- CONVERGENCE OF R.VS -

A seq. of r.vs (X_n) <u>converges</u> to a r.v. X

- <u>almost surely</u> if $\mathbb{P}\left(\{\omega\in\Omega,\ \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$

  notation: $X_n \xrightarrow{a.s.} X$

- <u>in probability</u> if $\forall \varepsilon$ $\mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

  notation $X_n \xrightarrow{\mathbb{P}} X$

- <u>in L_p</u> if $\mathbb{E}|X_n - X|^p \xrightarrow[n\to\infty]{} 0$.

  notation $X_n \xrightarrow{L_p} X$

<u>E.g.</u> $\Omega = \{1,2\}$, $\mathbb{P}(\{1\}) = \frac{1}{2} = \mathbb{P}(\{2\})$,

$X_n(1) = -\frac{1}{n}$, $X_n(2) = \frac{1}{n}$,

- $X_n \xrightarrow{a.s.} 0$ b/c $\forall \omega\in\Omega$ $X_n(\omega) \to 0$
- $X_n \xrightarrow{\mathbb{P}} 0$ $\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(\frac{1}{n} > \varepsilon) \xrightarrow[n\to\infty]{} 0$
- $\mathbb{E}|X_n|^p = 2\cdot\frac{1}{2}\frac{1}{n^p} = \frac{1}{n^p} \xrightarrow[n\to\infty]{} 0$, so $X_n \xrightarrow{L_p} 0$.

<u>Thm</u> If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{\mathbb{P}} X$, but <u>not</u> conversely!

<u>Proof</u>. • $X_n(\omega) \longrightarrow X(\omega) \iff \forall \ell \geq 1\ \exists N\ \forall n\geq N\ |X_n(\omega) - X(\omega)| < \frac{1}{\ell}$

• $\{\lim X_n = X\} = \bigcap_{\ell\geq 1} \bigcup_{N\geq 1} \bigcap_{n\geq N} \{|X_n - X| < 1/\ell\}$.

$$\bullet \quad \mathbb{P}(\lim X_n = X) = 1 = \mathbb{P}\Big(\bigcap_{l\ge 1} \bigcup_{N\ge 1} \bigcap_{n\ge N} \{|X_n - X| < 1/l\}\Big)$$

<u>iff</u> $\forall l \quad \mathbb{P}\Big(\bigcup_{N\ge 1} \bigcap_{n\ge N} \{|X_n - X| < 1/l\}\Big) = 1$

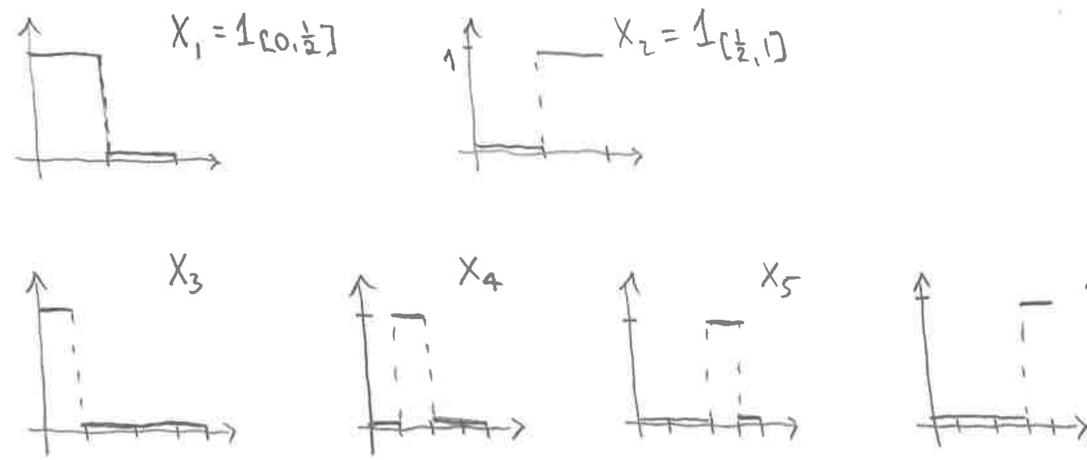
$$\Big\| $$

$$\lim_{N\to\infty} \underbrace{\mathbb{P}\Big(\bigcap_{n\ge N} \{|X_n - X| < 1/l\}\Big)}_{\le\ \mathbb{P}(|X_N - X| < 1/l)}$$

