- MULTIVARIATE DISTRIBUTIONS and INDEPENDENCE -

Two r.v.s $X, Y : \Omega \to \mathbb{R}$ give rise to a <u>random vector</u> $(X, Y) : \Omega \to \mathbb{R}^2$. Note that $\forall x, y \in \mathbb{R}$ $\{X \le x, Y \le y\} = \{X \le x\} \cap \{Y \le y\}$ are events. The <u>joint distribution function</u> is defined as

$$F_{(X,Y)}(x,y) = \mathbb{P}(X \le x, Y \le y), \quad x, y \in \mathbb{R}.$$

<u>Properties</u>

1) $\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} F_{(X,Y)}(x,y) = 0$

2) $\lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} F_{(X,Y)}(x,y) = 1$

3) right continuity

4) monotonicity : if $x_1 \le x_2$, $y_1 \le y_2$, then $F_{(X,Y)}(x_1, y_1) \le F_{(X,Y)}(x_2, y_2)$.

To find the marginal distributions we take limits

$$F_X(x) = \mathbb{P}(X \le x) = \lim_{y \to \infty} \mathbb{P}(X \le x, Y \le y) = \lim_{y \to \infty} F_{(X,Y)}(x,y),$$

$$F_Y(y) = \lim_{x \to \infty} F_{(X,Y)}(x,y).$$

X, Y are <u>independent</u> if

$\forall x, y \in \mathbb{R}$ events $\{X \le x\}$, $\{Y \le y\}$ are independent, i.e.

$$\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\,\mathbb{P}(Y \le y)$$

$$F_{(X,Y)}(x, y) = F_X(x)\,F_Y(y)$$

A family $\{X_i, i \in I\}$ is <u>independent</u> if

$\forall \underset{\text{finite}}{J \subset I}$ $\{X_j, j \in J\}$ independent, i.e.

$$\mathbb{P}\Big(\bigcap_{j \in J} \{X_j \le x_j\}\Big) = \prod_{j \in J} \mathbb{P}(X_j \le x_j), \quad x_j \in \mathbb{R}.$$

Of course for a random vector $\vec{X} = (X_1, \dots, X_n)$ in $\mathbb{R}^n$, we set

$$F_{\vec{X}}(x_1, \dots, x_n) = \mathbb{P}(X_1 \le x_1, \dots, X_n \le x_n).$$

<u>E.g.</u> $F_{(X,Y)}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y}, & x, y \ge 0 \\ 0, & \text{o/w} \end{cases}$

The marginals

$$F_X(x) = \lim_{y \to \infty} F_{(X,Y)}(x,y) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \ge 0 \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y}, & y \ge 0 \end{cases}$$

so $X, Y \sim \mathrm{Exp}(1)$ ← recap! ($\frac{d}{dx}(1 - e^{-x}) = e^{-x}$ ← $\mathrm{Exp}(1)$ density)

Are X and Y indep? Notice

$$\cancel{\ldots} \quad 1-e^{-x}-e^{-y}+e^{-x-y} = (1-e^{-x})(1-e^{-y})$$

so
$$F_{(X,Y)}(x,y) = F_X(x)\cdot F_Y(y),$$

so yes, they are indep.

Recall: a r.v. X is cts if $\displaystyle F_X(x) = \int_{-\infty}^{x} f_X$.

A r.vec (X,Y) is called <u>continuous</u> if $\exists f: \mathbb{R}^2 \to [0,\infty)$ s.t.

$$F_{(X,Y)}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,ds\,dt$$

<u>density</u> of (X,Y)

⚠

$$f(x,y) = \begin{cases} \dfrac{\partial^2}{\partial x\,\partial y} F_{(X,Y)}(x,y) & \text{if exists} \\ 0 & \text{o/w.} \end{cases}$$

<u>Properties</u>

1) $f(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$

2) $\int_{\mathbb{R}^2} f = 1$

3) $\forall A \subset \mathbb{R}^2$ Borel subset $\quad \mathbb{P}((X,Y)\in A) = \int_A f$

Interpretation: take $\delta, \varepsilon > 0$ small, $x,y \in \mathbb{R}$, consider

$$\mathbb{P}((X,Y) \in [x,x+\delta]\times[y,y+\varepsilon]) = \int_x^{x+\delta}\int_y^{y+\varepsilon} f(s,t)\,ds\,dt$$

$$\approx f(x,y)\cdot\delta\cdot\varepsilon.$$

Thm $f:\mathbb{R}^2 \to \mathbb{R}$ is the density of some r.vec (X,Y) in $\mathbb{R}^2$ iff

$\forall x,y \in \mathbb{R}$ $\quad f(x,y) \geq 0 \quad$ and $\quad \int_{\mathbb{R}^2} f = 1$.

