Combinatorics, Revision lecture

Term 3 2014/2015

Problems

1. Show that for a positive integer n we have

$$\sum_{k=0}^{n} \binom{n+k}{n} \frac{1}{2^k} = 2^n.$$

2. Prove that for a positive integer n we have

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

3. Determine the number of functions $f \colon \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ which are a) strictly increasing, b) nondecreasing, c) surjective.

4. Show the following formula for the exponential generating function of the Stirling numbers of the second kind

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{1}{k!} \left(e^x - 1\right)^k.$$

5. In how many ways u_n can one mount a staircase with n steps if every movement involves only one or two steps?

6. Let D_n be the number of sequences $(x_1, \ldots, x_{2n})$ such that $x_1, \ldots, x_{2n}$ take values ± 1, $x_1 + \ldots + x_k \geq 0$ for every $1 \leq k \leq 2n$ and $x_1 + \ldots + x_{2n} = 0$. Prove that

$$D_n = D_{n-1} + D_1 D_{n-2} + \ldots + D_{n-1}$$

and conclude $D_n = \frac{1}{n+1}\binom{2n}{n}$.

7. Let $T_1, \ldots, T_k$ be subtrees of a tree T with the property that each two of them have at least one vertex in common. Show that all of them has at least one vertex in common.

8. Let $d_1, \ldots, d_n$ be positive integers such that $d_1 \leq d_2 \leq \ldots \leq d_n$. Show that there exists a tree with n vertices of degrees $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2.$$

9. Suppose that the vertices of a maximal plane graph are coloured with 3 colours. Show that the number of faces whose vertices have all three colours is even.

10. Suppose that a plane graph on $n \geq 3$ vertices contains no triangle. Show that it has at most $2n-4$ edges.

11. Recall that $R_k(3)$ is the smallest number n such no matter how K_n is k-coloured, it contains a monochromatic triangle. It was shown in Assignment 4 that

$$R_k(3) \leq \lfloor k!e \rfloor + 1.$$

Prove that

$$R_k(3) \geq 2^k + 1.$$

Solutions

1. We will prove the desired identity by making up a story. Let us count the number of $0-1$ sequences of length $2n+1$ in a particular way. Notice that in every such sequence either 0 or 1 is repeated at least $n+1$ times. Thus for $k = 0, \dots, n$ let A_k be the set of all such sequences for which 1 is repeated the $n+1^{\text{st}}$ time only at the $n+1+k^{\text{th}}$ place. We have

$$|A_k| = \binom{n+k}{k} \cdot 2^{2n+1-(n+k+1)}$$

because every sequence in A_k looks like

$$\underbrace{\star\star\star\star\ldots\ldots\star}_{\substack{n+k \text{ terms}\\ \text{containing}\\ \text{exactly } n \text{ 1's}}}\ 1\ \underbrace{\star\star\star\star\ldots\ldots\star}_{2n+1-(n+k+1)}.$$

By symmetry we get

$$2^{2n+1} = 2\sum_{k=0}^{n} |A_k| = 2\sum_{k=0}^{n} \binom{n+k}{n} 2^{n-k}$$

which gives

$$2^n = \sum_{k=0}^{n} \binom{n+k}{n} 2^{-k}. \quad \square$$

2. Suppose we have n men and n women and from these $2n$ people we want to select a team of n people with a female captain. We can do it in $n\binom{2n-1}{n-1}$ ways by first selecting the female captain and then choosing $n-1$ people among the remaining $2n-1$. On the other hand, for $k = 1, \dots, n$ we can first choose k

women in $\binom{n}{k}$ ways, among them choose the captain in k ways and then choose $n-k$ men in $\binom{n}{n-k} = \binom{n}{k}$ ways.

