– BIRTH & DEATH PROCESS. QUEUEING MODEL –

Birth - death process

$L_t =$ size of population at time t, $t \geq 0$
(e.g. no. of bacteria)

(1) $L_0 = \ell$

(2) $L_t \in \{0, 1, 2, \dots\}$

(3) birth rate $\lambda > 0$: at each time t, each member of the population

- gives one offspring in time $(t, t+h]$ with prob. $\lambda h + o(h)$
- gives no offspring in time $(t, t+h]$ with prob. $1 - \lambda h + o(h)$

$\Leftarrow$ gives more than 1 offspring with prob. $o(h)$

(4) death rate $\mu > 0$: at each time t, each member of the population

- dies in time $(t, t+h]$ with prob. $\mu h + o(h)$

(5) births and deaths occur independently, independently for each member,

We have for $k = 0, 1, 2, \dots$

- $$\mathbb{P}(L_{t+h} = k \mid L_t = k) \underset{\substack{\text{no birth} \\ \text{no death in } (t,t+h]}}{=} (1 - \lambda h + o(h))^k (1 - \mu h + o(h))^k$$
$$\underset{(1+\varepsilon)^k \approx 1 + k\varepsilon}{=} 1 - (\lambda + \mu) k h + o(h)$$

- $$\mathbb{P}(L_{t+h} = k+1 \mid L_t = k) \underset{\substack{\text{one birth} \\ \text{no death}}}{=} \binom{k}{1} (1 - \lambda h + o(h))^{k-1} (\lambda h + o(h)) \times (1 - \mu h + o(h))^k$$
$$= \lambda k h + o(h)$$

• $\mathbb{P}(L_{t+h}=k-1 \mid L_t=k) \underset{\substack{\text{no birth} \\ \text{one death}}}{=} \cdots = \mu k h + o(h)$

• $\mathbb{P}(L_{t+h} > k+1 \text{ or } L_{t+h} < k-1 \mid L_t = k) = o(h)$

Like Poisson process, but • two competing rates λ, μ

• "current" rates depend on the current size λk, μk (for Poisson rate is uniform in time)

<u>Thm</u> If $L_0 = l_0$, then the p.g.f is

$$G(u,t) = \mathbb{E}(u^{L_t}) = \begin{cases} \left(\dfrac{\lambda t(1-u)+u}{\lambda t(1-u)+1}\right)^{l_0} & \text{if } \mu=\lambda \\ \left(\dfrac{\mu(1-u) - (\mu-\lambda u)e^{t(\mu-\lambda)}}{\lambda(1-u) - (\mu-\lambda u)e^{t(\mu-\lambda)}}\right)^{l_0} & \text{if } \mu \neq \lambda. \end{cases}$$

<u>Proof</u> $p_k(t) = \mathbb{P}(L_t = k)$, as for Poisson

$\sum u^k \cdot /$

$$p_k'(t) = \lambda(k-1)p_{k-1}(t) - (\lambda+\mu)kp_k(t) + \mu(k+1)p_{k+1}(t), \quad k=0,1,2,\ldots \quad (p_{-1} \equiv 0)$$

+ boundary $p_k(0) = \begin{cases} 1 & \text{if } k=l_0 \\ 0 & \text{o/w} \end{cases}$.

Unlike for Poisson, eq. for p_0' involves p_1, so cannot proceed recursively.

Instead, $G(u,t) = \mathbb{E}(u^{L_t}) = \sum_{k=0}^{\infty} u^k p_k(t)$

$$\begin{cases} \dfrac{\partial G}{\partial t} = (\lambda u - \mu)(u-1)\dfrac{\partial G}{\partial u} \\ G(u,0) = u^{l_0} \end{cases}$$

check that ok. □

Thm If $L_0 = \ell_0$, then

$$\mathbb{P}(L_t = 0) \xrightarrow[t\to\infty]{} \begin{cases} 1 & \text{if } \lambda \le \mu \\ \left(\frac{\mu}{\lambda}\right)^{\ell_0} & \text{if } \lambda > \mu. \end{cases}$$

(prob. of extinction by time t)

Proof

$$\mathbb{P}(L_t = 0) = p_0(t) = G(0,t)$$

$$= \begin{cases} \left(\frac{\lambda t}{\lambda t + 1}\right)^{\ell_0} & \text{if } \lambda = \mu \\ \left(\frac{\mu - \mu e^{t(\mu-\lambda)}}{\lambda - \mu e^{t(\mu-\lambda)}}\right)^{\ell_0} & \text{if } \lambda \neq \mu \end{cases}$$

$$\xrightarrow[t\to\infty]{} \begin{cases} 1 & \text{if } \lambda = \mu \\ 1 & \text{if } \mu - \lambda > 0 \\ \left(\frac{\mu}{\lambda}\right)^{\ell_0} & \text{if } \mu - \lambda < 0. \ \square \end{cases}$$

