1

Square Integrable Martingales, Local Martingales, Quadratic Variation

$(\Omega, \mathcal{F}, P)$, $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ given. Assume $\mathbb{F}$ satisfies the usual conditions

Let $X = \{X_t\}_{t \geq 0}$ be a right-continuous martingale.

If $E[X_t^2] < \infty \quad \forall\ t \geq 0$ we say that X is <u>square integrable</u>

- Additionally, if $X_0 = 0$ a.s. we say that $X \in \mathcal{M}_2$, i.e. we define

$$\mathcal{M}_2 = \{X \mid X \text{ is a right-continuous, square integrable martingale with } X_0 = 0 \text{ a.s.}\}.$$

- If X is continuous we say

$$X \in \mathcal{M}_2^c = \{X \in \mathcal{M}_2 \mid X \text{ is continuous}\}.$$

(2)

Metric Structure on $\mathcal{M}_2$:

For $X \in \mathcal{M}_2$ define

$$\|X\|_t^2 = \sqrt{E[X_t^2]} \qquad t > 0$$

$$\|X\| = \sum_{n=1}^{\infty} \frac{\|X\|_n \wedge 1}{2^n}$$

Claim: if we identify indistinguishable processes as being equal then $\|\cdot\|$ is a metric.

pf.

$$\|X - Y\| = 0 \quad \Rightarrow \quad \|X - Y\|_n = 0 \quad \forall n$$

$$\Rightarrow X_n = Y_n \quad \text{a.s.} \quad \forall n$$

$$\Rightarrow X_t = E[X_n | \mathcal{F}_t] = E[Y_n | \mathcal{F}_t] = Y_t$$

$$\forall \; 0 \le t \le n$$

$\Rightarrow X, Y$ indistinguishable since they are also right continuous.

③

Claim: M_2 is a complete metric space under $\|\cdot\|$ and M_2^c is a closed sub-space of M_2.

pf.

$\{X_n\}_{n=1,2,\ldots}$ Cauchy in $(M_2, \|\cdot\|)$

$\Rightarrow$ $\{X_n(t)\}_{n=1,2,\ldots}$ Cauchy in $L^2(\Omega, \mathcal{F}_t, P)$ $\forall t$.

$\Rightarrow$ $X_n(t) \to \cancel{X}(t)$ in $L^2(\Omega, \mathcal{F}_t, P)$

Now for $s \le t$, $A \in \mathcal{F}_s \subseteq \mathcal{F}_t$ we have

$$0 = \lim_{n\to\infty} E[1_A(X_n(t) - X(t))]$$
$$= \lim_{n\to\infty} E[1_A(X_n(s) - X(s))]$$

$\Rightarrow$ $X := \{X(t)\}_{t\ge 0}$ (collection of limits) is a square integrable martingale

$\Rightarrow$ $\mathbb{F}$ satisfies usual conditions implies we can take a right continuous modification, starting at 0 since $X_n(0) = 0$ $\forall n$

$\Rightarrow$ Using Dominated Convergence $\|X^n - X\| \to 0$

(4)

Thus, $(M_2, \|\cdot\|)$ is complete.

Now, Assume $\{X_n\}_{n=1,\ldots} \subseteq M_2^c$ converges to $X \in M_2$ in $\|\cdot\|$

$\{(X^n - X)_t\}_{t \geq 0}$ is a martingale

$\Rightarrow$ $|X^n - X|$ is a sub-martingale

$$\Rightarrow P\left(\sup_{t \leq T} |X^n - X|_t \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} E\left[|X^n - X|_T^2\right]$$

$$= \frac{1}{\varepsilon^2} \|X^n - X\|_T^2 \to 0$$

$\Rightarrow$ along some $\{n_k\}_{k=1,2,\ldots}$ we have

$$P\left(\sup_{t \leq T} |X^{n_k} - X|_t \geq \frac{1}{k}\right) \leq \frac{1}{2^k}$$

$\Rightarrow$ Borel-Cantelli gives $X^{n_k} \to X$ uniformly on $[0,T]$ almost surely.

$\therefore$ X is continuous on $[0,T]$ a.s. $\forall T \geq 0$, hence on $[0,\infty)$.

5

Before proceeding to quadratic variation we first introduce the fundamental concept of a local martingale.

Let $X = \{X_t\}_{t \geq 0}$ be an adapted process. If there exists a sequence of stopping times $\{T_n\}_{n=1,2,\ldots}$ s.t.

1) $T_n \to \infty$ a.s.

