Brownian Motion I
Solutions

Question 1. Let B be a standard linear Brownian motion. Show that for any $0 < t_1 < t_2 < \ldots < t_k$ the joint distribution of the vector $(B_{t_1}, \ldots, B(t_k))$ is Gaussian and compute the covariance matrix.

Solution. The vector $G = \left(\frac{B(t_1)}{\sqrt{t_1}}, \frac{B(t_2)-B(t_1)}{\sqrt{t_2-t_1}}, \ldots, \frac{B(t_n)-B(t_{n-1})}{\sqrt{t_n-t_{n-1}}}\right)$ has the standard Gaussian distribution. Thus, the vector $X = (B(t_1), \ldots, B(t_n))$, as a linear image of G, has a Gaussian distribution. Since $\mathbb{E}B(t_i)B(t_j) = t_i \wedge t_j$ (assuming that $B(t)$ is a standard Brownian motion, otherwise we have to subtract the mean), the covariance matrix of X equals $[t_i \wedge t_j]_{i,j \leq n}$ □

Question 2. (This exercise shows that just knowing the finite dimensional distributions is not enough to determine a stochastic process.) Let B be Brownian motion and consider an independent random variable U uniformly distributed on $[0, 1]$. Show that the process

$$\tilde{B}_t = \begin{cases} B_t, & t \neq U, \\ 0, & t = U \end{cases}$$

has the same finite dimensional distributions as B but a.s. it is not continuous.

Solution. Given $0 \leq t_1 < \ldots < t_n \leq 1$, on the even $\{U \neq t_i,\ i = 1, \ldots, n\}$, which has probability 1, we have that $(\tilde{B}(t_1), \ldots, \tilde{B}(t_n)) = (B(t_1), \ldots, B(t_n))$, so $\tilde{B}$ and B have the same finite dimensional distributions. Since, $\mathbb{P}\left(\lim_{t\to U} \tilde{B}_t = \tilde{B}_U\right) = \mathbb{P}(B_U = 0) = \int_0^1 \mathbb{P}(B_u = 0)\,du = 0$, the process $\tilde{B}$ is *not* continuous a.s. □

Question 3. Let $B(\cdot)$ be a standard linear Brownian motion. Prove that

$$\mathbb{P}\left(\sup_{s,t\in(0,1)} \frac{|B(s) - B(t)|}{|s-t|^{1/2}} = \infty\right) = 1.$$

Solution. Consider the events

$$A_n = \left\{\left|B\left(\frac{1}{n+1}\right) - B\left(\frac{1}{n}\right)\right| \geq \sqrt{2\ln n}\left|\frac{1}{n+1} - \frac{1}{n}\right|^{1/2}\right\}.$$

They are independent. Using a usual estimate for the tail of the standard Gaussian r.v. (see, e.g., Lemma 12.9 in [P. Mörters, Y. Peres, *Brownian Motion*]),

$$\mathbb{P}(A_n) = \mathbb{P}\left(|N(0,1)| \geq \sqrt{2\ln n}\right) \geq \frac{2}{\sqrt{2\pi}} \frac{\sqrt{2\ln n}}{\sqrt{2\ln n}^2 + 1} e^{-\sqrt{2\ln n}^2/2} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{n\sqrt{\ln n}},$$

so $\sum_n \mathbb{P}(A_n) = \infty$. By the Borel-Cantelli lemma, $\mathbb{P}(\limsup A_n) = 1$, i.e. with probability 1, infinitely many of A_n's occur. In particular, $\sup_{s,t\in(0,1)} |B(s) - B(t)|/|s-t|^{1/2} = \infty$ with probability 1. $\square$

Brownian Motion II
Solutions

Question 1. Show that a.s. linear Brownian motion has infinite variation, that is

$$V_B^{(1)}(t) = \sup \sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}| = \infty$$

with probability one, where the supremum is taken over all partitions (t_j), $0 = t_0 < t_1 < \ldots < t_k = t$, of the interval $[0, t]$.

Solution. It was shown in the lecture that

$$\sup \sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}|^2 \xrightarrow[\text{a.s.}]{k\to\infty} t,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = t$. We have

$$\sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}|^2 \leq V_B^{(1)}(t) \cdot \sup_j |B_{t_j} - B_{t_{j-1}}|.$$

By the uniform continuity of B on $[0, t]$ we get that as k goes to infinity, the supremum on the right hand side goes to 0 if the diameter of the partition (t_k) goes to zero. The left hand side goes to a positive t a.s., hence $V_B^{(1)}(t) = \infty$ a.s. □

Question 2. Let B be a standard linear Brownian motion. Define

$$D^*(t) = \overline{\lim}_{h\to 0} \frac{B_{t+h} - B_t}{h}, \qquad D_*(t) = \underline{\lim}_{h\to 0} \frac{B_{t+h} - B_t}{h}.$$

It was shown in the lecture that a.s., for every $t \in [0, 1]$ either $D^*(t) = +\infty$ or $D_*(t) = -\infty$ or both. Prove that

(a) for every $t \in [0, 1]$ we have $\mathbb{P}(B \text{ has a local maximum at } t) = 0$

(b) almost surely, local maxima of B exist

(c) almost surely, there exist $t_*, t^* \in [0, 1]$ such that $D^*(t^*) \leq 0$ and $D_*(t_*) \geq 0$.

Solution. Fix $t \in (0,1)$. We have

$$\begin{aligned}
\mathbb{P}(t \text{ is a local maximum of } B) &= \mathbb{P}(\exists \epsilon > 0 \ \forall 0 < |h| < \epsilon \ B_t - B_{t+h} \geq 0) \\
&\leq \mathbb{P}(\exists \epsilon > 0 \ \forall 0 < h < \epsilon \ B_t - B_{t+h} \geq 0) \\
&= \mathbb{P}(\exists \epsilon > 0 \ \forall 0 < h < \epsilon \ B_h \geq 0) \\
&= 1 - \mathbb{P}\left(\forall \epsilon > 0 \ \sup_{0<h<\epsilon} B_h\right) > 0 \\
&= 1 - \mathbb{P}\left(\forall n = 1, 2, \ldots \ \sup_{0<h<1/n} B_h > 0\right).
\end{aligned}$$

The event

$$A = \left\{\forall n = 1, 2, \ldots \ \sup_{0<h<1/n} B_h > 0\right\} = \bigcap_{n=1}^{\infty} \left\{\sup_{0<h<1/n} B_h > 0\right\}$$

belongs to $\mathcal{F}_{0+} = \bigcap_{t>0} \mathcal{F}_t$. By Blumenthal's 0-1 law, $\mathbb{P}(A) \in \{0,1\}$. But

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{0<h<1/n} B_h > 0\right) \geq \lim_{n\to\infty} \mathbb{P}\left(B_{1/(2n)} > 0\right) = \frac{1}{2}.$$

Hence, $\mathbb{P}(A) = 1$ and, consequently, $\mathbb{P}(t \text{ is a local maximum of } B) = 1$.

