Tomasz Tkocz 23/01/2013

Problem solving seminar
" Linear algebra "

1. MATRICES $AB - BA$

<u>Definition</u> For a matrix $A = [a_{ij}]_{i,j \le n}$ we define its <u>trace</u> as
$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

🔑 $\operatorname{tr} AB = \operatorname{tr} BA$, or $\operatorname{tr}(AB - BA) = 0$.

⚠ tr is a good invariant since plugging $B^{-1}A$ instead of A we get
$$\operatorname{tr} B^{-1}AB = \operatorname{tr} A.$$

<u>Question 1</u> Suppose $AB - BA = A$ for some $A, B \in M_{n\times n}(\mathbb{R})$. Prove that $\det A = 0$.

<u>Solution</u> Assume conversely that A is invertible. Then by the assumption
$$B - A^{-1}BA = I.$$
Taking trace yields a contradition. □

<u>Question 2</u> Let $A, B \in M_{2\times 2}(\mathbb{R})$ satisfy $(AB - BA)^n = I$ for some natural number n. Prove that

(a) $2 \mid n$

(b) $(AB - BA)^4 = I$.

<u>Solution</u> Since $\operatorname{tr}(AB - BA) = 0$ we can write
$$AB - BA = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = X.$$

Then
$$X^2 = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix} = \underbrace{(a^2+bc)}_{x} I,$$

$$X^3 = X^2 \cdot X = xI \cdot X = xX,$$
$$X^4 = x X^2 = x^2 I,$$
$$\vdots$$
$$X^{2k} = x^k I,$$
$$X^{2k+1} = x^k X.$$

We know that $I = X^n = \begin{cases} x^k I & \text{if } n = 2k \\ x^k X & \text{if } n = 2k+1 \end{cases}$.

Clearly the second case does not hold (take e.g. tr). So $n = 2k$, which proves (a).

For (b) notice that $I = x^k I$ implies $x^k = 1$, so $x = \pm 1$, hence $x^2 = 1$ and
$$X^4 = x^2 I = I. \quad \square$$

2. NILPOTENT MATRICES

<u>Definition</u> A matrix $A \in M_{n\times n}(\mathbb{R})$ is called nilpotent if $A^m = 0$ for some natural $m > 0$.

<u>Question 3</u> Let an $n \times n$ matrix A be nilpotent. Show that $A^n = 0$.

~~<u>Solution</u> Suppose $A^n \neq 0$. Then there exists a vector v such that $A^n v \neq 0$. Consider the vectors~~
~~$$v, Av, \dots, A^n v.$$~~
~~They are linearly independent which contradicts the fact that they are in $\mathbb{R}^n$ and there are $n+1$ of them. Indeed,~~
~~$$\alpha_0 v + \alpha_1 A v + \dots + \alpha_n A^n v = 0 \quad /$$~~

Solution Take the smallest m such that $A^m = 0$ (A is nilpotent so such m exists). Since $A^{m-1} \neq 0$ there is a vector v such that $A^{m-1}v \neq 0$. Consider the vectors

$$v, Av, \ldots, A^{m-1}v.$$

They are linearly independent. Indeed,

$$\alpha_0 v + \alpha_1 Av + \ldots + \alpha_{m-1} A^{m-1} v = 0 \quad / A^{m-1}$$

$$\alpha_0 \underbrace{A^{m-1}v}_{\neq 0} = 0 \Rightarrow \alpha_0, \text{ so}$$

$$\alpha_1 Av + \ldots + \alpha_{m-1} A^{m-1} v = 0 \quad / A^{m-1}$$

$$\alpha_1 \underbrace{A^{m-1}v}_{\neq 0} = 0 \Rightarrow \alpha_1 = 0, \quad \text{and so on.}$$

Therefore $m \leq \dim V = n$. □

Question 4 Let $n \times n$ matrices A, B be such that

$$A + t_i B$$

is nilpotent for some $n+1$ distinct reals $t_1, \ldots, t_{n+1}$. Show that A and B are nilpotent too.

Solution Consider the matrix

$$C(t) = (A + tB)^n = A^n + \ldots + t^n B^n.$$

We know that $C(t_i) = 0$, $i = 1, \ldots, n+1$. At each entry, $C(t)$ is a polynomial of degree n with respect to t. Since such a polynomial has got $n+1$ distinct roots, it must be the zero polynomial. In particular each entry of A^n and B^n equals 0 which means that $A^n = 0 = B^n$. □

Question* 5 $A \in M_{n \times n}(\mathbb{R})$ is nilpotent iff $\operatorname{tr} A^k = 0$, for $k = 1, \ldots,$

3. RANK

<u>Definition</u> The rank of a matrix $A \in M_{n \times n}(\mathbb{R})$ is the dimension of the subspace spanned by its columns or, equivalently, rows,

$$\begin{aligned} \operatorname{rank} A &= \dim \operatorname{span}\{\text{columns of } A\} \\ &= \dim \operatorname{span}\{\text{rows of } A\}. \end{aligned} \qquad \Rightarrow \operatorname{rank} A = \operatorname{rank} A^T.$$

In other words

$$\operatorname{rank} A = \dim \operatorname{im} A,$$

when we consider A as a linear mapping $\mathbb{R}^n \to \mathbb{R}^n$.