2) $X^{(n)} = \{X_{t \wedge T_n}\}_{t \geq 0}$ is a martingale $\forall$ $n = 1, 2, 3, \ldots$

then we say X is a local martingale

If additionally:

a) $X_0 = 0$ a.s., write $X \in \mathcal{M}^{loc}$

b) $X_0 = 0$ a.s., X continuous we write $X \in \mathcal{M}^{c,loc}$

6

Important Facts:

- ~~Local~~ Martingales are Local Martingales ($T_n = n$)
- Local Martingales need not be Martingales
  - we will come up with examples of uniformly integrable local martingales which are not martingales.

The Basic Issue:

X: (r-cont, cont) local martingale.

$\forall n$:

$$E[|X_{t \wedge T_n}|] < \infty \qquad \forall t \geq 0$$

$$E[X_{t \wedge T_n} 1_A] = E[X_{s \wedge T_n} 1_A]$$
$$\forall \quad 0 \leq s < t, A \in \mathcal{F}_s.$$

7

So, for X to be a martingale we need

$$\{X_{t \wedge T_n}\}_{n=1,2,\ldots} \text{ to be u.i.}$$

for each $t \geq 0$.

- not enough that $\{X_t\}_{t \geq 0}$ be u.i.

But, one can show:

1) Local Martingales of class DL are Martingales

2) Non-negative Local Martingales are super-Martingales

And, we can always assume, at least for continuous processes, that local martingales are uniformly bounded

1) $\{X_{t \wedge T_n}\}$ is a Martingale

2) set $\sigma_n = \inf[t \geq 0 \mid |X_t| = n]$ - stopping

with $\tilde{T}_n = \sigma_n \wedge T_n$ we have $\tilde{T}_n \to \infty$ a.s. and

$$|X_{t \wedge \tilde{T}_n}| \le n$$

and one can use optional sampling to show that $X_{t \wedge \tilde{T}_n}$ is a martingale.

- we can use $\tilde{T}_n$ as our sequence of stopping times.

Local Martingales are Ubiquitous!

Quadratic Variation.

Let $X \in M_2$. Then X^2 is a (r-cont) sub martingale which is non-negative (hence of class DL) so we can write

$$X_t^2 = M_t + A_t \qquad *$$

(9)

M: r-cont martingale

A: r-cont, $\nearrow$, natural

Also, if $X \in \mathcal{M}_2^c$ then we can take M, A to be continuous.

<u>Definition.</u>

$X \in \mathcal{M}_2$. Then the quadratic variation of X is the process

$$\langle X \rangle_t = A_t \qquad t \geq 0$$

where A is from $*$. Thus, $\langle X \rangle$ is the unique (up to indistinguishability) natural increasing process so that

$$X^2 - \langle X \rangle$$

is a martingale.

(10)

Q. Why "Quadratic" Variation?

Let X be any process, and fix $t \geq 0$.
Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,t]$

$$0 = t_0 < t_1 < \cdots < t_n = t.$$

The quadratic variation of X over Π is defined by

$$V_t^{(2)}(\Pi) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 \quad (*)$$

- this looks like "real" quadratic variation.

Define the "size" $\|\Pi\|$ of the partition by

$$\|\Pi\| = \max_{i \leq n} (t_i - t_{i-1})$$

The reason we say $\langle X \rangle$ is the quadratic variation is the following:

<u>Thm</u>

Let $X \in M_2^c$, $t \geq 0$. Then

$$\lim_{\|\Pi\| \to 0} V_t^{(2)}(\Pi) = \langle X \rangle_t \quad \text{in Probability.}$$

I.e. $\forall\ \varepsilon > 0, \eta > 0 \quad \exists\ \delta > 0$ s.t.

$$\|\Pi\| < \delta \implies P\left(\left|V_t^{(2)}(\Pi) - \langle X \rangle_t\right| > \varepsilon\right) < \eta.$$

Why is this true?

Key observation: $\quad 0 \leq s < t \leq u < v$

$$E\left[(X_t - X_s)(X_v - X_u)\right]$$

$$= E\left[(X_t - X_s)\, E\left[(X_v - X_u) \mid \mathcal{F}_u\right]\right]$$

$$= 0$$

- cross-product terms all go to zero in non-overlapping intervals.

(12)

In particular, for $0 \le t \le u < v$

$$E[(X_v - X_u)^2 \mid \mathcal{F}_t] = E[X_v^2 - 2X_vX_u + X_u^2 \mid \mathcal{F}_t]$$

$$= E[X_v^2 - 2X_u E[X_v \mid \mathcal{F}_u] + X_u^2 \mid \mathcal{F}_t]$$

$$= E[X_v^2 - X_u^2 \mid \mathcal{F}_t]$$

$$= E[X_v^2 - \langle X \rangle_v + \langle X \rangle_v - (X_u^2 - \langle X \rangle_u) - \langle X \rangle_u \mid \mathcal{F}_t]$$

$$= E[\langle X \rangle_v - \langle X \rangle_u \mid \mathcal{F}_t]$$

$\circ$ $X^2 - \langle X \rangle$ a martingale.

