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- EXPECTATION -

Prob space $(\Omega, \mathcal{F}, \mathbb{P})$ $\rightsquigarrow$ R.vs $X : \Omega \to \mathbb{R}$ $\rightsquigarrow$ CDF $F(x) = \mathbb{P}(X \le x)$

discrete	continuous
$X(\Omega)$ countable	$F(x) = \int_{-\infty}^{x} f(t)\,dt$
pmf $p(x) = \mathbb{P}(X = x)$	density $f(x) = F'(x)$
$\mathbb{E}X = \sum_{x \in X(\Omega)} x\, p(x)$	$\mathbb{E}X = \int x f(x)\,dx$

⚠ For cts r.vs $\forall x$ $\mathbb{P}(X = x) = 0$ — two seperate classes.
For discrete r.vs often $\mathbb{P}(X = x) > 0$

E.g. Cantor set → Devil's staircase → neither discrete nor cts r.v. (Cantor distribution)

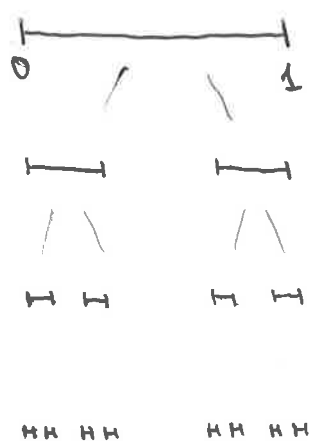
$C_0 = [0,1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup \dots$

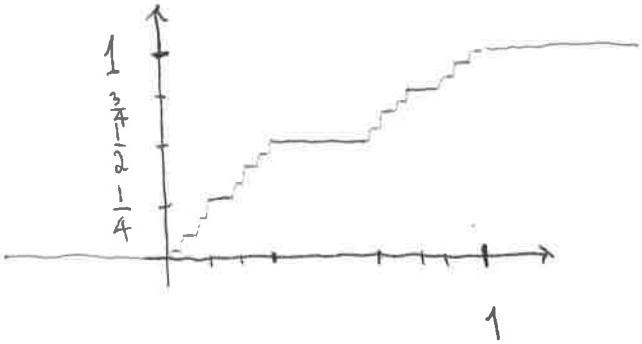
C_3

$\vdots$



$\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$ Cantor set

- uncountable
- doesn't contain any interval $|C_n| = (\frac{2}{3})^n$
- length (measure 0)

$F =$ 

Devil's staircase

nondecreasing

0 at $-\infty$, 1 at $+\infty$

so CDF of a r.v. X

- F is cts $\Rightarrow$ $\forall x \ \mathbb{P}(X=x)=0$ $\Rightarrow$ X not discrete

- If X was cts, $F(x) = \int_{-\infty}^{x} f(t)\,dt$ for some density f,

$$f(x) = F'(x) \underset{\substack{\uparrow \\ F \text{ is const outside } \mathcal{C}}}{=} 0 \quad , x \notin \mathcal{C}$$

$$1 = \int_{-\infty}^{\infty} f \underset{\substack{\uparrow \\ |\mathcal{C}|=0}}{=} \int_{\mathcal{C}^c} f = \int 0 = 0. \qquad \lightning$$

So, X neither discrete, nor cts.

We need a general def. of expectation.

Let $X : \Omega \to \mathbb{R}$ be a r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$

- if X is simple, that is $X(\Omega)$ is finite,

$$X = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \text{values}}}{x_k} \, \mathbb{1}_{\underset{\substack{\uparrow \\ \text{events}}}{A_k}} \qquad \text{for some } \begin{cases} \overset{\text{distinct}}{x_1, \dots, x_n} \in \mathbb{R} \\ \underbrace{A_1, \dots, A_n}_{\text{partition}} \in \mathcal{F} \end{cases}$$

$A_k = \{X = x_k\}$

we set

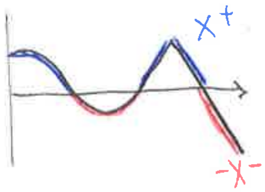
$$\mathbb{E}X = \sum_{k=1}^{n} x_k \, \mathbb{P}(A_k)$$

- if X is nonneg. $X(\omega) \geq 0 \ \forall \omega \in \Omega$, we set

⚠ It can be $\mathbb{E}X = +\infty$!

$$\mathbb{E}X = \sup \{ \mathbb{E}Z, \quad Z : \Omega \to \mathbb{R} \text{ simple}, \ Z \leq X \}$$

- if X is arbitrary, $X = X^+ - X^-$, where



$$X^+ = \max\{X, 0\} = X \mathbb{1}_{\{X \geq 0\}},$$

$$X^- = -\min\{X, 0\} = -X \mathbb{1}_{\{X \leq 0\}}$$

and we set

$$\mathbb{E}X = \mathbb{E}X^+ - \mathbb{E}X^-$$

provided that at least one of $\mathbb{E}X^+, \mathbb{E}X^-$ is finite (to avoid $\infty - \infty$)

