Problem solving seminar
Homework I - Solutions

1. Let $n \geq 2$ and let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$. Prove that there exists a subset $I \subset \{1, \ldots, n\}$ such that

$$4\left(\sum_{i\in I} x_i\right) \cdot \left(\sum_{i\notin I} x_i\right) \geq \sum_{i\neq j} x_i \cdot x_j,$$

where $\cdot$ denotes the standard scalar product. We adopt the convention that $\sum_{i\in\varnothing} x_i = 0$.
Solution. Suppose that there is no such subset, i.e. for every $I \subset \{1, \ldots, n\}$,

$$\sum_{i\neq j} x_i \cdot x_j > 4\left(\sum_{i\in I} x_i\right) \cdot \left(\sum_{i\notin I} x_i\right) = 4\sum_{i\neq j} x_i \cdot x_j \mathbf{1}_{\{i\in I, j\notin I\}}.$$

Adding up all these inequalities we get

$$2^n \sum_{i\neq j} x_i \cdot x_j > \sum_I 4 \sum_{i\neq j} x_i \cdot x_j \mathbf{1}_{\{i\in I, j\notin I\}} = 4\sum_{i\neq j} x_i \cdot x_j \sum_I \mathbf{1}_{\{i\in I, j\notin I\}}.$$

For $i \neq j$, $\sum_I \mathbf{1}_{\{i\in I, j\notin I\}} = 2^{n-2}$, so we get a contradiction. $\square$

2. Given positive numbers $t_1, \ldots, t_n$ let $a_{ij} = \min\{t_i, t_j\}$, $i, j = 1, \ldots, n$. Prove that for every real numbers $x_1, \ldots, x_n$ we have

$$\sum_{i,j=1}^n a_{ij} x_i x_j \geq 0.$$

Solution. Notice that $a_{ij} = \int_0^{\min\{t_i,t_j\}} \mathrm{d}x = \int_0^\infty \mathbf{1}_{[0,t_i]}(x)\mathbf{1}_{[0,t_j]}(x)\mathrm{d}x$. As a result,

$$\sum_{i,j=1}^n a_{ij} x_i x_j = \int_0^\infty \sum_{i,j=1}^n x_i \mathbf{1}_{[0,t_i]}(x) \cdot x_j \mathbf{1}_{[0,t_j]}(x)\mathrm{d}x = \int_0^\infty \left(\sum_{i=1}^n x_i \mathbf{1}_{[0,t_i]}(x)\right)^2 \mathrm{d}x \geq 0.$$

$\square$

3. Let $r \in (0,1)$ and denote $C_r = (1+r)/(1-r)$. Prove that for any real numbers $x_0, \ldots, x_n$ which are not all equal to zero

$$C_r^{-1} \sum_{k=0}^n x_k^2 < \sum_{0\leq k,l\leq n} x_k x_l r^{|k-l|} < C_r \sum_{k=0}^n x_k^2.$$

Solution. Notice that for an integer k,

$$r^{|k|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\left(1 + \sum_{n=1}^\infty r^n \left(e^{int} + e^{-int}\right)\right)\mathrm{d}t.$$

Introduce the function

$$f(t) = 1 + \sum_{n=1}^\infty r^n \left(e^{int} + e^{-int}\right) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

We get

$$\begin{aligned}\sum_{0\le k,l\le n} x_k x_l r^{|k-l|} &= \sum_{0\le k,l\le n} x_k x_l \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)t} f(t)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left|\sum_{k=0}^{n} x_k e^{ikt}\right|^2 \mathrm{d}t.\end{aligned}$$

Clearly,

$$\sum_{k=0}^{n} x_k^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n} x_k e^{ikt}\right|^2 \mathrm{d}t.$$

Therefore checking that $\inf_{[-\pi,\pi]} f = f(\pi) = C_r^{-1}$ and $\sup_{[-\pi,\pi]} f = f(0) = C_r$ finishes the proof of the inequality. □