Probability 21-325 **Homework 7 (due 28th March)** **TT**

1. Let X be a nonnegative random variable. Show that for $p > 0$ we have

$$\mathbb{E}X^p = \int_0^\infty p t^{p-1} \mathbb{P}(X > t)\,\mathrm{d}t.$$

2. Let X be a random variable such that $\mathbb{E}|X|^p < \infty$ for some $p > 0$. Show that $\lim_{t\to\infty} t^p \mathbb{P}(|X| > t) = 0$.

3. Show that the probability that in n throws of a fair die the number of sixes lies between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is at least $\frac{31}{36}$.

4. Let X be a random variable with values in an interval $[0, a]$. Show that for every t in this interval we have

$$\mathbb{P}(X \geq t) \geq \frac{\mathbb{E}X - t}{a - t}.$$

5. Prove the Payley-Zygmund inequality: for a nonnegative random variable X and every $\theta \in [0, 1]$ we have

$$\mathbb{P}(X > \theta\mathbb{E}X) \geq (1-\theta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

6. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent random signs. Prove that there is a positive constant c such that for every $n \geq 1$ and real numbers $a_1, \ldots, a_n$ we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n a_i\varepsilon_i\right| > \frac{1}{2}\sqrt{\sum_{i=1}^n a_i^2}\right) \geq c.$$

Hint. Use the Paley-Zygmund inequality and Q4 HW5.

7. Prove that for nonnegative random variables X and Y we have

$$\mathbb{E}\frac{X}{Y} \geq \frac{(\mathbb{E}\sqrt{X})^2}{\mathbb{E}Y}.$$

8. Suppose that $X = (X_1, \ldots, X_n)$ is a random vector uniformly distributed on the cube $[-\sqrt{3}, \sqrt{3}]^n$. Show that $X_1, \ldots, X_n$ are independent. Find $\mathbb{E}X_i$, $\mathbb{E}X_i^2$ and $\mathrm{Var}(X_i^2)$. Let $\|X\| = \sqrt{X_1^2 + \ldots + X_n^2}$ denote the distance from the point X to the origin. Show

$$\mathbb{E}\big|\|X\| - \sqrt{n}\big|^2 < 1.$$

Conclude that for $t > 0$,

$$\mathbb{P}\left(\big|\|X\| - \sqrt{n}\big| > t\right) < \frac{1}{t^2}.$$

In particular, $\mathbb{P}\left(\big|\|X\| - \sqrt{n}\big| > 10\right) \leq 1/100$, which implies that a random point X lands in the thin shell of width 20 around the sphere with radius $\sqrt{n}$ with probability greater than $99/100$ (think of n as being large).