Combinatorial Optimisation, Problems I

Term 2 2014/2015

Yuchen Pei, y (dot) pei (at) warwick (dot) ac (dot) uk & Tomasz Tkocz, t (dot) tkocz (at) warwick (dot) ac (dot) uk

The questions marked " [Npts]" will be assessed (N denotes the total mark). They are due on **January** 30^{th}, **2 pm**. You may use any results proved in the lecture as well as the support class. Please fill in and include the **cover sheet** as the front page of your submission (failure to do so may result in rejection of your work). It will help your TAs process the scripts.

1. Show that in a simple undirected graph $G = G(V, E)$ we have $\sum_{v \in V} \deg(v) = 2|E|$.

2. Show that if T is a tree on a vertex set V with $|V| \geq 2$, then the number of leaves (vertices of degree one) in T equals
$$2 + \sum_{\substack{v \in V \\ \deg(v) \geq 3}} (\deg(v) - 2).$$
Conclude that a tree which is not a single vertex has at least two leaves.

3. [10pts] Show that if Δ is the maximum degree of a vertex in a tree, then the tree contains at least Δ leaves.

4. Let G be a simple undirected graph. Show that G or its complement is connected.

5. Let $G(V, E)$ be a simple connected undirected graph, let (V, F) be a forest in G. Prove that there is a spanning tree (V, T) with $F \subset T \subset E$.

6. Let $(V, T_1), (V, T_2)$ be two trees on the same vertex set V. Prove that for every edge $e \in T_1$ there is an edge $f \in T_2$ such that $(V, (T_1 \setminus \{e\}) \cup \{f\})$ and $(V, (T_2 \setminus \{f\}) \cup \{e\})$ are trees.

7. [10pts] Let $G(V, E)$ be a simple connected undirected graph with weights $c \colon E \longrightarrow \mathbb{R}$ such that $c(e) \neq c(e')$ for all edges $e \neq e'$. Show that there is exactly one optimum spanning tree for G.

8. Suppose we wish to find a spanning tree T in an undirected graph such that the maximum weight of an edge in T is as small as possible. How can this be done?

9. [5pts] Give an example of a digraph and show that Dijkstra's algorithm might not work correctly when negative weights are allowed.

03/02/2015

Combinatorial Optimisation, Problems II

Term 2 2014/2015

Yuchen Pei, y (dot) pei (at) warwick (dot) ac (dot) uk & Tomasz Tkocz, t (dot) tkocz (at) warwick (dot) ac (dot) uk

The questions marked " [Npts]" will be assessed (N denotes the total mark). They are due on **February** 13^{th}, **2 pm**. You may use any results proved in the lecture as well as the support class. Please fill in and include the **cover sheet** as the front page of your submission (failure to do so may result in rejection of your work). It will help your TAs process the scripts.

1. Let f be an s-t-flow in a network (G, u, s, t). Recall that for a vertex v we define

$$\mathrm{ex}_f(v) = \sum_{e \in \delta^-(v)} u(e) - \sum_{e \in \delta^+(v)} u(e)$$

and

$$\mathrm{value}(f) = -\mathrm{ex}_f(s).$$

Prove that

$$\mathrm{value}(f) = \mathrm{ex}_f(t).$$

2. [5pts] Give examples of networks (G, u, s, t) possessing

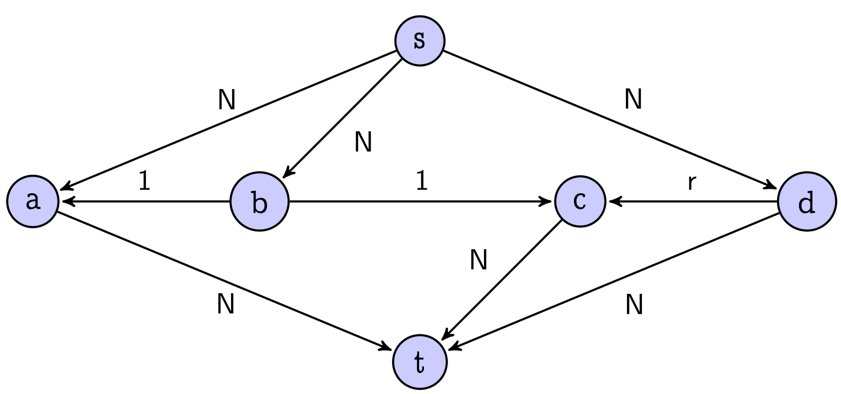
(a) more than one maximum s-t-flows and more than one minimum s-t-cuts,

(b) more than one maximum s-t-flows, and a unique minimum s-t-cut,

(c) a unique maximum s-t-flow, and more than one minimum s-t-cuts,

(d) a unique maximum s-t-flow, and a unique s-t-minimum cut.

