Problem solving seminar
IMC Preparation, Set IV — Solutions

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1. Consider the function f defined for positive real numbers x, y, z,

$$f(x,y,z) = \frac{(x+y+z)(xy+yz+zx)}{(x+y)(y+z)(z+x)}.$$

What is the image of f?

Solution. We will prove that $\mathrm{Im}(f) = (1, \frac{9}{8}]$. By expanding the multiplication one can check that the lower bound is equivalent to $0 < xyz$, and moreover $f(x,y,z)$ can get arbitrarily close to 1 by taking $z \to 0^+$.

The upper bound is equivalent to

$$xyz \le \frac{xyz + xz^2 + y^2z + yz^2 + xyz + xy^2 + x^2z + x^2y}{8}$$

which follows from the AM-GM inequality. □

2. Let A, $B \in M_{2\times 2}(\mathbb{R})$ such that $A^2 + B^2 = AB$. Show that $(AB - BA)^2 = 0$.

Solution. We will use the following result: If $X \in M_{2\times2}(\mathbb{R})$, with $\mathrm{tr}(X) = 0 = \det(X)$ then $X^2 = 0$. One way to see it is by recalling the identity[1] $X^2 - \mathrm{tr}(X)X + \det(X)I = 0$. Alternately, it is clear by using the Jordan Canonical form.

Now, $X = AB - BA$ is clearly traceless. To compute the determinant we use the cubic root of unity $\omega = e^{2\pi i/3}$, and note that

$$\begin{aligned}(A+\omega B)(A+\overline{\omega}B) &= A^2 + B^2 + \omega BA + \overline{\omega}AB \\ &= (1+\overline{\omega})AB + \omega BA \\ &= -\omega AB + \omega BA\end{aligned}$$

Hence

$$\begin{aligned}\omega^2 \det(AB - BA) &= \det(A+\omega B)\det(A+\overline{\omega}B) \\ &= \det(A+\omega B)\overline{\det(A+\omega B)} \\ &= |\det(A+\omega B)|^2\end{aligned}$$

The latter being purely real, therefore $\omega^2 \det(AB - BA = 0$ and we are done. □

[1] Cayley-Hamilton theorem

3. Let $n \ge 3$. Let $A_1A_2\ldots A_n$ be a regular n-gon inscribed in a circle with radius 1. Prove that

$$\prod_{k=1}^{n-1}(5 - |A_1A_{k+1}|^2) = F_n^2,$$

where the sequence (F_n) is defined recursively as $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \ge 1$ (the Fibonacci sequence).

Solution. Let $\epsilon_n = e^{2\pi i/n}$. Then ϵ_n^k, $0 \le k < n$ are consecutive vertices of the regular n-gon inscribed into the unit circle $\{z \in \mathbb{C},\ |z| = 1\}$. We can take $A_k = \epsilon_n^{k-1}$ and then $|A_1A_{k+1}|^2 = |1 - \epsilon_n^k|^2 = 2 - 2\mathfrak{Re}(\epsilon_n^k) = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$, $k \ge 1$. By the well-known Binet's formula we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Thus we want to show that

$$\prod_{k=1}^{n-1}\left(3 + 2\cos\left(\frac{2\pi k}{n}\right)\right) = \frac{1}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)^2. \qquad (\star)$$

To verify this identity observe that for any reals x, y we have

$$x^n - y^n = \prod_{k=0}^{n}(x - \epsilon_n^k y),$$

hence

$$\begin{aligned}(x^n - y^n)^2 = |x^n - y^n|^2 &= \prod_{k=0}^{n-1}|x - \epsilon_n^k y|^2 \\ &= \prod_{k=0}^{n-1}\left(x^2 + y^2 - 2xy\mathfrak{Re}(\epsilon_n^k)\right) \\ &= \prod_{k=0}^{n-1}\left(x^2 + y^2 - 2xy\cos(2\pi k/n)\right).\end{aligned}$$

For $x = (1+\sqrt{5})/2$, $y = (1-\sqrt{5})/2$ we have $x^2 + y^2 = 3$ and $xy = -1$, hence $(\star)$ □

4. Fix $1 \le k \le n$. Let $A_1, \ldots, A_m$ be distinct subsets of the set $\{1, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for all $i \ne j$. Prove that $m \le n$.

Here $|A|$ denotes the cardinality of A.

Solution. Let $v_i \in \mathbb{R}^n$, $i \le m$, be the characteristic vector of A_i, i.e. if $j \in A_i$, then its j^{th} coordinate is 1, otherwise it is 0. Then $\langle v_i, v_j \rangle = k$, for $i \ne j$, and $\langle v_i, v_i \rangle = |A_i|$.

If we show that v_i's are linearly independent, then necessarily $m \le n$ and we are done. Suppose $\sum_{i=1}^m \lambda_i v_i = 0$ with not all λ_i's being 0. Then there are (at least) two indices, say $s \ne t$ for which $\lambda_s \ne 0 \ne \lambda_t$. Observe that clearly $|A_i| \ge k$, hence

$$0 = \left\langle \sum_i \lambda_i v_i, \sum_j \lambda_j v_j \right\rangle = \sum_i \lambda_i^2 |A_i| + \sum_{i \ne j} \lambda_j \lambda_j \cdot k$$
$$= \sum_i \lambda_i^2 (|A_i| - k) + k \left(\sum_i \lambda_i \right)^2$$
$$\ge \lambda_s^2 (|A_s| - k) + \lambda_t^2 (|A_t| - k).$$

It follows that $|A_s| = |A_t| = k$ which together with $|A_s \cap A_t| = k$ implies that $A_s = A_t$, a contradiction. □

5. Let $v_0, v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors of length 1 such that $|v_i - v_j| > \sqrt{2}$ for all $i \ne j$. Prove that any n of them are linearly independent.

Solution. Observe that for $i \ne j$ we have

$$\langle v_i, v_j \rangle = \frac{-|v_i - v_j|^2 + |v_i|^2 + |v_j|^2}{2} < 0,$$

thus we conclude by the following lemma.

Lemma. *Let vectors $v_0, \ldots, v_n \in \mathbb{R}^n$ satisfy $\langle v_i, v_j \rangle < 0$ for all $i \ne j$. Then any n of them are linearly independent.*

Proof. Consider

$$\sum_{i=1}^n \lambda_i v_i = 0.$$

By taking the inner product with v_0 we see that to show that all λ_i's are zero it is enough to show that they have the same sign. Without loss of generality let $\lambda_1, \ldots, \lambda_k \ge 0$, $\lambda_{k+1}, \ldots, \lambda_n < 0$. If $k = n$ we are done. If not, consider

$$\sum_{i \le k} \lambda_i v_i = \sum_{j > k} (-\lambda_j) v_j$$

and take the inner product with $\sum_{i \le k} \lambda_i v_i$ to get

$$0 \le \left| \sum_{i \le k} \lambda_i v_i \right|^2 = \left\langle \sum_{i \le k} \lambda_i v_i, \sum_{j > k} (-\lambda_j) v_j \right\rangle$$
$$= \sum_{i \le k, j > k} \lambda_i \lambda_j (- \langle v_i, v_j \rangle).$$

As in the last sum $\lambda_i \lambda_j \le 0$, there is actually $\lambda_i \lambda_j = 0$ for every $i \le k$ and $j > k$. This is possible only if $\lambda_i = 0$ for every $i \le k$, so all λ_i's have the same sign.

□