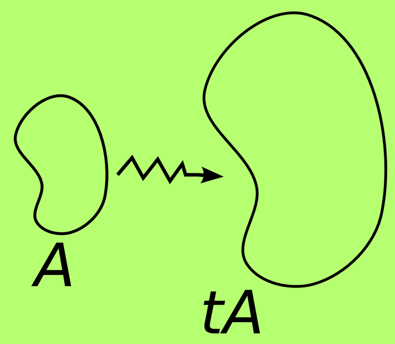
How fast does the Gaussian measure of a convex and symmetric set grow?

Tomasz Tkocz
University of Warsaw

t.tkocz@students.mimuw.edu.pl

Definition



Dilation of a Borel set $A \subset \mathbb{R}^n$ is

$$A \rightsquigarrow tA, \quad t > 0.$$

What happens with its measure?

- Lebesgue measure — it is trivial $|tA| = t^n|A|$.
- Gaussian measure — there is a question $\gamma_n(tA) = ?$

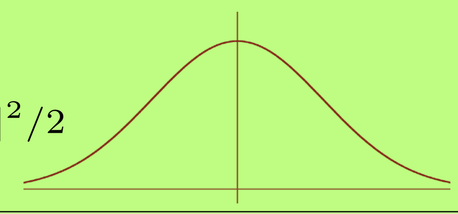
Problem

Give the optimal bounds from above and from below on the function

$$t \xrightarrow{f_A} \gamma_n(tA), \qquad t \geq 1.$$

Reminder

The Gaussian measure γ_n has the density $\frac{1}{\sqrt{2\pi}^n} e^{-|x|^2/2}$



Real case

Assumptions

We restrict ourself to the sets which are

- convex,
- symmetric ($A = -A$).

That is, A is a ball with respect to some norm on $\mathbb{R}^n$.

Figure. A convex and symmetric set.

Complex case

⚠ The isomorphism $\mathbb{C}^n \approx \mathbb{R}^{2n}$ equips the space $\mathbb{C}^n$ with the standard Gaussian measure ν_n.

Assumptions Let $A \subset \mathbb{C}^n$ be

- convex (in the usual real sense),
- rotationally symmetric ($A = \lambda A$, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$).

Question How fast does the function $t \xrightarrow{f_A} \nu_n(tA)$ grow?

Upper bound

Let $A \subset \mathbb{R}^n$ ($A \subset \mathbb{C}^n$) be convex and (rotationally) symmetric and let $B = \{|x| \leq R\}$ be an euclidean ball such that $\gamma_n(A) = \gamma_n(B)$ ($\nu_n(A) = \nu_n(B)$). Then

$$f_A(t) \leq f_B(t), \quad t \geq 1.$$

It means that the measure of balls grows the fastest.

Lower bound ([1])

Let $A \subset \mathbb{R}^n$ be convex and symmetric and $P = \{|x_1| \leq p\}$ be a strip such that $\gamma_n(A) = \gamma_n(P)$. Then

$$f_A(t) \geq f_P(t), \quad t \geq 1.$$

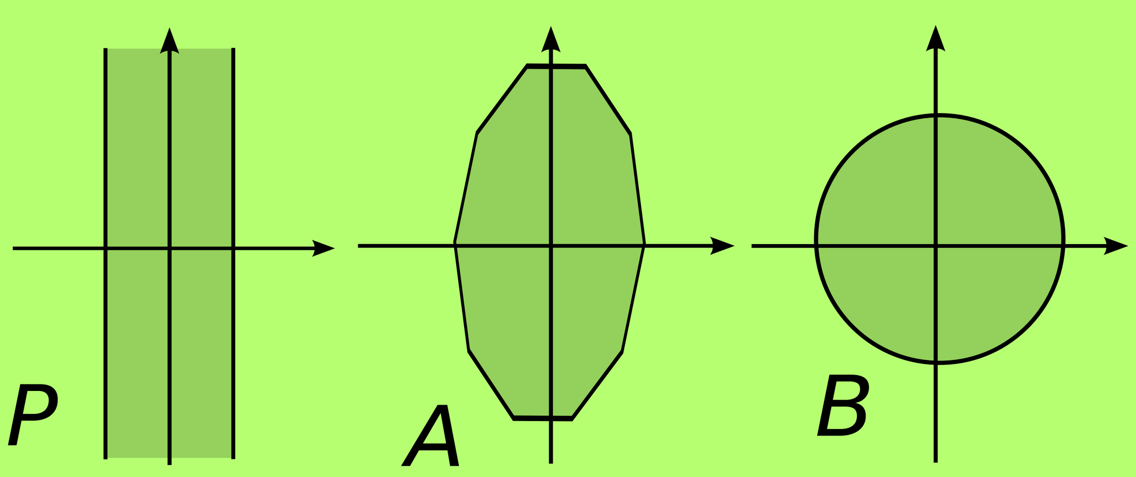
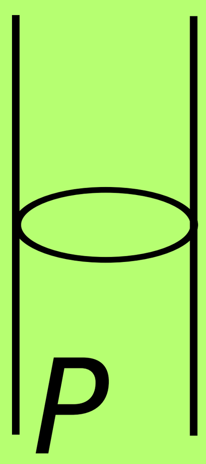


Figure. Bounds in the picture

Lower bound ([2])



Let $A \subset \mathbb{C}^n$ be convex and rotationally symmetric and let $P = \{|z_1| \leq p\}$ be a cylinder such that $\nu_n(A) = \nu_n(P)$. Then

$$f_A(t) \geq f_P(t), \quad t \in [1, t_0],$$

where $t_0 = t_0(A)$ is such that $\nu_n(t_0 A) = c$ for some absolute constant $c \approx 0.64$.

Conjecture

Cylinders are optimal, i.e. the above inequality holds for all $t \geq 1$.

References

[1] R. Latała and K. Oleszkiewicz, *Gaussian measures of dilatations of convex symmetric sets*, The Annals of Probability, **27** (1999), 1922–1938

[2] T. Tkocz, *Gaussian measures of dilations of convex rotationally symmetric sets in C^n*, arXiv:1007.2907v1 [math.PR] (2010)