21/04/2014

Problem solving seminar
IMC Preparation Set I

Instructions

1. Work independently.
2. Do not use any books, notes, nor calculators.
3. Please write down your solutions for each problem on **individual** sheets
4. Please submit your work via pigeonholes opposite room B1.38 or email (Problems 1, 2 & 3 to RT, 3 & 4 to TT) **by Friday, 25 April, 11:59pm**

Good luck!
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Problems

1. Let A be a $n \times n$ matrix such that Au is orthogonal to u for every vector $u \in \mathbb{R}^n$. Prove that

a) A is skew-symmetric, i.e., $A^t = -A$.

b) If n is odd, show that there exists $v \in \mathbb{R}^n$ such that $Av = 0$.

2. Consider 2014 points in general position (no three collinear) on the plane, and all the segments joining any two of them. Show that one of the following conditions always hold:

(i) It is possible to reach a point from any other by only using segments with rational length.

(ii) It is possible to reach a point from any other by only using segments with irrational length.

3. Any parabola P divides the plane into a convex region $A(P)$ and a non-convex $B(P)$. Is it possible to find a positive integer n and parabolas P_1, P_2, ..., P_n such that $A(P_1)$, $A(P_2)$, ..., $A(P_n)$ cover the whole plane?

4. Prove that for integers $1 \leq k \leq n$ we have

$$\sum_{j=0}^{k} \binom{n}{j} < \left(\frac{en}{k}\right)^k.$$

5. Using four colours, is it possible to colour the set of nonnegative real numbers (assign to each nonnegative number one of four colours) so that whenever $a + b = 2c + 2$ for some $a, b, c \geq 0$, then a, b, c will *not* be of the same colour?