– $6\frac{1}{2}$ BRANCHING –

Model

1) at time $n=0$ 1 bacterium is born

2) at time $n \geq 1$, each bacterium born at time $n-1$ gives (independently) a random number of children and dies

3) the number of children X of each bacterium follows the same dist., $\mathbb{P}(X=k) = p_k$, $k=0,1,2,\ldots$

4) Z_n = the no of bacteria at time n

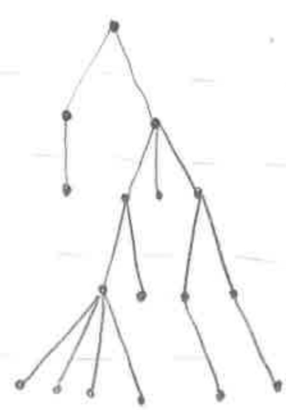
$n=0$ $\quad Z_0 = 1$

$n=1$ $\quad Z_1 = 2$

$n=2$ $\quad Z_2 = 4$

$n=3$ $\quad Z_3 = 4$

$n=4$ $\quad Z_4 = 6$



Q^s Mean, Var of Z_n,

What is Z_n like when $n \to \infty$?

In part., $\mathbb{P}(\lim_{n\to\infty} Z_n = 0) \overset{?}{>} 0$

(bacteria eventually become extinct with pos. prob), etc...

- $Z_0 = 1$
- $Z_1 = X$
- $Z_2 = X_1 + X_2 + \dots + X_{Z_1}$ $\qquad X_1, X_2, \dots$ i.i.d X

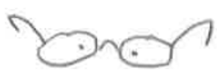
$\vdots$

- $Z_n = X_1 + X_2 + \dots + X_{Z_{n-1}}$

Let $\cdot$ $G(t) = \mathbb{E} t^X$ (mom. gen. fun of X)

$$= \sum_{k=0}^{\infty} p_k t^k$$

$\cdot$ $G_n(t) = \mathbb{E} t^{Z_n}$ (mom. gen. fun of Z_n)



$$G_n(t) = \mathbb{E} t^{X_1 + \dots + X_{Z_{n-1}}}$$

$$= \mathbb{E}\, t^{X_1} \cdot \dots \cdot t^{X_{Z_{n-1}}} = G_{n-1}(G(t))$$

So $$G_n(t) = \underbrace{G(G(\dots G(t)))}_{n \text{ times}} = \underbrace{G \circ \dots \circ G}_{n}(t).$$

__Thm__ $\mathbb{E} Z_n = (\mathbb{E} X)^n$, in part,

$$\lim_{n \to \infty} \mathbb{E} Z_n = \begin{cases} \infty, & \mathbb{E}X > 1 \\ 0, & \mathbb{E}X < 1 \\ 1, & \mathbb{E}X = 1 \end{cases}.$$

__Proof__. $\mathbb{E} Z_n = G_n'(1) = G_{n-1}'(G(t))\big|_{t=1} \cdot G'(t)\big|_{t=1}$

$$= G_{n-1}'(\underbrace{G(1)}_{1}) \cdot G'(1) = \mathbb{E} Z_{n-1} \cdot \mathbb{E} X. \quad \square$$

Thm $$\mathrm{Var}(Z_n) = \begin{cases} n\,\mathrm{Var}(X), & \mathbb{E}X = 1 \\ \mathrm{Var}(X)\cdot(\mathbb{E}X)^{n-1}\dfrac{(\mathbb{E}X)^n - 1}{\mathbb{E}X - 1}, & \mathbb{E}X \neq 1. \end{cases}$$

Exercise

Probability of extinction

$$A_n = \{Z_n = 0\} = \text{"bacteria have died by time } n\text{"}$$
$$\cap$$
$$\{Z_{n+1} = 0\}$$

$$\bigcup_{n=1}^{\infty} A_n = \text{"bacteria have died"}$$

$$\alpha = \mathbb{P}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mathbb{P}(A_n) \qquad \text{prob. of ultimate extinction}$$

Thm α = smallest nonnegative solution of $x = G(x)$.

Proof • $\alpha_n = \mathbb{P}(A_n) = \mathbb{P}(Z_n = 0) = G_n(0)$

• $G_n(t) = \underbrace{G \circ \dots \circ G}_{n}(t) = G(G_{n-1}(t))$

$t = 0 \rightsquigarrow$ $\alpha_n = G(\alpha_{n-1})$, $\alpha_0 = \mathbb{P}(Z_0 = 0) = 0$

$$\begin{array}{ccc} \alpha_n & = & G(\alpha_{n-1}) \\ {\scriptstyle n\to\infty}\downarrow & & \downarrow \\ \alpha & & G(\alpha) \end{array}$$

(G is cts! [even C^∞ on $(-1,1)$])

so $\alpha = G(\alpha)$.

• why is α the smallest ≥ 0 sol?

Let $\beta \geq 0$ be a sol, $\beta = G(\beta)$. We show $\alpha \leq \beta$.

$$0 \le \beta$$

$$\alpha_1 = G(0) \le G(\beta) = \beta$$

G is nondec (as given by a power series with ≥ 0 coeff.)

$$\alpha_2 = G(\alpha_1) \le G(\beta) = \beta$$

$$\vdots$$

$$\forall n \quad \alpha_n \le \beta \quad \rightsquigarrow \quad \alpha = \lim \alpha_n \le \beta. \ \square$$

Unless $\mathbb{E}X > 1$, with prob 1 the bacteria ev. become extinct!

Thm $\alpha = 1$ iff $\mathbb{E}X \le 1$ under the assump $\forall k \ge 0 \ \mathbb{P}(X=k) < 1$

rules out a silly situation when $\mathbb{P}(X=1)=1$, $\mathbb{E}X=1$, and $Z_n \equiv 1$.

Proof. • If $\underbrace{\mathbb{P}(X=0)}_{G(0)} = 0$, then $\alpha = 0$,

also $\mathbb{E}X = \sum_{k \ge 1} k\,\mathbb{P}(X=k) > 1$,

so thm OK in this case

• Suppose $\mathbb{P}(X=0) > 0$

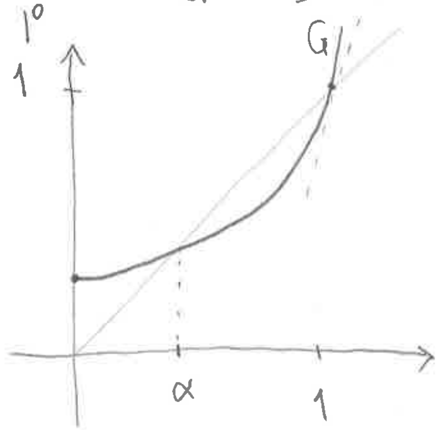
• G is on $[0,1]$

• $G(1) = 1$
• cts
• nondec. $(G' \ge 0)$
• convex $(G'' \ge 0)$

$x = G(x)$

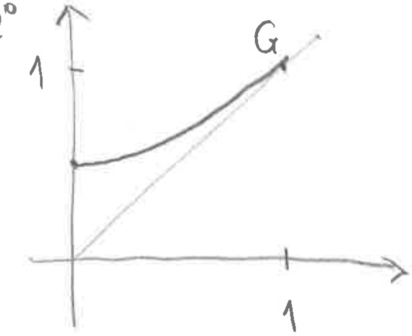
$G'(1) = \mathbb{E}X$

1°



$\mathbb{E}X = G'(1) > 1 \rightsquigarrow \alpha < 1$

2°



$\mathbb{E}X = G'(X) \le 1 \rightsquigarrow \alpha = 1$. $\square$