3. For example, by considering the following network and the augmenting paths $p_0 = (s, b, c, t)$, $p_1 = (s, d, c, b, a, t)$, $p_2 = (s, b, c, d, t)$ and $p_3 = (s, a, b, c, t)$, show that for nonintegral capacities, the Ford-Fulkerson algorithm might not terminate. ($r = \frac{\sqrt{5}-1}{2} = 1 - r^2$, N is a sufficiently large number.)



4. Let (G, u, s, t) be a network. If $X, Y \subset V(G)$ are such that $\delta^+(X)$ and $\delta^+(Y)$ are minimum s-t-cuts in (G, u), then so are $\delta^+(X \cup Y)$ and $\delta^+(X \cap Y)$.

5. Let G be an undirected graph with capacities $u \colon E(G) \longrightarrow \mathbb{R}_+$. For two vertices $s, t \in V(G)$ we define their *local edge connectivity*, λ_{st}, to be the minimum capacity of a cut separating s and t, that is the minimum of $\sum_{e \in \delta(A)} u(e)$ over all subsets A of $V(G)$ such that $s \in A$ but $t \notin A$.

   Show that for all vertices $r, s, t \in V(G)$ we have $\lambda_{rt} \geq \min\{\lambda_{rs}, \lambda_{st}\}$.

6. [10pts] Prove the converse: let λ_{ij}, $1 \leq i, j \leq n$, be nonnegative numbers with $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{ik} \geq \min\{\lambda_{ij}, \lambda_{jk}\}$ for all distinct indices $i, j, k \leq n$; show that there exists a graph with $V(G) = \{1, \ldots, n\}$ and capacities $u \colon E(G) \longrightarrow \mathbb{R}_+$ such that the local edge connectivities in G are λ_{ij}.

   Hint: Consider a maximum weight spanning tree in (K_n, c), where $c(\{i, j\}) = \lambda_{ij}$.

7. Prove Hoffman's circulation theorem: Given a digraph G and lower and upper capacities $l, u \colon E(G) \longrightarrow \mathbb{R}_+$ with $l(e) \leq u(e)$ for all $e \in E(G)$, there is circulation f with $l(e) \leq f(e) \leq u(e)$ for all $e \in E(G)$ if and only if

$$\sum_{e \in \delta^-(X)} l(e) \leq \sum_{e \in \delta^+(X)} u(e) \qquad \text{for all } X \subset V(G).$$

   Recall that $f \colon E(G) \longrightarrow \mathbb{R}_+$ is a circulation if $\mathrm{ex}_f(v) = 0$ for every vertex $v \in V(G)$.

8. Derive the Max-Flow-Min-Cut theorem from Hoffman's circulation theorem.

   Hint: Suppose m is the value of a minimum cut. Add the edge $e_0 = (t, s)$ to G and set l to be 0 everywhere but m at e_0 and set u to agree with given capacities everywhere but at e_0 where we set u to be m.

9. Let G be an undirected graph. Show how to compute an orientation G' of G such that for each $v, w \in V(G)$ the following holds: if G has two edge-disjoint v-w-paths, then G' has a (directed) v-w-path.

   Hint: Use DFS.

10. [10pts] Consider an undirected graph G with edge-connectivity $k \in \mathbb{N}$ and (not necessarily distinct) vertices $v_0, v_1, \ldots, v_k \in V(G)$. Prove that there are pairwise edge-disjoint paths $P_1, \ldots, P_k$ such that P_i is a v_0-v_i-path $(i = 1, \ldots, k)$.