Another solution. We have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$
$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Multiplying these identities and equating the coefficients at x^{n-1} yields the result. □

3. a) We want to choose $f(1),\dots,f(m)$ so that $1 \le f(1) < \dots < f(m) \le n$. Therefore we want to choose m *distinct* numbers among $1,\dots,n$. There are $\binom{n}{m}$ such choices.

b) Now we require $1 \le f(1) \le \dots \le f(m) \le n$. In other words, we want to select m numbers among $1,\dots,n$ allowing repetitions, or put m oranges into n boxes. Therefore, there are $\binom{n-1+m}{m}$ such choices.

c) For $i = 1,\dots,n$, let A_i be the set of functions $f\colon \{1,\dots,m\} \longrightarrow \{1,\dots,n\}$ *not* taking value i. We have $|A_i| = (n-1)^m$, $|A_i \cap A_j| = (n-2)^m$, $i < j$, etc. The number of surjective functions is $n^m - |A_1 \cup \dots \cup A_n|$ which by the exclusion-inclusion formula gives the answer

$$n^m - n(n-1)^m + \binom{n}{2}(n-2)^m - \dots + (-1)^{n-1}n. \quad \square$$

4. Using the explicit formula for the Stirling number

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n$$

we get (notice that the sums can be swapped because the series converges absolutely)

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \sum_{n=k}^{\infty}\sum_{j=0}^{k} \frac{1}{k!}\binom{k}{j}(-1)^{k-j} j^n \frac{x^n}{n!}$$
$$= \sum_{j=0}^{k} \frac{1}{k!}\binom{k}{j}(-1)^{k-j}\left(\sum_{n=k}^{\infty} \frac{(jx)^n}{n!}\right).$$

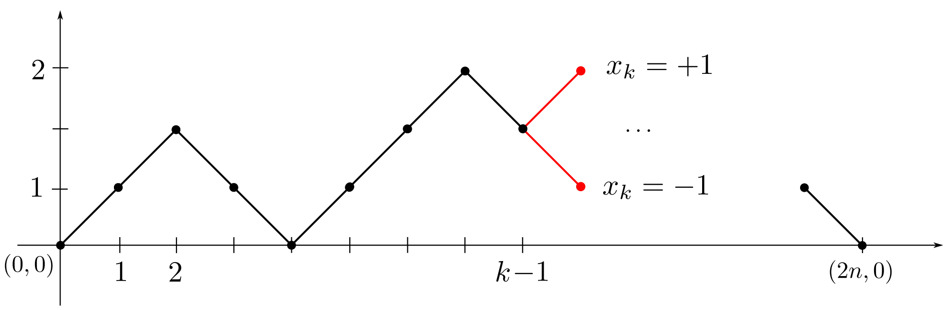
Notice that $\sum_{n=k}^{\infty}\frac{(jx)^n}{n!} = e^{jx} - \sum_{n=0}^{k-1}\frac{(jx)^n}{n!}$. Therefore,

$$\begin{aligned}\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}\frac{x^n}{n!} &= \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}(-1)^{k-j}\left(e^{jx} - \sum_{n=0}^{k-1}\frac{(jx)^n}{n!}\right)\\ &= \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}(-1)^{k-j}e^{jx} - \sum_{n=0}^{k-1}\frac{1}{k!}(-1)^k\frac{x^n}{n!}\sum_{j=0}^{k}\binom{k}{j}(-j)^n\\ &= \frac{1}{k!}\left(e^x - 1\right)^k,\end{aligned}$$

where in the last equality we used the orthogonality of the binomial coefficients $\binom{k}{j}$ to the sequence $\left((-j)^n\right)_{j=0,\ldots,k}$ for $n \leq k-1$. □

5. Clearly, $u_1 = 1$ and $u_2 = 1$ (we assume we start at the first step). We also have $u_n = u_{n-1} + u_{n-2}$ because if the first movement is by 1 step, then we still have to climb the remaining $n-1$ steps. If the first movement is by 2 steps, then we still have to climb the remaining $n-2$ steps. Therefore the u_n are the Fibonacci numbers. □

6. The number D_n is in fact the number of zigzag paths in the plane going from $(0,0)$ to $(2n,0)$ and staying nonnegative (see the picture).