It follows from the continuity of paths that a global maximum of B on $[0,1]$ always exists, which is also a local maximum.

If we take t^* to be a local maximum and t_* to be a local minimum, then $D^*(t^*) \leq 0$ and $D_*(t_*) \geq 0$. □

Question 3. Let B be a standard linear Brownian motion. Show that a.s.

$$\overline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} = +\infty \text{ and } \underline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} = -\infty.$$

You may want to use the Hewitt-Savage $0-1$ law which states that

Theorem (Hewitt-Savage). *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. variables. An event $A = A(X_1, X_2, \ldots)$ is called* exchangeable *if $A(X_1, X_2, \ldots) \subset A(X_{\sigma(1)}, X_{\sigma(2)}, \ldots)$ for any permutation σ of the set $\{1, 2, \ldots\}$ whose support $\{k \geq 1, \ \sigma(k) \neq k\}$ is a finite set. Then for every exchangeable event A we have $\mathbb{P}((A) \in \{0,1\})$.*

Solution. Fix $c > 0$ and take $A_c = \limsup_n \{B_n > c\sqrt{n}\}$. We want to show that $\bigcap_{c=1}^{\infty} A_c$ has probability one. Plainly, $\mathbb{P}\left(\bigcap_{c=1}^{\infty} A_c\right) = \lim_{c\to\infty} \mathbb{P}(A_c)$. Let $X_n = B_n - B_{n-1}$. They are i.i.d. Notice that

$$A_c = \limsup_n \left\{ \sum_{j=1}^{n} X_j > c\sqrt{n} \right\}$$

is an exchangeable event. By the Hewitt-Savage 0-1 law we obtain that $\mathbb{P}(A_c) \in \{0, 1\}$. Since

$$\mathbb{P}(A_c) \geq \limsup_{n\to\infty} \mathbb{P}\left(B_n > c\sqrt{n}\right) = \mathbb{P}(B_1 > c) > 0,$$

we conclude that $\mathbb{P}(A_c) = 1$.

The claim about lim inf can be proved similarly. □

Brownian Motion III
Solutions

Question 1. Let $f\colon [0,\infty)\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a smooth function and let B be standard Brownian motion in $\mathbb{R}^d$. Show that

$$M_t = f(t,B_t) - \int_0^t \left(f_t + \frac{1}{2}\Delta f\right)(s,B_s)ds$$

is a martingale. Using this, write a solution to the problem

$$\begin{cases} u_t &= \frac{1}{2}\Delta u, \qquad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) &= f(x), \qquad \text{on } \mathbb{R}^d, \end{cases}$$

where f is a given, smooth, compactly supported function (the initial condition)

Solution. In order to show that

$$M_t = f(t,B_t) - \int_0^t \left(f_t + \frac{1}{2}\Delta f\right)(s,B_s)ds$$

is a martingale, it is enough to follow closely the proof of Theorem 2.51 from [P. Mörters, Y. Peres, *Brownian Motion*].

Now we construct a solution to the problem

$$\begin{cases} u_t &= \frac{1}{2}\Delta u, \qquad t>0, x\in\mathbb{R}^d, \\ u(0,x) &= f(x), \qquad x\in\mathbb{R}^d, \end{cases}$$

where f is a given, smooth, compactly supported function (the initial condition). Let

$$u(t,x) = \mathbb{E}_x f(B_t).$$

Plainly, $u(0,x) = \mathbb{E}_x f(B_0) = f(x)$. Moreover, using the martingale property we just mentioned we have $\mathbb{E}_x M_t = \mathbb{E}_x M_0 = f(x)$, so

$$u(t,x) = f(x) + \mathbb{E}_x \int_0^t \frac{1}{2}(\Delta f)(B_s)ds.$$

Since f is compactly supported, f and all its derivatives are bounded. Therefore we can swap the integrals as well as the Laplacian and write

$$\begin{aligned} u(t,x) &= f(x) + \int_0^t \mathbb{E}_x \frac{1}{2}\Delta f(B_s)ds = f(x) + \int_0^t \mathbb{E}\frac{1}{2}(\Delta f)(x+B_s)ds \\ &= f(x) + \int_0^t \mathbb{E}\frac{1}{2}\Delta\big(f(x+B_s)\big)ds = f(x) + \int_0^t \frac{1}{2}\Delta\big(\mathbb{E}f(x+B_s)\big)ds \\ &= f(x) + \int_0^t \frac{1}{2}\Delta\big(\mathbb{E}_x f(B_s)\big)ds. \end{aligned}$$

Taking the time derivative yields

$$u_t = \frac{1}{2}\Delta\big(\mathbb{E}_x f(B_t)\big) = \frac{1}{2}\Delta u.$$

To come up with this solution, we could alternatively suppose that u solves the problem, observe that $M_t = u(t_0 - t, B_t)$ is a martingale which yields that

$$u(t_0, x) = \mathbb{E}_x u(t_0, B_0) = \mathbb{E}_x M_0 = \mathbb{E}_x M_{t_0} = \mathbb{E}_x u(0, B_{t_0}) = \mathbb{E}_x f(B_{t_0}),$$

so u is of the form $\mathbb{E}_x f(B_t)$ □

Question 2. Consider the problem

$$\begin{cases} u_t &= \frac{1}{2}\Delta u, & \text{in } \mathbb{R}_+ \times B(0,1), \\ u(0,x) &= f(x), & \text{on } B(0,1), \\ u(t,z) &= g(t,z), & \text{on } \mathbb{R}_+ \times \partial B(0,1). \end{cases}$$

(the heat equation in the cylinder $\mathbb{R}_+ \times B(0,1)$; $B(0,1) \subset \mathbb{R}^2$ is the unit disk centred at the origin), where f, g are smooth functions on $B(0,1$ and $\mathbb{R}_+ \times \partial B(0,1)$ respectively (the initial data). Show that a solution to this problem is of the form

$$u(t,x) = \mathbb{E}_x f(B_t)\mathbf{1}_{\{t \le \tau\}} + \mathbb{E}_x g(t-\tau, B_\tau)\mathbf{1}_{\{t > \tau\}}$$

where B is a standard planar Brownian motion and τ is the hitting time of $\partial B(0,1)$.