11. Let G be an undirected graph, x, y, z three vertices, and α, β nonnegative integers such that $\alpha \leq \lambda_{xy}$, $\beta \leq \lambda_{xz}$ and $\alpha + \beta \leq \max\{\lambda_{xy}, \lambda_{xz}\}$. Prove that there are α x-y paths, β x-z paths such that these $\alpha + \beta$ paths are pairwise edge-disjoint.

10/02/2015

Combinatorial Optimisation, Problems III

Term 2 2014/2015

Yuchen Pei, y (dot) pei (at) warwick (dot) ac (dot) uk & Tomasz Tkocz, t (dot) tkocz (at) warwick (dot) ac (dot) uk

The questions marked “ [Npts]” will be assessed (N denotes the total mark). They are due on **February** 27^{th}, **2 pm**. You may use any results proved in the lecture as well as the support class. Please fill in and include the **cover sheet** as the front page of your submission (failure to do so may result in rejection of your work). It will help your TAs process the scripts.

1. [5pts] Find α (the independence number), τ (the size of a minimum vertex cover), μ (the matching number), the edge and vertex connectivity for $K_{m,n}$, and justify your answers.

2. A matching M in G is called *awesome* if for every edge $e \in E(G)$, $M \cup \{e\}$ is not a matching (in other words, it is not possible to extend M to a bigger matching). Show that a maximum (cardinality) matching is awesome. Does the converse hold?

3. Let M be a matching and let M' be an awesome matching in G. Show that $|M| \le 2|M'|$.

4. Show that every tree has at most one perfect matching.

5. (A magic trick) A standard 52-card deck is divided into 13 piles of 4 cards each. Prove that it is possible to get all 13 ranks by selecting one card from every pile.

6. [10pts] Let $r \le n$. An $r \times n$ matrix is called a *brilliant rectangle* if it has the entries in the set $\{1, \dots, n\}$ and no number occurs more than once in any row or column. An $n \times n$ brilliant rectangle is called a *brilliant square*.

Prove that if $r < n$, then it is possible to append $n - r$ rows to an $r \times n$ brilliant rectangle to form a brilliant square.

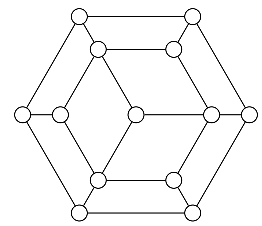
Hint: Hall's theorem.

7. Show how to solve the Maximum Independent Set problem for trees using the neighbour reduction lemma.

8. Let G be an undirected graph. Show that the number of edges in its line graph $L(G)$ equals

$$\frac{1}{2} \sum_{v \in V(G)} \big(\deg(v)\big)^2 - |E(G)|.$$

9. Decide whether it is possible for a spider to visit every vertex of the web shown in the picture exactly once and return to the starting point. (The spider can move only along the edges.)
Can you see how to generalise this observation?



10. [10pts] A graph is called *Hamiltonian* if it possesses a circuit that visits every vertex of the graph exactly once.

Let G be an undirected Eulerian graph. Show that the line graph $L(G)$ is both Hamiltonian and Eulerian.

If $L(G)$ is Hamiltonian, does G have to be Eulerian? (If *yes*, prove it, if *no*, provide an example well-explained.)

11. Let G be a bipartite graph with vertices $v_1, \ldots, v_n$ on one side and $w_1, \ldots, w_n$ on the other side. Let M be the $n \times n$ matrix with 1 at the entry (i, j) if v_i and w_j are adjacent in G, otherwise it is 0. This matrix is called the adjacency matrix of G.

For an $n \times n$ matrix $A = [a_{ij}]$ we define the *permanent* of A by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the sum is over all permutations σ of the set $\{1, \ldots, n\}$.

(a) Show that the number of perfect matchings in G equals $\operatorname{per}(A)$.

(b) Egoryčev (1980) and Falikman (1981) showed that for a nonnegative matrix A whose column sums and row sums are all 1 we have $\operatorname{per}(A) \geq n!/n^n$. (This was a famous conjecture of van der Waerden from 1926.) Using this result show that if G is k-regular, then it has at least $n!(k/n)^n$ perfect matchings.