E.g. • Uniform dist. on $[0,a]\times[0,b]$ has density

$$f(x,y) = \begin{cases} \frac{1}{ab} & \text{on } [0,a]\times[0,b] \\ 0 & \text{elsewhere} \end{cases}$$

$$= \frac{1}{ab} \mathbb{1}_{[0,a]\times[0,b]}(x,y)$$

• In gen., $K \subset \mathbb{R}^2$, $(X,Y) \sim \mathrm{Unif}(K)$ has density

$$f(x,y) = \frac{1}{|K|} \mathbb{1}_K(x,y)$$

$$\mathbb{P}((X,Y) \in A) = \int_A f = \frac{1}{|K|} \int_A \mathbb{1}_K = \frac{|A \cap K|}{|K|}.$$

How to find densities of marginals?

$(X,Y) \rightsquigarrow$ density f

$? \swarrow \quad \searrow ?$

$$\mathbb{P}(X \in A) = \mathbb{P}((X,Y) \in A \times \mathbb{R}) = \int_{A\times\mathbb{R}} f(x,y)\,dx\,dy$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} f(x,y)\,dy \right)}_{f_X(x)} dx,$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy, \quad f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx.$$

Recap: X,Y discrete r.vs indep iff $P_{(X,Y)}(x,y)=f(x)g(y)$

Thm A continuous r.vec (X,Y) has independent components iff

$$f_{(X,Y)}(x,y) = f(x)\,g(y) \quad , x,y \in \mathbb{R}$$

for some functions $f,g:\mathbb{R}\to\mathbb{R}$.

Proof The same as in the discrete case — exercise.

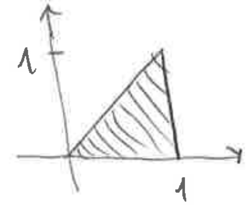
E.g. • $f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$.

Marginals
$$f_X(x) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dy$$
$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_{\mathbb{R}} \underbrace{\boxed{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}}_{\text{density of } N(0,1)}\,dy$$
$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2},$$

so $X,Y \sim N(0,1)$. Moreover,

$$f(x,y) = f_X(x)\,f_Y(y),$$

so X,Y are indep.



• $K = \{(x,y)\in\mathbb{R}^2, 0\le y\le x\le 1\}$

$(X,Y) \sim \mathrm{Unif}(K)$

$$f_X(x) = \int_{\mathbb{R}} \frac{1}{|K|} 1_K(x,y)\,dy = \frac{1}{1/2}\int_{0\le y\le x\le 1} dy = 2x\,1_{[0,1]}(x).$$

The same for Y. X,Y are not indep — check!

Sums of r.vs

Thm If (X,Y) is a cts r.vec with density f, then $Z=X+Y$ has density

$$f_Z(z) = \int_{\mathbb{R}} f(x, z-x)\,dx = \int_{\mathbb{R}} f(z-y, y)\,dy.$$

In particular, if X, Y are indep.

$$f_Z(z) = \int_{\mathbb{R}} f_X(x) f_Y(z-x)\,dx = \int_{\mathbb{R}} f_X(z-y) f_Y(y)\,dy$$

$$= (f_X * f_Y)(z)$$

$*$ convolution.

Proof

$$\mathbb{P}(X+Y \in A) = \int_{\{(x,y)\in\mathbb{R}^2,\ x+y\in A\}} f(x,y)\,dx\,dy$$

$$\underset{\begin{cases} x' = x \\ z = x+y \end{cases}}{=} \int_{(x',z)\in \mathbb{R}\times A} f(x, z-x)\,dx'\,dz$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} f(x, z-x')\,dx' \right)}_{f_Z(z)} dz. \qquad \square$$

The most important example

$Z = aX + Y$, $a \in \mathbb{R}$, $X, Y \sim N(0,1)$ indep

$$f_Z(z) = \int_{\mathbb{R}} f_{aX}(x) f_Y(z-x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}^2 |a|} \exp\left(-\frac{x^2}{2a^2} - \frac{(z-x)^2}{2}\right) dx$$

$$= \frac{1}{2\pi|a|} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left((1+a^{-2})x^2 - 2zx + z^2 \right)\right) dx$$

$$= \frac{1}{2\pi|a|} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}(1+a^{-2})\left(x - \frac{z}{1+a^{-2}}\right)^2 - \frac{1}{2}z^2 + \frac{1}{2(1+a^{-2})} z^2 \right) dx$$

