Extra questions for first year maths students, Term 2 2012/2013

Tomasz Tkocz

Question 1. Prove that for a natural number $n \geq 1$ we have

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}.$$

Conclude that

$$\left(\frac{(1+1/n)^n}{e}\right)^n \xrightarrow[n\to\infty]{} 1/\sqrt{e}.$$

Question 2. Find all differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying for all reals $x \neq y$

$$\frac{f(y) - f(x)}{y - x} = f'\left(\frac{x+y}{2}\right).$$

What is the geometric interpretation of this equation? Let $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be bounded below, say $f(x), g(x) \geq 0$ for all $x \in \mathbb{R}$. Suppose that $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the function $f \Box g$ defined by

$$(f \Box g)(x) = \inf_{t\in\mathbb{R}} \{f(x) + g(x-t)\}.$$

is continuous.

Question 3. Given distinct real numbers $\lambda_1, \lambda_2, \ldots$ prove that in the linear space $V = \{f\colon \mathbb{R} \longrightarrow \mathbb{R}\}$ of all real valued functions on $\mathbb{R}$ the following sets of functions are linearly independent

(a) $\{e^{\lambda_1 x}, e^{\lambda_2 x}, e^{\lambda_3 x}\}$

(b) $\{e^{\lambda_n x}\}_{n=1,2,\ldots}$

Remark. A subset (possibly infinite) $A \subset V$ of a linear space V is called *linearly independent* if for any for any finite subset $\{v_1, v_2, \ldots, v_n\} \subset A$ the vectors $v_1, \ldots, v_n$ are linearly independent.

Question 4. Let V be a real n dimensional vector space. For a linear map $f\colon V \longrightarrow V$ satisfying $f \circ f = \mathrm{id}$ prove that $\mathrm{rank}(f + \mathrm{id}) + \mathrm{rank}(f - \mathrm{id}) = n$.

Question 5. Find all differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying for all reals $x \neq y$

$$\frac{f(y) - f(x)}{y - x} = f'\left(\frac{x+y}{2}\right).$$

What is the geometric interpretation of this equation?

Question 6. Let V be an n dimensional vector space and let $S, T\colon V \longrightarrow V$ be linear maps. Prove that

$$\mathrm{rank}\, TS \geq \mathrm{rank}\, T + \mathrm{rank}\, S - n.$$

Question 7. Does there exist a differentiable function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f'(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0\,? \\ 1, & x > 0 \end{cases}$$

Question 8. Let V be a real vector space, let $\varphi\colon V \longrightarrow V$ be a linear map, and let $v_1, \ldots, v_k \in V$ be non-zero distinct vectors such that $\varphi(v_1) = v_1$, $\varphi(v_i) = v_i + v_{i-1}$ for $i = 2, \ldots, k$. Prove that $v_i's$ are linearly independent.

Question 9. Prove that

$$\det \begin{bmatrix} 1233 & 3046 & 5098 & 347 \\ 2013 & 4327 & 5832 & 8743 \\ 496 & 5439 & 1205 & 9731 \\ 7655 & 7095 & 2341 & 5402 \end{bmatrix} \neq 0.$$

Question 10. Given $n \geq 3$ and real numbers $t_1, \ldots, t_n$ prove that

$$\det[\cos(t_i - t_j)]_{i,j=1}^n = 0.$$