Solution. Suppose we have a solution u and we want to determine its form. Fix (t_0, x) in $\mathbb{R}_+ \times B(0,1)$ and consider

$$M_t = u(t_0 - t, B_t).$$

From Question 1 we know that this is a martingale as long as B_t stays in $B(0,1)$ so that u solves the heat equation. Therefore, defining τ to be the hitting time of $\partial B(0,1)$,

$(M_t)_{t\in[0,\tau]}$ is a martingale. Using Doob's optional stopping theorem for $0 \leq t \wedge \tau$ we thus get

$$u(t_0, x) = \mathbb{E}_x M_0 = \mathbb{E}_x M_{t\wedge\tau} = \mathbb{E}_x u(t_0 - t, B_t)\mathbf{1}_{\{t\leq\tau\}} + \mathbb{E}_x u(t_0 - \tau, B_\tau)\mathbf{1}_{\{t>\tau\}}.$$

Letting t go to t_0 yields

$$u(t_0, x) = \mathbb{E}_x u(0, B_{t_0})\mathbf{1}_{\{t_0\leq\tau\}} + \mathbb{E}_x u(t_0 - \tau, B_\tau)\mathbf{1}_{\{t_0>\tau\}}.$$

Given initial and boundary conditions this rewrites as

$$u(t_0, x) = \mathbb{E}_x f(B_{t_0})\mathbf{1}_{\{t_0\leq\tau\}} + \mathbb{E}_x g(t_0 - \tau, B_\tau)\mathbf{1}_{\{t_0>\tau\}},$$

so a solution has to be of this form.

Verifying directly that this solves the problem might not be easy. To bypass it, we could refer to the existence and uniqueness of the solutions of the heat equation.

□

Question 3. Let B be a d-dimensional standard Brownian motion. For which dimensions, does it hit a single point different from its starting location?

Solution. When $d = 1$, we know the density of the hitting time of a single point. Particularly, this stopping time is a.s. finite.

Let $d \geq 2$ and fix two different points $a, x \in \mathbb{R}^d$. We will show that B_t starting at $a \in \mathbb{R}^d$, with probability one, never hits x. Let τ_r be the hitting time of the sphere $\partial B(x, r)$ (r small), let τ_R be the hitting time of the sphere $\partial B(0, R)$ (R large) and let $\tau_{\{x\}}$ be the hitting time of x. Notice that $\lim_{r\to 0} \tau_r = \tau_{\{x\}}$ and $\lim_{R\to\infty} \tau_R = \infty$. From the lecture we know that (see also Theorem 3.18 in [P. Mörters, Y. Peres, *Brownian Motion*])

$$\mathbb{P}_a(\tau_r < \tau_R) = \begin{cases} \frac{\ln R - \ln |a|}{\ln R - \ln r}, & d = 2, \\ \frac{R^{2-d} - |a|^{2-d}}{R^{2-d} - r^{2-d}}, & d \geq 3. \end{cases}$$

Therefore letting r go to zero yields

$$\mathbb{P}_a(\tau_{\{x\}} < \tau_R) = 0,$$

hence letting R go to infinity we obtain $\mathbb{P}_a(\tau_{\{x\}} < \infty) = 0$. □

Question 4. Let f be a function compactly supported function in the upper half space $\{x_d \geq 0\}$ of $\mathbb{R}^d$. Show that

$$\int G(x,y)f(y)dy - \int G(x,\overline{y})f(y)dy = \mathbb{E}_x \int_0^\tau f(B_t)dt,$$

where B is a standard d-dimensional Brownian motion, τ is the hitting time of the hyperplane $H = \{x_d = 0\}$, $G(x,y)$ is the Green's function for $\mathbb{R}^d$, and $\bar{y}$ means the reflection of a point $y \in \mathbb{R}^d$ about the hyperplane H.

This shows that $G(x,y) - G(x,\bar{y})$ is the Green's function in the upper half-space.

Solution. We have (see Theorem 3.32 in [P. Mörters, Y. Peres, *Brownian Motion*])

$$\begin{aligned}
\int G(x,y)f(y)dy &= \mathbb{E}_x \int_0^\infty f(B_t)dt, \\
\int G(x,\overline{y})f(y)dy &= \int G(x,y)f(\overline{y})dy \\
&= \mathbb{E}_x \int_0^\infty f(\overline{B_t})dt.
\end{aligned}$$

The key observation is that the processes $\{B_t,\ t \geq \tau\}$ and $\{\overline{B_t},\ t \geq \tau\}$ have the same distribution. Therefore, breaking each integral on the right hand side into two, on $[0,\tau]$ and on $[\tau,\infty)$ and subtracting the above equalities, we see that two of the integrals will cancel each other, one will be zero as f is compactly supported in the upper half plane, and we will get

$$\int G(x,y)f(y)dy - \int G(x,\overline{y})f(y)dy = \mathbb{E}_x \int_0^\tau f(B_t)dt,$$

as required. □

Brownian Motion IV
Solutions

Question 1. Show that Donsker's theorem can be applied to bounded functions which are continuous only a.s. with respect to the Wiener measure.

Solution. To fix the notation, by $(S_n^*(t))_{t\in[0,1]}$ we mean piecewise linear paths constructed from a standard simple random walk S_n by rescaling time by n and space by $\sqrt{n}$ (that is, from $S_{\lfloor nt\rfloor}/\sqrt{n}$). By $(B(t))_{t\in[0,1]}$ we denote standard Brownian motion in $\mathbb{R}$. Donsker's principle states that

$$S_n^* \text{ convergent in distribution to } B \quad (\text{as } (C[0,1], \|\cdot\|_\infty) \text{ valued random variables}),$$

which means that for every bounded continuous function $f\colon C[0,1] \longrightarrow \mathbb{R}$ we have

$$\mathbb{E}f(S_n^*) \underset{n\to\infty}{\longrightarrow} \mathbb{E}f(B). \qquad (\star)$$

In applications this might be insufficient. Consider for instance the function $f(u) = \sup\{t \le 1,\ u(t) = 0\}$, $u \in C[0,1]$, that is $f(u)$ is the last zero of a path u. Plainly, f is bounded but not continuous. Indeed, looking at the piecewise linear paths u_ϵ with $u_\epsilon(0) = 0$, $u_\epsilon(1/3) = u_\epsilon(1) = 1$ and $u_\epsilon(2/3) = \epsilon$, we have that u_ϵ converges to u_0 but $f(u_\epsilon) = 0$ for $\epsilon > 0$, but $f(u_0) = 2/3$. However, if u is a path such that it changes sign in each interval $(f(u) - \delta, f(u))$, as a generic path of B does!, then f is continuous at u (why?).

This example motives the following strengthening of Donsker's principle:

for every function $f\colon C[0,1] \longrightarrow \mathbb{R}$ *which is bounded and continuous for almost every Brownian path, that is,* $\mathbb{P}(f \text{ is continous at } B) = 1$, *we have* $(\star)$.