$$\underset{x - \frac{z}{1+a^2} = x'}{=} \quad \frac{1}{2\pi|a|} \int_{\mathbb{R}} \exp\left(-\frac{1+a^{-2}}{2} x'^2\right) \exp\left(-\frac{z^2}{2}\left(1 - \frac{1}{1+a^{-2}}\right)\right) dx'$$

$$\underset{\sqrt{(1+a^{-2})}\,x' = t}{=} \quad \frac{1}{2\pi|a|} \exp\left(-\frac{z^2}{2(1+a^2)}\right) \int_{\mathbb{R}} e^{-t^2/2} \frac{dt}{\sqrt{1+a^{-2}}}$$

$$= \frac{1}{2\pi|a|\sqrt{1+a^{-2}}} \exp\left(-\frac{z^2}{2(1+a^2)}\right) \sqrt{2\pi} = \frac{1}{\sqrt{2\pi(1+a^2)}} e^{-\frac{z^2}{2(1+a^2)}},$$

so $\quad X + Y \sim N(0, 1+a^2)$.

In gen, if $\quad X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, then

$$X = \sigma_1 N_1 + \mu_1, \quad Y = \sigma_2 N_2 + \mu_2, \qquad N_1, N_2 \sim N(0,1)$$

$$\begin{aligned}
X + Y &= \sigma_1 N_1 + \sigma_2 N_2 + \mu_1 + \mu_2 \\
&= \sigma_2 \left(\frac{\sigma_1}{\sigma_2} N_1 + N_2\right) + \mu_1 + \mu_2 \\
&= \sigma_2 \cdot \underbrace{Z}_{\sim N(0,\ 1 + \frac{\sigma_1^2}{\sigma_2^2})} + \mu_1 + \mu_2 \\
&\sim N(0, \sigma_1^2 + \sigma_2^2) + \mu_1 + \mu_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2),
\end{aligned}$$

so $\quad \boxed{N(\mu_1, \sigma_1^2) * N(\mu_2, \sigma_2^2) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)}$

__E.g.__ $\quad X_1, \dots, X_n \sim$ i.i.d. $N(0,1)$

$$\sum_{i=1}^n a_i X_i \sim N(\mu, \sigma^2)$$

$$\mu = \mathbb{E}\left(\sum a_i X_i\right) = 0$$

$$\sigma^2 = \mathrm{Var}\left(\sum a_i X_i\right) = \sum a_i^2 \,\mathrm{Var} X_i = \sum a_i^2.$$

Change of variables

E.g. X, Y indep, $\mathrm{Exp}(1)$ $\qquad f_X(x) = f_Y(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x \le 0 \end{cases}$

$$f_{(X,Y)}(x,y) = \begin{cases} e^{-x-y}, & x,y>0 \\ 0, & o/w \end{cases}$$

Find the density of $\quad (U,V) = \left(X+Y, \frac{X}{X+Y}\right)$.

$$\mathbb{P}((U,V) \in A) = \int_A f_{(U,V)}(u,v)\, du\, dv$$

$$\parallel$$

$$\mathbb{P}\left(\left(X+Y, \frac{X}{X+Y}\right) \in A\right) = \int_{\{x,y>0,\ (x+y, \frac{x}{x+y}) \in A\}} e^{-x-y}\, dx\, dy$$

$$= \int_{\{(u,v) \in (0,\infty)\times(0,1),\ (u,v)\in A\}} e^{-uv-(u-uv)}\, u\, du\, dv$$

$x = v(x+y) = u \cdot v$

$y = u - x = u - uv$

$$\begin{cases} u = x+y \in (0,\infty) \\ v = \frac{x}{x+y} \in (0,1) \end{cases}$$

$$dx\, dy = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| du\, dv$$

$$= \left| \begin{vmatrix} v & 1-v \\ u & -u \end{vmatrix} \right| = |-uv - u(1-v)| = |u| = u$$

$$= \int_A e^{-u} u\, du\, dv$$

So $\quad f_{(U,V)}(u,v) = \begin{cases} ue^{-u}, & u>0,\ 0<v<1, \\ 0, & o/w \end{cases}$

$$= \underbrace{f(u)}_{ue^{-u}} \cdot \underbrace{g(v)}_{1},$$

so U, V indep, $\quad U = X+Y \sim \mathrm{Gamma}(2)$

$\qquad\qquad\qquad\quad V = \frac{X}{X+Y} \sim \mathrm{Unif}([0,1])$.