(c) Brègman (1973) showed that for a 0-1 matrix A with row sums $r_1, \ldots, r_n$ we have $\operatorname{per}(A) \leq (r_1!)^{1/r_1} \cdot \ldots \cdot (r_n!)^{1/r_n}$. Using this result show that if G is k-regular, then it has at most $k!^{n/k}$ perfect matchings.

(d) Can these bounds be attained?

(e) Show that for a nonnegative matrix A whose column sums and row sums are all 1 we have $\operatorname{per}(A) \leq 1$. Can this bound be attained?

14/03/2015

Combinatorial Optimisation, Problems IV

Term 2 2014/2015

Yuchen Pei, y (dot) pei (at) warwick (dot) ac (dot) uk & Tomasz Tkocz, t (dot) tkocz (at) warwick (dot) ac (dot) uk

The questions marked " [Npts]" will be assessed (N denotes the total mark). They are due on **March 13th, 2 pm**. You may use any results proved in the lecture as well as the support class. Please fill in and include the **cover sheet** as the front page of your submission (failure to do so may result in rejection of your work). It will help your TAs process the scripts.

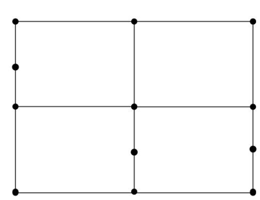
1. [10pts] Consider the Gale-Shapley algorithm.

(a) Suppose a man M and a woman W both have each other ranked as the first in their lists, does the output matching produced by the algorithm have to contain (M, W)?

(b) Suppose a man M_0 is the second on every woman's list, is it possible that the male-proposing algorithm will produce a matching of which M_0 is matched with the woman ranked at the bottom of his list?

(c) Suppose there are N men and N women in total, and all the men have the identity list. How many proposals is required for the male-proposing algorithm to produce an output?

2. Solve the Chinese Postman Problem for the following graph by solving the corresponding minimum T-join problem, where each edge has weight 1. Write down steps including specifying T, the minimum T-join, and the auxiliary graph G^*.



3. Let $G = (V, E)$ be an undirected graph. Given a vertex set $T \subset V$, a T-**cut** is a cut $\delta(X)$ such that $|X \cap T|$ is odd. Suppose $|T|$ is even and $F \subset E$. Show that

(a) $F \cap J \neq \emptyset$ for any T-join J if and only if F contains a T-cut.

(b) $F \cap C \neq \emptyset$ for any T-cut C if and only if F contains a T-join.

4. [5pts] Given a graph $G = (V, E)$ (directed or undirected), and $c : E \to \mathbb{R}$ and $s, t \in V$ such that t is reachable from s, let $\mathcal{F} = \{F \subset E : F \text{ is a subset of edges of an } s\text{-}t\text{-path}\}$. Show that $(E, \mathcal{F})$ is an indepdence system but not necessarily a matroid.

5. For an independence system $(E, \mathcal{F})$, define its **dual** $(E, \mathcal{F}^*)$ such that $\mathcal{F}^*$ is the set of all $F \subset E$ such that there is a basis B of $(E, \mathcal{F})$ with $F \cap B = \emptyset$. Show that $(E, \mathcal{F}^*)$ is also an independence system.

6. Show that in the following $(E, \mathcal{F})$ are metroid.

   (a) Let G be a digraph, $E = V(G)$ and fix $s \in V(G)$. Let $\mathcal{F}$ be the set of all the vertex sets $I \subset E$ such that there are edge-disjoint paths from s to each vertex in I.

   (b) Let G be a graph and $E = V(G)$. Let $\mathcal{F}$ be the set of vertex sets $X \subset E$ such that a maximum matching exists that covers no vertex in X.

7. [10pts] Let E be a finite set and $\mathcal{B} \subset 2^E$. Show that a nonempty set $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if for any $B_1, B_2 \in \mathcal{B}$ and $y \in B_2 \setminus B_1$ there exists an $x \in B_1 \setminus B_2$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

8. (Tricky) Let $A_1, \ldots, A_{n+1}$ be nonempty subsets of the set $\{1, \ldots, n\}$. Prove that there exist two nonempty disjoint subsets I and J of the set $\{1, \ldots, n+1\}$ such that

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$