This is however the portmanteau theorem. We shall show that for a sequence $X, X_1, X_2, \ldots$ of random variables taking values in a metric space (E, ρ) we have that the condition

$$\lim_{n\to\infty} \mathbb{P}(X_n \in A) = \mathbb{P}(X \in A) \text{ for every Borel subset } A \text{ of } E \text{ with } \mathbb{P}(X \in \partial A) = 0 \quad (1)$$

implies

$$\mathbb{E}f(X_n) \longrightarrow \mathbb{E}f(X) \quad \text{for every bounded function } f\colon E \longrightarrow \mathbb{R} \text{ such that } \mathbb{P}(f \text{ is continuous at } X) = 1. \quad (2)$$

This suffices as (1) is equivalent to the convergence in distribution of X_n to X. To show that (1) implies (2) the idea will be to approximate f with a piecewise constant function which expectation will be expressed easily in terms of probabilities that we will know converge. We assume that f is bounded, say $|f(x)| \leq K$ for every $x \in E$. Fix ϵ and choose $a_0 < \ldots < a_l$ such that $a_0 < -K$, $a_l > K$ and $a_i - a_{i-1} < \epsilon$ for $i = 1, \ldots, l$ but also $\mathbb{P}(f(X) = a_i) = 0$ for $0 \leq i \leq l$ (this is possible as there are only countably many a's for which $\mathbb{P}(f(X) = a) > 0$.) This sequence sort of discretises the image of f. Now let $A_i = f^{-1}((a_{i-1}, a_i])$ for $1 \leq i \leq l$. Then we get that $\partial A_i \subset f^{-1}(\{a_{i-1}, a_i\}) \cup D$, where D is the set of discontinuity points of f. Therefore $\mathbb{P}(X \in \partial A_i) \leq \mathbb{P}\left(X \in f^{-1}(\{a_{i-1}, a_i\}) \cup D\right) = 0$. Hence,

$$\sum_{i=1}^{l} a_i \mathbb{P}(X_n \in A_i) \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^{l} a_i \mathbb{P}(X \in A_i)$$

By the choice of the a_i

$$\left| \mathbb{E}f(X_n) - \sum_{i=1}^{l} a_i \mathbb{P}(X_n \in A_i) \right| = \left| \mathbb{E} \sum_{i=1}^{l} (f(X_n) - a_i) \mathbf{1}_{\{X_n \in A_i\}} \right| \leq \epsilon$$

and the same holds with X in place of X_n. Combining these inequalities yields

$$\limsup_{n\to\infty} |\mathbb{E}f(X_n) - \mathbb{E}f(X)| \leq 2\epsilon.$$

□

Question 2. Let $(S_n)_{n\geq 0}$ be a symmetric, simple random walk.

(i) Show that there are positive constants c and C such that for every $n \geq 1$ we have

$$\frac{c}{\sqrt{n}} \leq \mathbb{P}(S_i \geq 0 \text{ for all } i = 1, 2, \ldots, n) \leq \frac{C}{\sqrt{n}}.$$

(ii) Given $a \in \mathbb{R}$ find the limit

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{i=1}^{n} S_i > a \right).$$

Solution. Let $S_0 = 0$ and $S_n = \varepsilon_1 + \ldots + \varepsilon_n$, where the ε_i are i.i.d. Bernoulli random variables, $\mathbb{P}(\varepsilon_i = 1) = 1/2 = \mathbb{P}(\varepsilon_i = -1)$.

To compute the probability

$$p_n = \mathbb{P}(\forall\ 1 \leq i \leq n,\ S_i \geq 0)$$

we look at the stopping time $\tau = \inf\{k \geq 1,\ S_k = -1\}$. Note that

$$\begin{aligned} p_n &= \mathbb{P}(S_n \geq 0, \tau > n) = \mathbb{P}(\{S_n \geq 0\} \setminus \{S_n \geq 0, \tau < n\}) \\ &= \mathbb{P}(S_n \geq 0) - \mathbb{P}(S_n \geq 0, \tau < n). \end{aligned}$$

Let $\widetilde{S}_n$ be the random walk S_n reflected at time τ with respect to the level -1, that is

$$\widetilde{S}_j = \begin{cases} S_j, & j \le \tau, \\ -2 - S_j, & j > \tau. \end{cases}$$

If $\tau < n$ then $S_n \ge 0$ if equivalent to $\widetilde{S}_n \le -2$, so $\mathbb{P}(S_n \ge 0, \tau < n) = \mathbb{P}\left(\widetilde{S}_n \le -2, \tau < n\right)$, but $\{\widetilde{S}_n \le -2\} \subset \{\tau < n\}$, therefore we get

$$p_n = \mathbb{P}(S_n \ge 0) - \mathbb{P}\left(\widetilde{S}_n \le -2\right),$$

which by symmetry and the reflection principle becomes

$$p_n = \mathbb{P}(S_n \ge 0) - \mathbb{P}(S_n \ge 2) = \mathbb{P}(S_n \in \{0,1\}) = \begin{cases} \binom{n}{n/2} 2^{-n}, & n \text{ is even}, \\ \binom{n}{(n-1)/2} 2^{-n}, & n \text{ is odd}. \end{cases}.$$

Using Stirling's formula we easily find that $cn^{-1/2} \le p_n \le Cn^{-1/2}$ for some positive constants c and C.

To find the limit

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{j=1}^{n} S_j > a\right)$$

we shall use the central limit theorem. Notice that

$$\sum_{j=1}^{n} S_j = n\varepsilon_1 + (n-1)\varepsilon_2 + \ldots + \varepsilon_n$$

which of course has the same distribution as $\sum_{j=1}^{n} j\varepsilon_j$. This is a sum of independent random variables. The variance is

$$\mathrm{Var}\left(\sum_{j=1}^{n} j\varepsilon_j\right) = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Call it σ_n^2 and let $X_j = j\varepsilon_j/\sigma_n$. We want to find

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_j > an^{3/2}/\sigma_n\right).$$

It is readily checked that the variables X_j satisfy Lindeberg's condition

$$\sum_{j=1}^{n} \mathbb{E}X_j \mathbf{1}_{\{|X_j|>\epsilon\}} = \sum_{j=1}^{n} j^2 \mathbf{1}_{\{j>\epsilon\sigma_n\}} \underset{n\to\infty}{\longrightarrow} 0$$

(in fact this sequence is eventually zero, precisely for n such that $\sigma_n/n > 1/\epsilon$). Therefore, by the central limit theorem we get that

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{j=1}^{n} S_j > a\right) = \lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_j > an^{3/2}/\sigma_n\right) = \mathbb{P}\left(G > a\sqrt{3}\right),$$

where G is a standard Gaussian random variable.

