21-801 Convex Geometry **TT**

Homework 1 (due 21st Sep)

1. Does the set of all extremal points of a compact convex set in $\mathbb{R}^n$ have to be closed?

2. Show that a local minimum of a convex function on $\mathbb{R}^n$ is also its global minimum.

3. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a convex function. Then f is continuous in the interior of its domain and Lipschitz continuous on any compact subset of that interior.

4. Let K be a symmetric convex body in $\mathbb{R}^n$. Show that its Minkowski functional defined as $p_K(x) = \inf\{t > 0, x \in tK\}$ is a norm on $\mathbb{R}^n$.

5. Show that if K is a closed convex set in $\mathbb{R}^n$ containing the origin, then $(K^\circ)^\circ = K$.

6. Show that for a closed convex set K in $\mathbb{R}^n$ containing the origin and an invertible linear map $A \in GL(n)$, we have $(AK)^\circ = (A^T)^{-1}K^\circ$.

7. Let K and L be closed convex sets in $\mathbb{R}^n$ containing the origin. Show that

$$\frac{1}{2}(K^\circ \cap L^\circ) \;\subset\; (K+L)^\circ \;\subset\; K^\circ \cap L^\circ.$$

8. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Show that for $x \in \mathbb{R}^n$, we have

$$\frac{1}{2}\inf_{y\in\mathbb{R}^n}\{\|y\|_K + \|x-y\|_L\} \le \|x\|_{K+L} \le \inf_{y\in\mathbb{R}^n}\{\|y\|_K + \|x-y\|_L\}.$$

Homework 2 (due 5th Oct)

1. Let X be a random variable with density $\frac{1}{2}e^{-|x|}$ on $\mathbb{R}$. Show that for every $p \geq 1$, $c_1 p \leq \|X\|_p \leq c_2 p$ for some absolute constants $c_1, c_2 > 0$.

2. Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with a finite positive integral. Then there are positive constants A, α such that $f(x) \leq Ae^{-\alpha|x|}$, for all $x \in \mathbb{R}^n$. In particular, for every $p > -n$, $\int_{\mathbb{R}^n} |x|^p f(x)\mathrm{d}x < \infty$.

3. Let $f : \mathbb{R} \to [0, \infty)$ be a centred log-concave function. Then,

$$\frac{1}{12e^2} \leq \frac{f(0)^2 \int x^2 f(x)\mathrm{d}x}{\left(\int f\right)^3} \leq 2.$$

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex and 1-Lipschitz function (with respect to the Euclidean distance). Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a random vector of independent random signs. Then for $t > 0$,

$$\mathbb{P}\left(|f(\varepsilon) - \mathrm{Med}(f(\varepsilon))| > t\right) \leq 4e^{-t^2/8}.$$

5. This is Talagrand's example showing that the concentration result above is really specific to convex functions: let $A = \{x \in \{-1, 1\}^n,\ \sum_{i=1}^n x_i \leq 0\}$ and define $f(x) = \inf\{|x - y|, y \in A\}$, which is 1-Lipschitz. Show that the median of $f(\varepsilon)$ is 0, but $\mathbb{P}\left(f(\varepsilon) > cn^{1/4}\right) \geq c$, for some absolute constant $c > 0$.

6. Show that $\cosh x \leq e^{x^2/2}$, for $x \in \mathbb{R}$. Show also that $\cos x \leq e^{-x^2/2}$, for $x \in [0, \frac{\pi}{2}]$.

7. This exercise shows how to deduce concentration on the sphere from concentration on Gaussian space. Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function with $\int_{S^{n-1}} f\mathrm{d}\sigma = 0$.

1) Show that $\|f\|_\infty \leq 2$.

2) Show that $F : \mathbb{R}^n \to \mathbb{R}$, $F(x) = |x|f\left(\frac{x}{|x|}\right)$ is 4-Lipschitz and $\int_{\mathbb{R}^n} F\mathrm{d}\gamma_n = 0$.

3) Using Gaussian concentration for Lipschitz functions (Corollary 3.5), deduce that $\sigma\{f > t\} \leq e^{-cnt^2}$, $t > 0$, with a universal constant c.

Hints: Theorem A.2. It may also help to first show that $\gamma_n(|x| < \frac{1}{100}\sqrt{n}) \leq e^{-c\sqrt{n}}$.

8. Let H be a k-dimensional subspace of $\mathbb{R}^n$. Show that $\mathbb{E}(\mathrm{dist}(\varepsilon, H))^2 = n - k$. Show that for $1 \leq k \leq n - 1$,

$$\mathbb{P}\left(|\mathrm{dist}(\varepsilon, H) - \sqrt{n-k}| > t\right) \leq Ce^{-ct^2}, \qquad t > 0,$$

with universal constants $c, C > 0$. (This concentration result, established by Tao and Vu, is a cornerstone in the study of singularity of random ± 1 matrices.)