Conditional density function

Let (X,Y) have density $f(x,y)$. No info about X

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

What if we know $X=x$? The event $\{X=x\}$ has prob. 0 so we cannot condition on $\{X=x\}$, but

$$\mathbb{P}(y\leq Y \leq y+\varepsilon \mid x \leq X \leq x+\delta) = \frac{\mathbb{P}(y\leq Y \leq y+\varepsilon,\ x \leq X \leq x+\delta)}{\mathbb{P}(x \leq X \leq x+\delta)}$$

$$\approx \frac{f_X(x,y)\cdot\delta\cdot\varepsilon}{f_X(x)\cdot\delta} = \frac{f_{X,Y}(x,y)}{f_X(x)}\cdot\varepsilon$$

This motives: the <u>conditional density</u> of Y given $X=x$ is

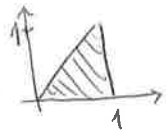
$$f_{Y|X}(y\mid x) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}, \quad y \in \mathbb{R},\ x \text{ s.t. } f_X(x)>0.$$

This is a density because it is ≥ 0 and

$$\int f_{Y|X}(y\mid x)\,dy = \int \frac{f_{(X,Y)}(x,y)}{f_X(x)}\,dy = \frac{f_X(x)}{f_X(x)} = 1.$$

⚠ If $X \perp\!\!\!\perp Y$, then

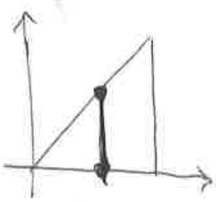
$$f_{Y|X}(y\mid x) = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y).$$

__E.g.__ $(X,Y) \sim \mathrm{Unif}\Big($  $\Big)$

$$f_{(X,Y)}(x,y) = \frac{1}{|K|} \mathbb{1}_K(x,y) = \begin{cases} 2, & 0<y<x<1 \\ 0, & o/w \end{cases}$$

$$f_X(x) = \int f_{(X,Y)}(x,y)\, dy = \int_{0<y<x<1} 2\, dy$$

$$= 2\int_0^x dy = 2x \mathbb{1}_{(0,1)}(x).$$



$X=x$

$$f_Y(y) = 2(1-y)\mathbb{1}_{(0,1)}(y)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2x}\mathbb{1}_{\{0<y<x\}}(x,y) = \frac{1}{x}\mathbb{1}_{(0,x)}(y)$$

so const on $(0,x)$ equal to $\frac{1}{x}$, so "$Y|X \sim \mathrm{Unif}(0,X)$"

__Conditional expectation__

$$\mathbb{E}(Y \mid X=x) = \int_{\mathbb{R}} y f_{Y|X}(y|x)\, dy.$$

__Thm__ $$\mathbb{E}Y = \int \mathbb{E}(Y|X=x) \underbrace{f_X(x)\, dx}_{\text{"}\approx \mathbb{P}(X=x)\text{"}}$$

__Proof__

$$\text{RHS} = \int_{\mathbb{R}}\int_{\mathbb{R}} y \underbrace{f_{Y|X}(y|x) f_X(x)}_{f_{(X,Y)}(x,y)}\, dy\, dx$$

$$= \int_{\mathbb{R}} y \Big(\int_{\mathbb{R}} f_{(X,Y)}(x,y)\, dx\Big) dy = \int y f_Y(y) = \mathbb{E}Y. \square$$

Thm For a cts r.vec $X=(X_1,\dots,X_n)$ in $\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}$ we have

$$\mathbb{E}g(X) = \int_{\mathbb{R}^n} g(x) f_X(x)\,dx$$

Cor. $\mathbb{E}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i \mathbb{E}X_i$.

Proof $g(x) = \sum a_i x_i$,

$$\mathbb{E}\left(\sum a_i X_i\right) = \int_{\mathbb{R}^n} \left(\sum a_i x_i\right) f_X(x)\,dx$$

$$= \sum a_i \int_{\mathbb{R}^n} x_i f_X(x)\,dx$$

$$= \sum a_i \int_{\mathbb{R}} x_i \left(\int_{\mathbb{R}^{n-1}} f_X(x)\, \underbrace{d\widehat{x_i}}_{dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_n} \right) dx_i$$

$$= \sum a_i \int_{\mathbb{R}} x_i f_{X_i}(x_i)\,dx_i$$

$$= \sum a_i \mathbb{E}X_i. \quad \square$$

Important quantities for r.vecs

Let $X=(X_1, \dots, X_n)$ be a r.vec in $\mathbb{R}^n$. Its mean is the vector

$$\mathbb{E}X = \begin{bmatrix} \mathbb{E}X_1 \\ \vdots \\ \mathbb{E}X_n \end{bmatrix} \in \mathbb{R}^n$$

