Problem solving seminar

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Linear algebra'

Remark. $M_{n\times n}(\mathbb{R})$ denotes the set of all $n \times n$ real matrices. I is the identity matrix.

Question 1. Let $S \in M_{n\times n}(\mathbb{R})$ be a skew symmetric matrix, i.e. $S^T = -S$. Prove that

(a) eigenvalues of S are purely imaginary, and go in pairs $\lambda, \overline{\lambda}$

(b) S is diagonalizable in an orthonormal basis, i.e. there exists an orthogonal matrix T such that TST^T is diagonal.

Question 2. Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A + A^T = I$. Prove that $\det A > 0$.

Question 3. Let V a finite-dimensional vector space and let $f\colon V \longrightarrow V$ be a linear map such that $f \circ f = f$. Prove that f is diagonalizable with eigenvalues $0, 1$.

Question 4. Let V a finite-dimensional vector space and let $f_1, \ldots, f_m\colon V \longrightarrow V$ be linear maps which commute and are diagonalizable. Prove that they are simultaneously diagonalizable.

Question 5. Let V a finite-dimensional vector space. A linear map $f\colon V \longrightarrow V$ is called an *involution* if $f \circ f = \mathrm{id}$.

(a) Prove that every involution is diagonalizable

(b) Find the maximal number of distinct commuting involutions on V in terms of $n := \dim V$.

Question 6. Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that $AB - BA = \alpha A$ for some $\alpha \neq 0$. Prove that

(a) $A^kB - BA^k = \alpha k A^k$

(b) $A^m = 0$ for some $m > 0$.

Question 7. (a) Show that for every n there exists $A \in M_{n\times n}(\mathbb{R})$ such that $A^3 = A + I$

(b) Show that $\det A > 0$ for every $A \in M_{n\times n}(\mathbb{R})$ satisfying $A^3 = A + I$.

Question 8. Given $A, B \in M_{n\times n}(\mathbb{R})$ such that $\operatorname{rank}(AB - BA) = 1$ prove that $(AB - BA)^2 = 0$.

Question 9. Prove that for an $n \times n$ complex matrix A there exist a unitary matrix U such that UAU^* is upper-triangular.

Question 10. Let $X \in M_{n\times n}(\mathbb{R})$. Prove that $\operatorname{tr} X^2 \leq \operatorname{tr} XX^T$.

Question 11 (†)**.** Prove that for an $n \times n$ complex matrix A and a positive integer m we have

$$|\operatorname{tr} A^{2m}| \leq \operatorname{tr}(AA^*)^m.$$

Question 12. Let $A, B \in M_{n\times n}(\mathbb{R})$ be symmetric. Prove that $\operatorname{tr} ABAB \leq \operatorname{tr} A^2B^2$.

Question 13. Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric and positive definite. Prove that $\det(I + A) \geq 1 + \det A$.

Remark. † questions may be slightly harder.