For $k = 2, 4, \ldots, 2n$ consider the paths which hit the $0x$ axis for the first time at k. When $k = 2$ there are D_{n-1} such paths, when $k = 4$ there are D_1D_{n-2} such paths, when $k = 6$ there are D_2D_{n-3} such paths, and so on, when $k = 2n$ there are D_{n-1} such paths. Therefore

$$D_n = D_{n-1} + D_1D_{n-2} + \ldots + D_{n-2}D_1 + D_{n-1}.$$

Since $D_1 = 1$, D_n is the Catalan number, hence $D_n = \frac{1}{n+1}\binom{2n}{n}$. □

7. We proceed by induction on the number of vertices of T. If T is a single vertex, then the statement is clear. Suppose T has more than 1 vertex, choose its leaf, say x, connected to, say y and consider the tree $T \setminus \{x\}$. If for some i,

T_i is the single vertex x, then since T_i shares a vertex with every other T_j, all the subtrees have x as a common vertex. Now consider the other case when $T_i \setminus \{x\} \neq \varnothing$ for every i. The subtrees $T_i \setminus \{x\}$ of the tree $T \setminus \{x\}$ also satisfy the property that every two of them share a vertex (If T_i and T_j share x, they also share y, so do $T_i \setminus \{x\}$, $T_j \setminus \{x\}$). By induction, they all share a vertex, so the T_i as well. $\square$

8. If there is such a tree, then the formula follows from the hand shaking lemma as T has $n-1$ edges.

The other implication will be shown inductively on n. The case $n = 1$ is trivial. Suppose $d_1 + \ldots + d_n = 2n - 2$. Then $d_1 = 1$ (otherwise $d_1 + \ldots + d_n \geq 2n$) and $d_n \leq n - 1$ (otherwise $d_1 + \ldots + d_{n-1} + d_n \geq n - 1 + n = 2n - 1$). So

$$d_2 + \ldots + (d_n - 1) = 2n - 4$$

and applying the inductive assumption to $d_1, \ldots, d_n - 1$ we get a tree on $n-1$ vertices with degrees $d_1, \ldots, d_n - 1$. Add a leaf to it at the vertex with degree $d_n - 1$. $\square$

9. Suppose the colours are b, r, y (blue, red, yellow). If the vertices of a face are coloured with three different colours, then the number of edges at this face of type $\{b, r\}$ equals 1. If not, then this number equals 0 or 2. Double-count the number of pairs (a face f, an edge of type $\{b, r\}$ in f). On one hand, it is even because each edge belongs to two faces. On the other hand, it equals

$$\sum_{f\text{-face}} |\{\text{edges } \{b, r\} \text{ on the boundary of } f\}|$$
$$= \underbrace{(1 + 1 + \ldots + 1)}_{\text{faces with all colours}} + (0 + 0 + \ldots + 0) + (2 + 2 + \ldots + 2),$$

hence the number of faces with all 3 colours is even. $\square$

10. Suppose that the number of edges is e and the number of faces is f. Euler's formula gives $n + f = e + 2$. Let e' be the number of edges which are on the boundary between exactly two faces. If $e' = 0$, then our graph is a tree, hence $e = n - 1 \leq 2n - 4$ as $n \geq 3$. If $e' > 0$, we can double-count

$$2e \geq 2e' + (e - e') \geq |\{(\gamma, F), \gamma \text{ is an edge on the boundary of a face } F\}|$$
$$\geq 4f = 4(e + 2 - n),$$

so

$$e \leq 2(n - 2). \quad \square$$

11. Let n_k be the largest n such that there is a colouring of K_n without a monochromatic triangle. We want to show that $n_k \geq 2^k$. Obviously, $n_1 = 2$. Now we show inductively on k that $n_k \geq 2n_{k-1}$, $k \geq 2$. We take two copies G and G' of $K_{n_{k-1}}$, colour each one with $k-1$ colours so that none contains a monochromatic triangles. Now we build $K_{2n_{k-1}}$ by adding all possible edges across G, G', that is we add the edges $\{v, v'\}$ for every $v \in V(G)$, $v \in V(G')$. We colour these edges with the k^{th} colour obtaining a $K_{2n_{k-1}}$ which is k-coloured without a monochromatic triangle. Therefore, $n_k \geq 2n_{k-1}$. □

References

T I. Tomescu, Problems in combinatorics and graph theory. Translated from the Romanian by R. A. Melter. A Wiley-Interscience Publication. *John Wiley & Sons, Ltd., Chichester*, 1985.