Properties:

- $\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$ for any two r.vecs X, Y in $\mathbb{R}^n$
- $\mathbb{E}(AX) = A\,\mathbb{E}X$, A $m\times n$ matrix
- $\mathbb{E}XB = (\mathbb{E}X)B$, B $n\times m$ matrix

The covariance matrix

$$\mathrm{Cov}(X) = \Big[\underbrace{\mathrm{Cov}(X_i, X_j)}_{\mathbb{E}(X_i-\mathbb{E}X_i)(X_j-\mathbb{E}X_j)}\Big]_{i,j=1}^{n} = \mathbb{E}(X-\mathbb{E}X)(X-\mathbb{E}X)^T$$

Properties:

- $\mathrm{Cov}(X)$ is a symmetric matrix
- $\mathrm{Cov}(X)$ is a positive semi-definite matrix
- $\mathrm{Cov}(AX) = \mathbb{E}A(X-\mathbb{E}X)(X-\mathbb{E}X)^T A^T$
  $= A\,\mathrm{Cov}(X)\,A^T.$
- $\mathrm{Cov}(\underbrace{X+Y}_{\text{indep}}) = \mathrm{Cov}(X) + \mathrm{Cov}(Y).$

Important example: multivariate Gaussian distribution

Let $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix (in part. $\det A > 0$)

The Gaussian ~~r.vec~~ distribution in $\mathbb{R}^n$ with mean b and covariance A has density

Notation: $X \sim N(b,A)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}^n \sqrt{\det A}} \exp\Big(-\frac{1}{2} \overbrace{\langle A^{-1}(x-b), x-b\rangle}^{\text{scalar product}}\Big), \quad x \in \mathbb{R}^n.$$

A standard Gaussian r.vec in $\mathbb{R}^n$: $b=0$, $A = I_{n\times n}$,

$$f_X(x) = \frac{1}{\sqrt{2\pi}^n} \exp\Big(-\frac{1}{2} \underbrace{\langle x, x\rangle}_{\sum x_i^2}\Big) = \frac{1}{\sqrt{2\pi}^n} e^{-\|x\|_2^2/2}$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2}$$

$$X = (X_1, \dots, X_n)$$

indep, $N(0,1)$

⚠ $f_X(x) = \frac{1}{\sqrt{2\pi}^n} e^{-\|x\|_2^2/2}$ is rotationally invariant,

In particular, X = standard Gaussian, V = orthogonal $n\times n$ matrix

$$X \sim VX$$

E.g. $\underbrace{(X_1, X_2)}_{\text{iid } N(0,1)} \sim \Big(\frac{X_1+X_2}{\sqrt{2}}, \frac{X_1-X_2}{\sqrt{2}}\Big)$

indep! $N(0,1)$.

E.g. Let $X \sim N(0, Id)$, A $n\times n$ invertible matrix, $b \in \mathbb{R}^n$

$$Y = AX + b \quad ?$$

$$\mathbb{E}Y = \mathbb{E}(AX + b) = A(\mathbb{E}X) + b = A0 + b = b.$$

$$\mathrm{Cov}(Y) = \mathrm{Cov}(AX) = A I A^T = A A^T = \Sigma$$

Density of Y: $\quad \mathcal{U} \subset \mathbb{R}^n$

$$\mathbb{P}(Y \in \mathcal{U}) = \int_{\mathcal{U}} f_Y(y)\, dy$$

$$\mathbb{P}(AX + b \in \mathcal{U}) = \int_{x \in \mathbb{R}^n :\, Ax + b \in \mathcal{U}} f_X(x)\, dx$$

$$\underset{y = Ax + b}{=} \int_{\mathcal{U}} f_X(A^{-1}(y-b)) \frac{dy}{|\det A|}$$

$$f_Y(y) = \frac{1}{|\det A|} f_X(A^{-1}(y-b))$$

$$= \frac{1}{\sqrt{2\pi}^n \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} \langle A^{-1}(y-b), A^{-1}(y-b) \rangle\right)$$

$\Sigma^{-1} = (A^{-1})^T A^{-1}$

$$= \frac{1}{\sqrt{2\pi}^n \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} \langle (A^{-1})^T A^{-1}(y-b), (y-b) \rangle\right)$$

$$= \frac{1}{\sqrt{2\pi}^n \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} \langle \Sigma^{-1}(y-b), y-b \rangle\right).$$

So $Y \sim N(b, \Sigma)$.