Homework 3 (due 22nd Oct)

1. Show the following analogue of the Brunn-Minkowski inequality: for $n \times n$ positive semi-definite real matrices A and B, we have

$$\big[\det(A+B)\big]^{1/n} \geq \big[\det(A)\big]^{1/n} + \big[\det(B)\big]^{1/n}.$$

2. Find the isotropic constant of an n-simplex in $\mathbb{R}^n$.

3. Show that for every $n \geq 3$ and every $\varepsilon > 0$, there is a symmetric convex body K in $\mathbb{R}^n$ with volume 1 such that

$$\int_{S^{n-1}} \mathrm{vol}_{n-1}(K \cap \theta^{\perp}) \mathrm{d}\sigma(\theta) < \varepsilon.$$

This shows that in general it is not possible to prove the slicing conjecture by a simple probabilistic argument. What about $n \leq 2$?

4. Let X be a random variable. Show that the function $t \mapsto \log \mathbb{E} e^{tX}$ is convex on $\mathbb{R}$ and the function $p \mapsto \log \|X\|_{1/p}$ is convex on $(0, \infty)$.

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Homework 4 (due 12th Nov)

1. Show that for an n-dimensional space $X = (\mathbb{R}^n, \|\cdot\|)$, we have

$$\frac{M}{b} \geq \frac{c}{\sqrt{n}}$$

with a universal constant $c > 0$. Here $M = \int_{S^{n-1}} \|\theta\| \mathrm{d}\sigma(\theta)$ is the mean norm and $b = \max_{\theta \in S^{n-1}} \|\theta\|$ is the Lipschitz constant of $\|\cdot\|$.

2. Let P be a polytope with facets $F_1, \ldots, F_m$. Show that $P = \bigcap_{j=1}^m H_j$, where the H_j are half-spaces corresponding to the supporting hyperplanes of the facets F_j. Show that such a representation of P is minimal in the sense that if $P = \bigcap_{i \in I} E_i$ for some half-spaces E_i and a finite set of indices I, then $\{F_1, \ldots, F_m\} \subset \{E_i, i \in I\}$.

3. Let $P = \mathrm{conv}\{\pm x_i\}_{i=1}^N$ be a symmetric polytope in $\mathbb{R}^n$. Show that $P^\circ = \bigcap_{i=1}^N \{x \in \mathbb{R}^n,\ |\langle x, x_i \rangle| \leq 1\}$ and conclude that the number of facets of P is equal to the number of vertices of the dual P°.

4. Show that $d_{BM}(\ell_1^n, \ell_\infty^n) \leq \sqrt{n}$ for every $n = 2^m$, $m = 1, 2, \ldots$. Show also that $d_{BM}(\ell_1^n, \ell_\infty^n) \geq \frac{\sqrt{n}}{e}$ for every $n = 1, 2, \ldots$.

Final exam (take-home, due 7th Dec)

1. Let g be a standard Gaussian random variable. Show that for every $p \geq 1$, we have $c_1\sqrt{p} \leq \|g\|_p \leq c_2\sqrt{p}$ for some universal constants $c_1, c_2 > 0$.

2. Show that for $n \times n$ positive definite real matrices A, B and $\lambda \in [0,1]$, we have

$$\log\big[\det(\lambda A + (1-\lambda)B)\big] \geq \lambda \log\big[\det(A)\big] + (1-\lambda)\log\big[\det(B)\big]$$

and for $\alpha > 0$,

$$\big[\det(\lambda A + (1-\lambda)B)\big]^{-\alpha} \leq \lambda\big[\det(A)\big]^{-\alpha} + (1-\lambda)\big[\det(B)\big]^{-\alpha}.$$

3. For $p \in [1,\infty]$, find the ellipsoid of maximal volume in B_p^n.

4. Let X be a log-concave random variable. Let a be such that $\mathbb{P}(X > a) \leq e^{-1}$. Then $\mathbb{E}X \leq a$.

5. Let K be a centred convex body of volume 1 in $\mathbb{R}^n$. For a unit vector θ consider $K_\theta = K \cap \{x \in \mathbb{R}^n,\ \langle x, \theta\rangle \geq 0\}$. Show that $e^{-1} \leq |K_\theta| \leq 1 - e^{-1}$. (In words, any hyperplane passing through the barycentre of a convex body divides in into two pieces of roughly the same volume – Grünbaum's lemma.)

6. Let f and g be densities of two independent symmetric log-concave random variables X and Y. Let $\lambda \in [0,1]$ and let h be the density of $\lambda X + (1-\lambda)Y$. Is it true that

$$-\int h \log h \leq \max\left\{-\int f \log f, -\int g \log g\right\}\ ?$$

Remark. This is an open problem.