<u>Question 6</u> Prove that for any $A, B \in M_{n \times n}(\mathbb{R})$ we have

(a) $\operatorname{rank} AB \leq \operatorname{rank} A, \operatorname{rank} B$

(b) $\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - n$.

<u>Solution</u> (a) Consider A, B as linear maps $f, g : \mathbb{R}^n \to \mathbb{R}^n$. Clearly

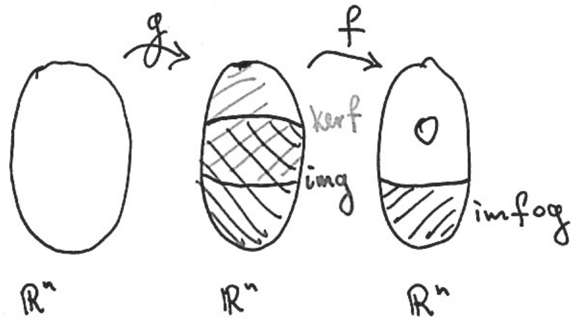
$$\operatorname{im} f \circ g \subset \operatorname{im} f,$$

so $\operatorname{rank} AB \leq \operatorname{rank} A$. Moreover

$$\operatorname{rank}(AB)^T = \operatorname{rank} B^T A^T \leq \operatorname{rank} B^T = \operatorname{rank} B,$$

which proves (a).

(b)



$$\begin{aligned} \dim \operatorname{im} f \circ g &\geq \dim \operatorname{im} g - \underbrace{\dim \ker f}_{n - \dim \operatorname{im} f} \\ &= \dim \operatorname{im} g + \dim \operatorname{im} f - n \end{aligned}$$

which proves (b). $\square$

As a corollary we find

⚠ rank is a good invariant too

$$\operatorname{rank} A = \operatorname{rank} B^{-1} A B.$$

Question 7 Let $A_1, \dots, A_k$ be $n \times n$ matrices of rank $n-1$. Prove that if $k < n$ then $A_1 \cdot \ldots \cdot A_k \neq 0$.

Solution By Question 6 (b) we get

$$\operatorname{rank}(A_1 \cdot \ldots \cdot A_k) \geq \sum_{i=1}^{k} \operatorname{rank} A_i - (k-1)n$$

$$= k \cdot (n-1) - (k-1) \cdot n = n-k > 0. \square$$

4. VARIA a k a MISCELLANEA

Question 8 Let $A, B \in M_{n \times n}(\mathbb{R})$ satisfy $AB - BA = \alpha A$. Prove that

(a) $A^k B - BA^k = \alpha k A^k$ for all integers $k \geq 0$,

(b) A is nilpotent.

Solution (a) easy induction.

(b) Consider the linear map $L : M_{n \times n}(\mathbb{R}) \circlearrowleft$

$L(X) = XB - BX$.

(a) says that A^k is an eigenvector of L with eigenvalue αk. Since L mustn't have infinitely many distinct eigenvalues, there must be $A^k = 0$ for $k \geq m$ for certain m. $\square$

Question 9 There are $n \geq 2$ people sitting at the round table. Each person has got a prime number written down on a sticky note. At each minute a certain person modifies her or his number multiplying it by a number of a neighbour. Is it possible that at some point two people have got the same number?

Solution At time t, the i-th person has got a number $p_1^{\alpha_{i,1}(t)} \cdot \ldots \cdot p_n^{\alpha_{i,n}(t)}$, where p_j is the prime attached at the very beginning to the j-th person. Consider the matrix

$$A_t = [\alpha_{i,j}^{(t)}]_{i,j\le n}.$$

Clearly $A_0 = I$. Moreover, A_{t+1} is obtained from A_t by adding to the k-th column the k-1 or k+1-th one. Therefore

$$\operatorname{rank} A_t = \text{const} = n.$$

Yet, if there were two people at time t_* with the same number, two columns of A_{t_*} would be the same. That would contradict the above. $\square$

5. HOMEWORK

<u>Question 1</u> Suppose that 2×2 real matrices A and B satisfy $AB - BA = B^2$. Prove that $AB = BA$.

<u>Question 2</u> Let X be an $n\times n$ real invertible matrix with columns $X_1, X_2, \dots, X_n$. Let Y be the matrix with columns $X_2, X_3, \dots, X_n, 0$. Prove that YX^{-1} is a rank $n-1$ matrix.