Probability 21-325 **Homework 11 (due 30th April)** **TT**

1. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate λ. Show that for $s < t$, $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$.

2. Suppose you get at some point $t = t_0$ to a bus stop where buses arrive at according to a Poisson process $N = (N_t)_{t\geq 0}$ with rate λ. Show that your waiting time X for the next bus is an exponential random variable with parameter λ, independent of N_{t_0}.

3. Let X be a Poisson random variable with parameter $\lambda > 0$. Find $\sup_{k\geq 0} \mathbb{P}(X = k)$ and show that it goes to 0 as $\lambda \to \infty$.

4. Consider a simple random walk on $\{0, 1, \dots, N\}$ with absorbing barriers at 0 and N. Find the probability u_k that the walk is absorbed at N if it begins at a point k, $0 \leq k \leq N$. Why is this called the Gambler's Ruin problem?

5. Show that for an asymmetric simple random walk on the integers, the number of revisits of the walk to its starting point is a geometric random variable.

6. Let $(S_n^{(1)})_{n\geq 0}, \dots, (S_n^{(1)})_{n\geq 0}$ be independent symmetric random walks on the integers, each starting at 0. Consider the random walk $S_n = (S_n^{(1)}, \dots, S_n^{(1)})$ on the lattice $\mathbb{Z}^d$. In which dimensions d is this walk recurrent and in which transient?

7. Show that a Gaussian random vector in $\mathbb{R}^n$ has independent components if and only if they are uncorrelated.

8. Let X be an integrable random variable. Show that the function $a \mapsto \mathbb{E}|X - a|$ attains its minimum at $a = \mathrm{Med}(X)$.

9. Show that for any random variable X we have $|\mathbb{E}X - \mathrm{Med}(X)| \leq \sqrt{\mathrm{Var}(X)}$.

10. We flip a biased coin showing heads with probability $0 < p < 1$ a random number of times which is a Poisson random variable with parameter λ, independent of the coin tosses. Let X and Y be the number of times heads and tails show up. Find the distribution of X and Y. Prove that X and Y are independent. What is the conditional distribution of N given $X = k$?