We say X is <u>integrable</u> if $\mathbb{E}|X| < \infty$

($|X| = X^+ + X^-$, so X integrable iff $\mathbb{E}X^+$ & $\mathbb{E}X^- < \infty$)

<u>Nonnegative r.vs</u>

<u>Properties</u>

(a) if $0 \leq X \leq Y$, then $\mathbb{E}X \leq \mathbb{E}Y$

(b) if $X \geq 0$, $a \in \mathbb{R}$, then $\mathbb{E}(aX) = a\mathbb{E}X$

(c) if $X \geq 0$ and $\mathbb{E}X = 0$, then $\underbrace{X = 0 \text{ a.s.}}_{\mathbb{P}(X=0)=1}$

(d) if $X \geq 0$, $A \subset B$, $A, B \in \mathcal{F}$, then $\mathbb{E}X\mathbb{1}_A \leq \mathbb{E}X\mathbb{1}_B$.

Proof (a) Let $Z \le X$ be a simple r.v. such that

$$\mathbb{E}Z > \mathbb{E}X - \varepsilon$$

Since $Z \le Y$, $\mathbb{E}Z \le \mathbb{E}Y$, so $\mathbb{E}X - \varepsilon < \mathbb{E}Y$.

(b) exercise

(d) follows from (a) : $X 1_A \le X 1_B$

(c) WTS $\mathbb{P}(X > 0) = 0$, $\{X > 0\} = \bigcap_{n=1}^{\infty} \{X \ge \frac{1}{n}\}$,

$$X \ge X 1_{\{X \ge \frac{1}{n}\}} \ge \frac{1}{n} 1_{\{X \ge \frac{1}{n}\}}$$

(a)

$$0 = \mathbb{E}X \ge \frac{1}{n} \mathbb{E} 1_{\{X \ge \frac{1}{n}\}} = \frac{1}{n} \mathbb{P}(X \ge \tfrac{1}{n}),$$

so $\mathbb{P}(X \ge \frac{1}{n}) = 0$, so $\mathbb{P}(X > 0) = \lim_{n \to \infty} \mathbb{P}(X \ge \frac{1}{n}) = 0$. $\square$

Lm If $X \ge 0$, then there is a seq. (Z_n) of simple r.vs such that $\forall \omega \in \Omega$ $Z_n(\omega) \nearrow X(\omega)$.

Proof

$$Z_n = \sum_{k=1}^{n 2^n} \frac{k-1}{2^n} 1_{\{\frac{k-1}{2^n} \le X < \frac{k}{2^n}\}} + n \cdot 1_{\{X \ge n\}}$$

Fix $\omega \in \Omega$. Then $Z_n(\omega)$ is a nondec. seq. (check!), since eventually $n > X(\omega)$, we have for such n,

$$0 \le X(\omega) - Z_n(\omega) \le \frac{1}{2^n}. \quad \square$$

Thm (Lebesgue's monotone convergence) If X_n is a seq. of r.vs such that $\begin{cases} X_n \geq 0 \\ X_n \leq X_{n+1} \\ X_n \to X \text{ a.s.} \end{cases}$, then $\mathbb{E}X_n \nearrow \mathbb{E}X$.

Monotone bdd seqs have limits

Proof By Prop (a) $\mathbb{E}X_n \leq \mathbb{E}X_{n+1}$, $\mathbb{E}X_n \leq \mathbb{E}X$,

so $\lim \mathbb{E}X_n$ exists and $\leq \mathbb{E}X$. WTS $\mathbb{E}X \leq \lim_n \mathbb{E}X_n$.

Take a simple r.v. $0 \leq Z \leq X$. Z is bdd, say by K.

$$\forall n, \varepsilon \qquad Z - X_n \leq K \cdot 1_{\{Z \geq X_n + \varepsilon\}} + \varepsilon$$

so $$\mathbb{E}Z \leq \mathbb{E}X_n + K \cdot \mathbb{P}(Z \geq X_n + \varepsilon) + \varepsilon$$

As $n \to \infty$, $\{Z \geq X_n + \varepsilon\} \searrow \{Z \geq X + \varepsilon\} = \emptyset$, so

$$\mathbb{E}Z \leq \lim \mathbb{E}X_n + \varepsilon$$

so $$\mathbb{E}X \leq \lim \mathbb{E}X_n + \varepsilon. \quad \square$$

Thm (linearity) If $X, Y \geq 0$, then $\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$.