HW to show $\sum_{n\ge 1} \mathbb{P}(|X_n - X| > \varepsilon) < \infty \Downarrow X_n \xrightarrow{a.s.} X$

so $\lim_{N\to\infty} \mathbb{P}(|X_N - X| < 1/l) = 1$. To finish, take the complement.

<u>E.g.</u> $\Omega = [0,1]$, $\mathbb{P}(A) = |A|$



...

$\bullet \quad X_n \xrightarrow{\mathbb{P}} 0$ b/c $\mathbb{P}(|X_n| > \varepsilon) \le \frac{1}{2^{k_n}}$, $k_n \xrightarrow[n\to\infty]{} \infty$.

If $X_n \xrightarrow{a.s} X$, by the thm. $X = 0$

$\bullet \quad \forall \omega \in \Omega \quad X_n(\omega) \not\to 0$ b/c $X_n(\omega)$ contains ∞ many 0s as well as 1s.

so $X_n \stackrel{a.s.}{\not\to} 0$

⚠ In this example $\mathbb{E}|X_n|^p = \frac{1}{2^{k_n}} \xrightarrow[n\to\infty]{} 0$, so

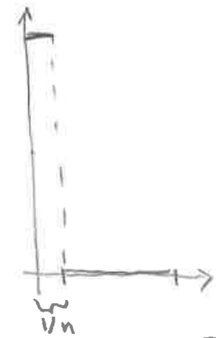
$X_n \xrightarrow{L_p} 0$

Thm If $X_n \xrightarrow{L_p} X$ for some $p>0$, then $X_n \xrightarrow{\mathbb{P}} X$
but not conversely!

Proof $\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(|X_n - X|^p > \varepsilon^p) \le \frac{1}{\varepsilon^p}\mathbb{E}|X_n - X|^p \xrightarrow[n\to\infty]{} 0$. □

E.g. $\Omega = [0,1]$, $\mathbb{P}(A) = |A|$,

$X_n = n^{1/p} \mathbb{1}_{[0, 1/n]}$



$\mathbb{P}(|X_n| > \varepsilon) = \frac{1}{n} \to 0$, so $X_n \xrightarrow{\mathbb{P}} 0$, but

$X_n \overset{L_p}{\not\to} 0$ b/c $\mathbb{E}|X_n|^p = \mathbb{E}(n^{1/p})^p \mathbb{1}_{[0, 1/n]} = n \cdot \frac{1}{n} = 1$.

⚠ In this example $X_n \xrightarrow{a.s.} 0$.

Summary convergence:

$$\begin{array}{ccc} \text{a.s.} & \overset{\not\Leftarrow}{\Rightarrow} & \text{in } \mathbb{P} \\ \not\Downarrow \ \not\Uparrow & & \\ \text{in } L_p & \overset{\Rightarrow}{\not\Leftarrow} & \end{array}$$

Properties

- if $X_n \xrightarrow[L_p]{\substack{a.s.\\ \mathbb{P}}} X$, $Y_n \xrightarrow[L_p]{\substack{a.s.\\ \mathbb{P}}} Y$, then $X_n + Y_n \xrightarrow{\substack{a.s.\\ \mathbb{P}\\ L_p}} X + Y$

- if $X_n \xrightarrow{\substack{a.s.\\ \mathbb{P}}} X$, $Y_n \xrightarrow{\substack{a.s.\\ \mathbb{P}}} Y$, then $X_n \cdot Y_n \xrightarrow{\substack{a.s.\\ \mathbb{P}}} X \cdot Y$

It's more "difficult" to converge in higher L_p

- if $0 < p < q$, $X_n \xrightarrow{L_q} X$, then $X_n \xrightarrow{L_p} X$.