Alternatively, using Donsker's principle we get that

$$\mathbb{P}\left(n^{-3/2}\sum_{j=1}^{n} S_j > a\right) = \mathbb{P}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{S_{\frac{j}{n}\cdot n}}{\sqrt{n}} > a\right)$$

tends to

$$\mathbb{P}\left(\int_0^1 B_t dt > a\right).$$

To finish, we notice that $\int_0^1 B_t dt$ is a Gaussian random variable with mean zero and variance

$$\mathbb{E}\left(\int_0^1 B_t dt\right)^2 = \mathbb{E}\int_0^1\int_0^1 B_s B_t ds dt = \int_0^1\int_0^1 \min\{s,t\} ds dt = 1/3.$$

□

Question 3. This question discusses Doob's h-transform.

(i) Let $(X_n)_{\neq 0}$ be an irreducible Markov chain on a finite state space S with a transition matrix $[p_{ij}]_{i,j\in S}$. Let D be a subset of S, $\hat{S} = S \setminus D$ and τ the hitting time of D. Define the function

$$h(i) = \mathbb{P}(\tau = \infty \mid X_0 = i), \quad i \in \hat{S}.$$

Show that h is harmonic, that is

$$h(i) = \sum_{j\in\hat{S}} p_{ij} h(j), \qquad i \in \hat{S}.$$

Define $\hat{p}_{ij} = \frac{h(j)}{h(i)} p_{ij}$, $i, j \in \hat{S}$. Show that $[\hat{p}_{ij}]_{i,j\in\hat{S}}$ is a stochastic matrix of the transition probabilities of the chain (X_n) conditioned on never hitting D.

(ii) Let B be a standard linear Brownian motion and let τ be the hitting time of 0. Set $h(x) = \mathbb{P}_x(\tau = \infty)$. Show that B conditioned on never hitting 0 is a Markov chain with transition densities

$$\hat{p}(s,x;t,y) = \frac{h(y)}{h(x)}\big(p(s,x;t,y) - p(s,-x;t,y)\big), \qquad x, y > 0 \text{ or } x, y < 0$$

where p is the transition density of B.

(iii) Let B be a 3-dimensional standard Brownian motion. Show that the transition densities of the process $|B|$, the Euclidean norm of B, are given by $\hat{p}$ defined

Solution. (i) To get right intuitions, we shall discuss a discrete version of Doob's h transform first.

Let $X_0, X_1, \dots$ be a Markov chain on a finite state space S with transition matrix $P = [p_{ij}]_{i,j \in S}$. Suppose it is irreducible. Fix a subset D in S and define the reaching time $\tau = \inf\{n \geq 1, X_n \in D\}$ of D. Denote $\hat{S} = S \setminus D$. We would like to know how the process (X_n) conditioned on never reaching D behaves. We define the function on $\hat{S}$

$$h(i) = \mathbb{P}(\tau = \infty \mid X_0 = i), \quad i \in \hat{S}$$

First we check that this function is *harmonic* in the sense that

$$h(i) = \sum_{j \in \hat{S}} p_{ij} h(j)$$

for every $i \in \hat{S}$. Notice that

$$\begin{aligned}
h(i) &= \mathbb{P}(\tau = \infty \mid X_0 = i) \\
&= \sum_{j \in \hat{S}} \mathbb{P}(\tau = \infty, X_1 = j \mid X_0 = i) \\
&= \sum_{j \in \hat{S}} \frac{\mathbb{P}(\tau = \infty, X_1 = j, X_0 = i)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{j \in \hat{S}} \frac{\mathbb{P}(\tau = \infty \mid X_1 = j, X_0 = i)\, \mathbb{P}(X_0 = i, X_1 = j)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{j \in \hat{S}} \mathbb{P}(\tau = \infty \mid X_1 = j)\, \mathbb{P}(X_1 = j \mid X_0 = i) \\
&= \sum_{j \in \hat{S}} \mathbb{P}(\tau = \infty \mid X_0 = j)\, \mathbb{P}(X_1 = j \mid X_0 = i) \\
&= \sum_{j \in \hat{S}} h(j) p_{ij}.
\end{aligned}$$

This harmonicity of h is equivalent to saying that the matrix $\hat{P} = [\hat{p}_{ij}]_{i,j \in \hat{S}}$ is a transition matrix of a Markov chain on $\hat{S}$, where

$$\hat{p}_{ij} = \frac{h(j)}{h(i)} p_{ij}.$$

Now we will show that this Markov chain has the same distribution as (X_n) conditioned on never reaching D. To this end, it is enough to check that for every $j_0, j_1, \dots, j_n \in \hat{S}$ we have

$$\mathbb{P}(X_1 = j_1, \dots, X_n = j_n \mid \tau = \infty, X_0 = j_0) = \hat{p}_{j_0, j_1} \cdot \ldots \cdot \hat{p}_{j_{n-1}, j_n}.$$

Clearly,

$$\mathbb{P}(X_1 = j_1, \dots, X_n = j_n \mid \tau = \infty, X_0 = j_0) = \frac{\mathbb{P}(X_1 = j_1, \dots, X_n = j_n, \tau = \infty, X_0 = j_0)}{\mathbb{P}(\tau = \infty, X_0 = j_0)}.$$

The denominator is simply $\mathbb{P}(\tau = \infty \mid X_0 = j_0)\mathbb{P}(X_0 = j_0) = h(j_0)\mathbb{P}(X_0 = j_0)$. Conditioning consecutively, the numerator becomes

$$\begin{aligned} p = \mathbb{P}(\tau = \infty \mid X_n = j_n, \dots, X_0 = j_0) \cdot \mathbb{P}(X_n = j_n \mid X_{n-1} = j_{n-1}, \dots, X_0 = j_0) \\ \cdot \mathbb{P}(X_{n-1} = j_{n-1} \mid X_{n-2} = j_{n-2}, \dots, X_0 = j_0) \\ \cdot \dots \\ \cdot \mathbb{P}(X_1 = j_1 \mid X_0 = j_0)\mathbb{P}(X_0 = j_0). \end{aligned}$$