Proof • $X = \sum_{k=1}^{m} x_k 1_{F_k}$, $Y = \sum_{l=1}^{n} y_l 1_{G_l}$ simple r.vs

$$X + Y = \sum_{k,l} (x_k + y_l) 1_{F_k \cap G_l}$$

$$\begin{aligned} \text{LHS} = \mathbb{E}(X+Y) &= \sum_{k,l} (x_k + y_l) \mathbb{P}(F_k \cap G_l) \\ &= \sum_{k,l} x_k \mathbb{P}(F_k \cap G_l) + \sum_{k,l} y_l \mathbb{P}(F_k \cap G_l) \\ &= \sum_k x_k \mathbb{P}(F_k) + \sum_l y_l \mathbb{P}(G_l) = \mathbb{E}X + \mathbb{E}Y. \end{aligned}$$

• $X, Y \geq 0$ arbitrary, by OTT Lm there are $0 \leq Z_n \nearrow X$, $0 \leq V_n \nearrow Y$ simple

then $Z_n + V_n \nearrow X + Y$

and we know
$$\begin{array}{ccccc} \mathbb{E}(Z_n + V_n) & = & \mathbb{E}Z_n & + & \mathbb{E}V_n \\ \downarrow{\scriptstyle n\to\infty} & & \downarrow & & \downarrow \\ \mathbb{E}(X+Y) & = & \mathbb{E}X & + & \mathbb{E}Y. \ \square \end{array}$$
By Lebesgue's

<u>Thm</u> (Fatou's Lm) If $X_n \geq 0$, then $\mathbb{E} \liminf X_n \leq \liminf \mathbb{E}X_n$.

<u>Proof</u> $Y_n = \inf_{k \geq n} X_k$, $Y_n \nearrow \liminf_{m\to\infty} X_m$, $Y_n \leq X_n$,

so $\liminf \mathbb{E}X_n \geq \liminf \mathbb{E}Y_n = \lim \mathbb{E}Y_n$

$$\underset{\text{Leb.}}{=} \mathbb{E} \lim Y_n = \mathbb{E} \liminf X_n$$
$\square$

$\mathbb{E}X = \mathbb{E}X^+ - \mathbb{E}X^-$
X integrable if $\mathbb{E}|X| < \infty$

<u>General r.vs</u> (not nec. nonneg.)

<u>Properties</u> if X, Y are integrable, then

(a) $X+Y$ is integrable and $\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$

(b) if $a \in \mathbb{R}$, $\mathbb{E}(aX) = a\mathbb{E}X$

(c) if $X \leq Y$, then $\mathbb{E}X \leq \mathbb{E}Y$

(d) $|\mathbb{E}X| \leq \mathbb{E}|X|$

<u>Proof</u> (a) $|X+Y| \le |X| + |Y|$

$\mathbb{E}|X+Y| \le \mathbb{E}|X| + \mathbb{E}|Y| < \infty$, so $X+Y$ integrable

$$(X+Y)^+ - (X+Y)^- = X+Y = X^+ - X^- + Y^+ - Y^-$$

rearr. $$(X+Y)^+ + X^- + Y^- = (X+Y)^- + X^+ + Y^+$$

so $$\mathbb{E}(X+Y)^+ + \mathbb{E}X^- + \mathbb{E}Y^- = \mathbb{E}(X+Y)^- + \mathbb{E}X^+ + \mathbb{E}Y^+$$

rearr. $$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y.$$

(b) exercise

(c) $X \le Y$ iff $X^+ \le Y^+$ and $X^- \ge Y^-$

(d) $-|X| \le X \le |X|$ so by (c) $-\mathbb{E}|X| \le \mathbb{E}X \le \mathbb{E}|X|$. $\square$

<u>Thm</u> (Lebesgue's dominated convergence) If X_n is a seq. of r.vs such that $X_n \to X$ and $|X_n| \le Y$ for some integrable Y, then

$$\mathbb{E}X_n \to \mathbb{E}X.$$

<u>Proof</u> $|X_n| \le Y \Rightarrow |X| \le Y$ so X also integrable,

$$|X_n - X| \le 2Y$$

By Fatou's Lm

$$
\begin{aligned}
\mathbb{E}(2Y) &= \mathbb{E}\, \underline{\lim}\,(2Y - |X_n - X|) \\
&\le \underline{\lim}\; \mathbb{E}(2Y - |X_n - X|) \\
&= 2\mathbb{E}Y - \overline{\lim}\; \mathbb{E}|X_n - X|,
\end{aligned}
$$

so $\quad \overline{\lim}\; \mathbb{E}|X_n - X| = 0, \quad$ so $\quad \mathbb{E}|X_n - X| \to 0.$

In part., $\quad |\mathbb{E}(X_n - X)| \le \mathbb{E}|X_n - X| \to 0,$

so $\quad \mathbb{E}X_n \to \mathbb{E}X. \quad \square$