log\|X\|_{1/p}$ is convex on $(0, \infty)$.

11* Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. Show that there is a constant $c > 0$ such that for every $n \ge 1$ and reals $a_1, \ldots, a_n$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n a_i\varepsilon_i\right| \ge \sqrt{\sum_{i=1}^n a_i^2}\right) \ge c.$$