Notice that $\{\tau = \infty\} = \{\forall m \geq n+1\ X_m \notin \hat{S}\} \cap \{\forall m \leq n\ X_m \notin \hat{S}\}$. Moreover, $\{\forall m \leq n\ X_m \notin \hat{S}\} \supset \{X_n = j_n, \dots, X_0 = j_0\}$. Since (X_n) is stationary, this yields

$$\begin{aligned} \mathbb{P}(\tau = \infty \mid X_n = j_n, \dots, X_0 = j_0) &= \frac{\mathbb{P}(\tau = \infty, X_n = j_n, \dots, X_0 = j_0)}{\mathbb{P}(X_n = j_n, \dots, X_0 = j_0)} \\ &= \frac{\mathbb{P}\left(\{\forall m \geq n+1\ X_m \notin \hat{S}\}, X_n = j_n, \dots, X_0 = j_0\right)}{\mathbb{P}(X_n = j_n, \dots, X_0 = j_0)} \\ &= \mathbb{P}\left(\forall m \geq n+1\ X_m \notin \hat{S} \mid X_n = j_n, \dots, X_0 = j_0\right) \\ &= \mathbb{P}\left(\forall m \geq 1\ X_m \notin \hat{S} \mid X_0 = j_n\right) \\ &= \mathbb{P}(\tau = \infty \mid X_0 = j_n) \\ &= h(j_n). \end{aligned}$$

Therefore, the numerator p equals

$$\begin{aligned} h(j_n)p_{j_{n-1},j_n} \cdot \dots \cdot p_{j_0,j_1}\mathbb{P}(X_0 = j_0) &= \frac{h(j_n)}{h(j_{n-1})}p_{j_{n-1},j_n} \cdot \dots \cdot \frac{h(j_1)}{h(j_0)}p_{j_0,j_1} \cdot h(j_0)\mathbb{P}(X_0 = j_0) \\ &= \hat{p}_{j_0,j_1} \cdot \dots \cdot \hat{p}_{j_{n-1},j_n} \cdot h(j_0)\mathbb{P}(X_0 = j_0), \end{aligned}$$

where we used the fact that

(ii) Consider standard 1-dimensional Brownian and let τ be the hitting time of $D = \{0\}$, $\tau = \inf\{t > 0, B_t = 0\}$. We would like to understand the process (B_t) conditioned on never hitting 0. Call this process $\hat{B}$. Set

$$h(x) = \mathbb{P}_x(\tau = \infty).$$

We know that this is a harmonic function on $\mathbb{R} \setminus \{0\}$. Define new probability $\hat{\mathbb{P}}_x$ by

$$\mathbb{P}_x(A) = \frac{\mathbb{P}_x(A, \tau = \infty)}{h(x)}.$$

($\hat{\mathbb{P}}_x$ is absolutely continuous with respect to $\mathbb{P}_x$, that is $\hat{\mathbb{E}}_x Y = \frac{1}{h(x)}\mathbb{E}Y\mathbf{1}_{\{\tau=\infty\}}$ for any bounded random variable Y). We will show that the process $\hat{B}$ is a Markov process with respect to $\hat{\mathbb{P}}_x$ with the transition probabilities

$$\hat{p}(s, x; t, y) = \frac{h(y)}{h(x)}\big(p(s, x; t, y) - p(s, -x; t, y)\big), \qquad x, y > 0 \text{ or } x, y < 0.$$

That is, the Markov property is

$$\mathbb{P}_x(\hat{B}_t \in A \mid \mathcal{F}_s) = \int_A \hat{p}(s, \hat{B}_s; y, t) dy$$

or, equivalently, for any bounded measurable f,

$$\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) = \int f(y)\hat{p}(s, \hat{B}_s; y, t) dy. \tag{3}$$

To see why the transition probabilities for $\hat{B}$ look as claimed, let us look at the following heuristic computation

$$\mathbb{P}_x(B_t = y \mid \tau = \infty) = \frac{\mathbb{P}_x(\tau = \infty \mid B_t = y, \tau > t)\mathbb{P}_x(B_t = y, \tau > t)}{\mathbb{P}_x(\tau = \infty)}.$$

By the Markov property of B we have that $\mathbb{P}_x(\tau = \infty \mid B_t = y, \tau > t) = h(y)$, so $\hat{p}(0, x; t, y)$ should be

$$\frac{h(y)}{h(x)}\mathbb{P}_x(B_t = y, \tau > t).$$

We have $\mathbb{P}_x(B_t = y, \tau > t) = \mathbb{P}_x(B_t = y) - \mathbb{P}_x(B_t = y, \tau \le t)$. The first term has the meaning of $p(0, x; t, y)$. The second term by the reflection principle is $\mathbb{P}_{-x}(B_t = y, \tau \le t) = \mathbb{P}_{-x}(B_t = y)$ which gives $p(0, -x; t, y)$.

Let us now formally prove (3). The trick is as always to first condition on $\mathcal{F}_t$. Doing this we obtain

$$\begin{aligned}\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) &= \frac{1}{h(x)}\mathbb{E}_x\big(f(B_t)\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_s\big) = \frac{1}{h(x)}\mathbb{E}_x\Big(f(B_t)\mathbb{E}_x\big(\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_t\big)\Big|\mathcal{F}_s\Big) \\ &= \frac{1}{h(x)}\mathbb{E}_x\Big(f(B_t)\mathbf{1}_{\{\tau>t\}}h(B_t)\Big|\mathcal{F}_s\Big)\end{aligned}$$

as using the strong Markov property for B we get

$$\mathbb{E}_x\big(\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_t\big) = \mathbb{E}_x\big(\mathbf{1}_{\{\forall u>t\ B_u\neq 0\}}\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t\big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_x\big(\mathbf{1}_{\{\forall u>t\ B_u\neq 0\}}|\mathcal{F}_t\big) = \mathbf{1}_{\{\tau>t\}}h(B_t).$$

We write $\mathbf{1}_{\{\tau>t\}} = 1 - \mathbf{1}_{\{\tau\le t\}}$ and using the reflection principle as well as the Markov property for B we get ($\tilde{B}$ is the reflected Brownian motion at τ)

$$\begin{aligned}\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) &= \frac{1}{h(x)}\Big(\mathbb{E}_x\big(f(B_t)h(B_t)\big|\mathcal{F}_s\big) - \mathbb{E}_{-x}\big(f(\tilde{B}_t)h(\tilde{B}_t)\big|\mathcal{F}_s\big)\Big) \\ &= \frac{1}{h(x)}\int \big(f(y)h(y)(p(s, B_s; t, y) - f(y)h(y)p(s, -B_s; t, y)\big)dy,\end{aligned}$$

which shows (3).