Queueing model

(1) customers arrive according to a Poisson process with rate $\lambda > 0$

(2) there is one server, the service time of each customer is $\mathrm{Exp}(\mu)$, indep. for each customer

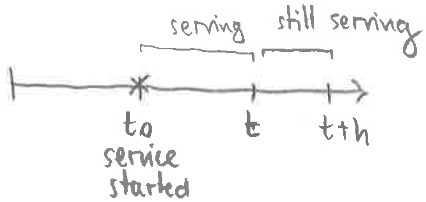
(3) service times ($S_1, S_2, \dots$) are independent of arrival times ($T_1, T_2, \dots$)

(4) first come, first served

Q_t = size of the queue at time t (inc. the person being served)

⚠ Service times are memory-less,

$$\mathbb{P}(S > t+h-t_0 \mid S > t-t_0) = \mathbb{P}(S>h) = e^{-\mu h}$$
$$= 1-\mu h + o(h).$$



$$\Rightarrow \quad \mathbb{P}(\text{Tom out in } (t,t+h] \mid \text{Tom being served at } t) = \mu h + o(h).$$

- $\mathbb{P}(Q_{t+h} = k \mid Q_t = k) \overset{\text{no arrival / no departure}}{=} [1-\lambda h + o(h)][1-\mu h + o(h)]$

  $= 1-(\lambda+\mu)h + o(h) \qquad k \geq 1$

- $\mathbb{P}(Q_{t+h} = k-1 \mid Q_t = k) = [1-\lambda h + o(h)][\mu h + o(h)]$

  $= \mu h + o(h) \qquad k \geq 1$

- $\mathbb{P}(Q_{t+h} = k+1 \mid Q_t = k) = [\lambda h + o(h)][1-\mu h + o(h)]$

  $= \lambda h + o(h) \qquad k \geq 0$

- $\mathbb{P}(Q_{t+h} = 0 \mid Q_t = 0) \overset{\text{no arrival | no one served}}{=} 1-\lambda h + o(h).$

Usual procedure $\underset{p_k(t) = \mathbb{P}(Q_t = k)}{\rightsquigarrow}$
$$\begin{cases} p_k'(t) = \lambda p_{k-1}(t) - (\lambda+\mu) p_k(t) + \mu p_{k+1}(t), & k \geq 1 \\ p_0'(t) = -\lambda p_0(t) + \mu p_1(t) \end{cases}$$

Difficult to solve

Long-time (asymptotic) behaviour, $t \to \infty$:

suppose
- $\lim_{t\to\infty} p_k(t)$ exists $\forall k \geq 0$, call it π_k
- $\lim_{t\to\infty} p_k'(t) = 0$

Then
$$\begin{cases} 0 = \lambda \pi_{k-1} - (\lambda+\mu)\pi_k + \mu \pi_{k+1}, & k \geq 1 \\ 0 = -\lambda \pi_0 + \mu \pi_1 \end{cases}$$

Let $\rho = \frac{\lambda}{\mu}$ $\left(= \frac{\text{arrival rate}}{\text{service rate}} = \text{intensity} \right)$.
$$\begin{cases} \pi_{k+1} = (1+\rho)\pi_k - \rho \pi_{k-1} \\ \pi_1 = \rho \pi_0 \end{cases}$$

$$\pi_1 = \rho \pi_0$$

$$\pi_2 = (1+\rho)\pi_1 - \rho \pi_0 = \rho^2 \pi_0$$

$$\vdots$$

$$\pi_k = \rho^k \pi_0$$

$(\pi_k)_{k\geq 0}$ is a prob. distribution iff $\pi_k \geq 0$, $\sum_{k=0}^{\infty} \pi_k = 1$

possible iff $\rho < 1$, $\pi_0 = \frac{1}{1-\rho}$

so, if $\rho = \frac{\lambda}{\mu} < 1$, the queue has a steady state
(settles for $t \to \infty$)

$$\mathbb{P}(Q_\infty = k) = \pi_k = (1-\rho)\rho^k \qquad (\text{Geom}(\rho))$$

E.g., in part. $\mathbb{E}\, Q_\infty = \mathbb{E}\, \text{Geom}(\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$.