Probability 21-325 **Homework 4 (due 21st Feb)** **TT**

1. There are n different coupons and each time you obtain a coupon it is equally likely to be any of the n types. Let Y_i be the additional number of coupons collected, after obtaining i distinct types, before a new type is collected (including the new one). Show that Y_i has the geometric distribution with parameter $\frac{n-i}{n}$ and find the expected number of coupons collected before you have a complete set.

2. Suppose a mathematician carries two matchboxes at all times: one in his left pocket and one in his right. Each time he needs a match, he is equally likely to take it from either pocket. Suppose he reaches into his pocket and discovers for the first time that the box picked is empty. If each of the matchboxes originally contained N matches, what is the probability that there are exactly k matches in the other box?

3. Let X be a discrete random variable with finite second moment. Show that the minimum of the function $a \mapsto \mathbb{E}(X-a)^2$ is attained at $a = \mathbb{E}X$.

4. A fair die is thrown 7 times. What is the probability that the sum of the obtained numbers is 14?

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables uniformly distributed on the set $\{1, \ldots, m\}$. For $n \le k \le mn$ show that

$$\mathbb{P}\left(X_1 + \ldots + X_n = k\right) = \frac{1}{m^n} \sum_{0 \le l \le (k-n)/m} (-1)^l \binom{n}{l} \binom{k-ml-1}{n-1}.$$

Hint: The general binomial series $(1+z)^\alpha = \sum_{l=0}^{\infty} \binom{\alpha}{l} z^l$, $|z| < 1$.

6. A biased coin showing heads with probability $p \in (0,1)$ is tossed n times. Find the probability that the total number of heads is a) even b) divisible by 3.

7. Let G_X be the generating function of a random variable X taking values $0, 1, 2, \ldots$. Let $t_n = \mathbb{P}(X > n)$, $n = 0, 1, 2, \ldots$. Show that the generating function T of the sequence $t_0, t_1, t_2, \ldots$ satisfies $T(s) = \frac{1 - G_X(s)}{1-s}$, $|s| < 1$.

8. Show that two disjoint events are independent if and only if at least one of them is impossible. Show that a constant random variable is independent of any other discrete random variable.

9. Give an example of a discrete random variable whose expectation does not exist.

10.* Let $A_1, A_2, \ldots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The event *infinitely many A_i occur* is, of course, defined as

$$\limsup A_n = \bigcap_{k \geq 1} \bigcup_{l \geq k} A_l$$

and called the limit supremum of the sequence of events A_n. Prove the so-called Borel-Cantelli lemmas saying that

a) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$,

b) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent, then $\mathbb{P}(\limsup A_n) = 1$.

11.* Prove the so-called infinite monkey theorem: when we toss a fair coin infinitely many times then the event that a given finite sequence of heads/tails occurs infinitely many times is certain.

12.* Decide whether there exists a probability measure $\mathbb{P}$ on the σ-field of all subsets of positive integers such that for every $k = 1, 2, \ldots$, we have

$$\mathbb{P}(\text{the set of positive integers divisible by } k) = 1/k.$$