Notice that the ratio $h(y)/h(x) = \mathbb{P}_y(\tau = \infty)/\mathbb{P}_x(\tau = \infty)$ can be computed explicitly. Define the hitting time τ_a of $\partial(-a, a) = \{-a, a\}$. Fix $R > r$. Since $\mathbb{P}_x(\tau_R < \tau_r)$ is a harmonic function in $\{r < |x| < R\}$ with the boundary conditions: 0 on $\{|x| = r\}$, 1 on $\{|x| = R\}$, we get that

$$\mathbb{P}_x(\tau_R < \tau_r) = \frac{|x| - r}{R - r}.$$

Taking the ratio and letting $r \to 0$ and $R \to \infty$ yield

$$\frac{h(y)}{h(x)} = \frac{\mathbb{P}_y(\tau = \infty)}{\mathbb{P}_x(\tau = \infty)} = \frac{|y|}{|x|}.$$

(iii) As an application, we will heuristically convince ourselves that the 3 dimensional Bessel process $(|B_t|)$ (the magnitude of a standard 3 dimensional Brownian motion) is the linear Brownian motion conditioned on never hitting 0. To this end, we will find a one point density of the Bessel process

$$p(s, x; t, y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathbb{P}\left(|B_t| \in (y - \epsilon, y + \epsilon) \mid |B_s| \in (x - \epsilon, x + \epsilon)\right)$$

and check that it matches the transition probabilities found in (ii). By the Markov property of Brownian motion as well as rotational invariance we get that $p(s, x; t, y)$ is the density g_Y of the variable

$$Y = \sqrt{(B^{(1)}_{t-s} + x)^2 + (B^{(2)}_{t-s})^2 + (B^{(3)}_{t-s})^2}$$

at y. To find it, it will suffice to find the density g_Z of the variable

$$Z = \frac{Y^2}{t-s} \sim \left(G_1 + \frac{x}{\sqrt{t-s}}\right)^2 + G_2^2 + G_3^2$$

where G_1, G_2, G_2 are i.i.d. $N(0, 1)$ random variables because then

$$g_Y(y) = \frac{d}{dy} \mathbb{P}\left(Y \le y\right) = \frac{d}{dy} \mathbb{P}\left(Z \le \frac{y^2}{t-s}\right) = g_Z\left(\frac{y^2}{t-s}\right) \frac{2y}{t-s}.$$

We know that the distribution of $G_2^2 + G_3^2$ is $\chi^2(2)$ with density $\frac{1}{2}e^{-u/2}\mathbf{1}_{(0,\infty)}(u)$. If we denote by φ the density of G_1, then the density of $(G_1 + x/\sqrt{t-s})^2$ at $u > 0$ equals

$$\begin{aligned}
\psi(u) &= \frac{d}{du} \mathbb{P}\left(\left(G_1 + \frac{x}{\sqrt{t-s}}\right)^2 \le u\right) = \frac{d}{du} \mathbb{P}\left(-\sqrt{u} - \frac{x}{\sqrt{t-s}} \le G_1 \le \sqrt{u} - \frac{x}{\sqrt{t-s}}\right) \\
&= \frac{d}{du} \int_{-\sqrt{u}-x/\sqrt{t-s}}^{\sqrt{u}-x/\sqrt{t-s}} \varphi = \frac{1}{2\sqrt{u}} \left(\varphi\left(\sqrt{u} - \frac{x}{\sqrt{t-s}}\right) + \varphi\left(-\sqrt{u} - \frac{x}{\sqrt{t-s}}\right)\right).
\end{aligned}$$

Therefore

$$g_Z(z) = \int_0^z \frac{1}{2} e^{-u/2} \psi(z-u) du,$$

which after computing the integral becomes

$$g_Z(z) = \varphi\left(\frac{x}{\sqrt{t-s}}\right) e^{-z/2} \frac{\sqrt{t-s}}{x} \sinh\left(\frac{x}{\sqrt{t-s}}\sqrt{z}\right).$$

Thus

$$g_Y(y) = \frac{y}{x} \frac{1}{\sqrt{t-s}} \left(\varphi\left(\frac{y-x}{\sqrt{t-s}}\right) - \varphi\left(\frac{y+x}{\sqrt{t-s}}\right)\right)$$

which agrees with the transition probabilities found in (ii). □

Brownian Motion V
Solutions

Question 1. Let L_t be the local time at zero of linear Brownian motion. Show that $\mathbb{E}L_t = \sqrt{2t/\pi}$.

Solution. We know from the lecture that the local time at zero L_t has the same distribution as the maximum $M_t = \max_{0\le s\le t} B_t$. Therefore

$$\mathbb{E}L_t = \mathbb{E}M_t = \int_0^\infty \mathbb{P}\left(M_t > u\right) du.$$

Moreover, using the reflection principle it has been shown that $\mathbb{P}\left(M_t > u\right) = 2\mathbb{P}\left(B_t > u\right) = \mathbb{P}\left(|B_t| > u\right)$, so we get

$$\mathbb{E}L_t = \int_0^\infty \mathbb{P}\left(|B_t| > u\right) du = \mathbb{E}|B_t| = \sqrt{t}\mathbb{E}|N(0,1)| = \sqrt{\frac{2t}{\pi}}.$$

□

Question 2. Let $a < 0 < b < c$ and τ_a, τ_b, τ_cbe the hitting times of these levels for one dimensional Brownian motion. Compute

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right).$$

Solution. Notice that

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right) = \mathbb{P}\left(\tau_b < \tau_a,\ \tau_a < \tau_c\right) = \mathbb{P}\left(\tau_a < \tau_c \mid \tau_b < \tau_a\right)\mathbb{P}\left(\tau_b < \tau_a\right).$$

Take the stopping time $\tau = \tau_a \wedge \tau_b = \inf\{t > 0, B_t \in \{a, b\}\}$ and observe that $\{\tau_b < \tau_a\} = \{B_\tau = b\}$. Therefore using the strong Markov property we obtain

$$\mathbb{P}\left(\tau_a < \tau_c \mid \tau_b < \tau_a\right) = \mathbb{P}\left(\tau_a < \tau_c \mid B_\tau = b\right) = \mathbb{P}_b\{\tau_a < \tau_c\} = \mathbb{P}\left(\tau_{a-b} < \tau_{c-b}\right).$$

Recall that using Wald's lemma it is easy to find that $\mathbb{P}\left(\tau_a < \tau_b\right) = \frac{b}{b-a}$ (basically we combine the equations $\mathbb{E}B_\tau = 0$ and $\mathbb{P}\left(\tau_a < \tau_b\right) = 1 - \mathbb{P}\left(\tau_a > \tau_b\right)$). Thus we get

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right) = \frac{c-b}{c-b-(a-b)}\frac{-a}{b-a} = \frac{-a(c-b)}{(b-a)(c-a)}.$$

□

Brownian Motion VI
Solutions

Question 1. Let (B_t) be a standard one dimensional Brownian motion and τ_1 the hitting time of level 1. Show that

$$\mathbb{E}\int_0^{\tau_1} \mathbf{1}_{\{0\le B_s\le 1\}} ds = 1.$$

Solution. Notice that using Fubini's theorem

$$\mu = \mathbb{E}\int_0^{\tau_1} \mathbf{1}_{\{0\le B_s\le 1\}} ds = \mathbb{E}\int_0^{\infty} \mathbf{1}_{\{0\le B_s\le 1, s<\tau_1\}} ds = \int_0^\infty \mathbb{P}\left(0 < B_s < 1, s < \tau_1\right) ds.$$

The integrand equals

$$\mathbb{P}\left(0 < B_s < 1\right) - \mathbb{P}\left(0 < B_s < 1, s > \tau_1\right).$$

Let B^* be the reflected Brownian motion at τ_1. Since $B^*_s = 2 - B_s$ for $s > \tau_1$, we get from the reflection principle that

$$\mathbb{P}\left(0 < B_s < 1, s > \tau_1\right) = \mathbb{P}\left(1 < B^*_s < 2, s > \tau_1\right) = \mathbb{P}\left(1 < B^*_s < 2\right) = \mathbb{P}\left(1 < B_s < 2\right).$$

Let φ be the density of the standard Gaussian distribution and let Φ be its distribution function, $\Phi(x) = \int_{-\infty}^x \varphi$. We obtain

$$\mathbb{P}\left(0 < B_s < 1\right) - \mathbb{P}\left(1 < B_s < 2\right) = \Phi\left(\frac{1}{\sqrt{s}}\right) - \Phi(0) - \left(\Phi\left(\frac{2}{\sqrt{s}}\right) - \Phi\left(\frac{1}{\sqrt{s}}\right)\right),$$

so integrating by substitution ($t = 1/\sqrt{s}$) gives

$$\mu = \int_0^\infty \big(\mathbb{P}\left(0 < B_s < 1\right) - \mathbb{P}\left(1 < B_s < 2\right)\big) ds = \int_0^\infty \big(2\Phi(t) - \Phi(2t) - \Phi(0)\big)\left(\frac{-1}{t^2}\right)' dt.$$

Integrating by parts twice yields (one has to check that the boundary term vanishes each time; recall also that $\varphi'(x) = -x\varphi(x)$)

$$\mu = \int_0^\infty \big(2\varphi(t) - 2\varphi(2t)\big)\left(\frac{-1}{t}\right)' dt = \int_0^\infty \big(-2t\varphi(t) + 4t\varphi(t)\big)\frac{1}{t} dt = 2\int_0^\infty \varphi = 1.$$

□

Question 2. Let H be a hyperplane in $\mathbb{R}^d$ passing through the origin. Let B be a d-dimensional Brownian motion and let τ be the hitting time of H. Show that for every $x \in \mathbb{R}^d$

$$\sup_{t>0} \mathbb{E}_x |B_t| \mathbf{1}_{\{t<\tau\}} < \infty.$$

Solution. We can assume that B starts at 0 and H passes through x. Moreover, by rotational invariance, we can assume that $x = (a, 0, \ldots, 0)$ for some $a > 0$ so that $H = \{y \in \mathbb{R}^d,\ y_1 = a\}$. Then τ is in fact the hitting time of the first coordinate $W = B^{(1)}$ of B of level a. Write $B = (W, \bar{B})$, where $\bar{B}$ denotes the process of the last $d-1$ coordinates of B. W and $\bar{B}$ are independent standard Brownian motions. We have

$$\mathbb{E}|B_t|\mathbf{1}_{\{t<\tau\}} \leq \mathbb{E}|W_t|\mathbf{1}_{\{t<\tau\}} + \mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}}.$$

The second term is easy to handle because of independence

$$\mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}} = \mathbb{E}|\bar{B}_t|\mathbb{E}\mathbf{1}_{\{t<\tau\}} = C\sqrt{t}\,\mathbb{P}\left(t < \tau\right),$$

where C is some positive constant which depends only on d. Using the reflection principle we get that

$$\mathbb{P}\left(t < \tau\right) = 1 - \mathbb{P}\left(|B_t| > a\right) = \mathbb{P}\left(|B_t| < a\right) = 2\int_0^{a/\sqrt{t}} \varphi < 2\frac{a}{\sqrt{t}}\varphi(0)$$

(by φ we denote the density of the standard Gaussian distribution). Therefore

$$\sup_{t>0} \mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}} = C\sup_{t>0}\sqrt{t}\,\mathbb{P}\left(t < \tau\right) < 2Ca.$$

To handle the first term, notice that

$$\mathbb{E}|W_t|\mathbf{1}_{\{t<\tau\}} = \int_0^\infty \mathbb{P}\left(|W_t| > u, t < \tau\right) du \leq a + \int_a^\infty \mathbb{P}\left(|W_t| > u, t < \tau\right) du.$$

Reflecting W at τ, we can rewrite the integrand as follows (bear in mind that $u > a$)

$$\begin{aligned}
\mathbb{P}\left(|W_t| > u, t < \tau\right) &= \mathbb{P}\left(W_t > u, t < \tau\right) + \mathbb{P}\left(W_t < -u, t < \tau\right) \\
&= \mathbb{P}\left(\varnothing\right) + \mathbb{P}\left(W_t < -u\right) - \mathbb{P}\left(W_t < -u, t > \tau\right) \\
&= \mathbb{P}\left(W_t > u\right) - \mathbb{P}\left(W_t^* > 2a + u, t > \tau\right) \\
&= \mathbb{P}\left(W_t > u\right) - \mathbb{P}\left(W_t^* > 2a + u\right) \\
&= \mathbb{P}\left(u < W_t < 2a + u\right) = \int_{u/\sqrt{t}}^{(2a+u)/\sqrt{t}} \varphi(v)dv.
\end{aligned}$$

Hence, our integral becomes

$$\int_a^\infty \mathbb{P}\left(|W_t| > u, t < \tau\right) du = \int_a^\infty \int_{u/\sqrt{t}}^{(2a+u)/\sqrt{t}} \varphi(v) dv.$$

Using Fubini's theorem we get that this equals ($|\cdot|$ of course denotes Lebesgue measure)

$$\int_0^\infty \left|\left\{u > a,\ v\sqrt{t} - 2a < u < v\sqrt{t}\right\}\right| \varphi(v) dv \leq 2a \int_0^\infty \varphi(v) dv = a.$$

Putting these together yields

$$\mathbb{E}|W_t| \mathbf{1}_{\{t<\tau\}} \leq a + a = 2a$$

and finally

$$\sup_{t>0} \mathbb{E}|B_t| \mathbf{1}_{\{t<\tau\}} \leq 2a(1 + C).$$

□

Question 3. This is question 3.17 from [P. Mörters, Y. Peres, *Brownian Motion*]. It is left to the diligent student.