~~Summary~~

Summarizing: 3 equalities

1) $E[(X_v - X_u)^2 \mid \mathcal{F}_t] = E[X_v^2 - X_u^2 \mid \mathcal{F}_t]$

2) $E[(X_v - X_u)^2 - (\langle X \rangle_v - \langle X \rangle_u) \mid \mathcal{F}_t] = 0$

3) $E[X_v^2 - X_u^2 - (\langle X \rangle_v - \langle X \rangle_u) \mid \mathcal{F}_t] = 0$

(13)

Using these equalities the proof is straight forward, using a localization technique:

Step 1: Assume $|X|_s \le K$ for $s \le t$.

Then $E[(V_t^{(2)}(\pi))^2] \le 6K^4$

pf

$$E[(V_t^{(2)}(\pi))^2] = \overset{①}{E\left[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^4\right]}$$

$$+ 2\overset{②}{E\left[\sum_{i=1}^{n}\sum_{j=1+i}^{n}(X_{t_i}-X_{t_{i-1}})^2(X_{t_j}-X_{t_{j-1}})^2\right]}$$

① $\le 4K^2 E\left[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^2\right]$

$|X_{t_i}-X_{t_{i-1}}| \le 2K.$

$= 4K^2 E\left[\sum_{i=1}^{n} X_{t_i}^2 - X_{t_{i-1}}^2\right] = 4K^2 E[X_t^2 - X_0^2]$

$\le 4K^4.$

(14)

$$E\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n}(X_{t_i}-X_{t_{i-1}})^2(X_{t_j}-X_{t_{j-1}})^2\right]$$

$$= E\left[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^2\, E\left[\sum_{j=i+1}^{n}(X_{t_j}-X_{t_{j-1}})^2 \,\middle|\, \mathcal{F}_{t_i}\right]\right]$$

$$= E\left[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^2\, \underbrace{E\left[\sum_{j=i+1}^{n}(X_{t_j}^2-X_{t_{j-1}}^2) \,\middle|\, \mathcal{F}_{t_i}\right]}_{= X_t^2 - X_{t_i}^2 \le K^2}\right]$$

$$\le K^2\, E\left[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^2\right]$$

$$\le K^4 \quad \text{(previous calculation)}$$

$\therefore$ (1) + 2(2) $\le 4K^4 + 2K^4$ ▨.

Step 2: Assume $|X_s| \le K \quad s \le t$

Then $\lim_{\|\Pi\|\to 0} E\left[V_t^{(4)}(\Pi) \triangleq \sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^4\right] = 0$

pf.

Note $V_t^{(4)}(\Pi) \le V_t^{(2)}(\Pi) \cdot \max_{i=1,\dots,n} |X_{t_i}-X_{t_{i-1}}|^2$

Set $m_t(X;\delta) = \sup\limits_{\substack{0 \le u < v \le t \\ v-u \le \delta}} |X_v - X_u|$

-mbl r.v. b/c X continuous

$$\Rightarrow V_t^{(4)}(\Pi) \le V_t^{(2)}(\Pi) \cdot m_t(X;\|\Pi\|)^2$$

Now.

1) $m_t(X;\|\Pi\|) \le 2K$ $\quad |X_s| \le K$

2) $\lim\limits_{\|\Pi\| \downarrow 0} m_t(X;\|\Pi\|) = 0$ a.s. b/c X is continuous, hence uniformly continuous on $[0,t]$.

$$\therefore E\left[V_t^{(4)}(\Pi)\right] \le E\left[\left(V_t^{(2)}(\Pi)\right)^2\right]^{1/2} E\left[m_t^4(X;\|\Pi\|)\right]^{1/2}$$

$$\le \sqrt{6K^4}\, E\left[m_t^4(X;\|\Pi\|)\right]^{1/2}$$

$\to 0$ as $\|\Pi\| \to 0$ by the dominated convergence thm. $\blacksquare$

Step 3: Assume $|X|_s \le K$, $\langle X \rangle_s \le K$ for $s \le t$. Then

$$\lim_{\|\pi\| \to 0} E\left[\left(V_t^{(2)}(\pi) - \langle X \rangle_t\right)^2\right] = 0$$

so $V_t^{(2)}(\pi) \to \langle X \rangle_t$ in L^2, Prob.

pf

$$E\left[\left(V_t^{(2)}(\pi) - \langle X \rangle_t\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 - (\langle X \rangle_{t_i} - \langle X \rangle_{t_{i-1}})\right)^2\right]$$

$$= \sum_{i=1}^{n} E\left[\left((X_{t_i} - X_{t_{i-1}})^2 - (\langle X \rangle_{t_i} - \langle X \rangle_{t_{i-1}})\right)^2\right]$$

- equality 2) before

$$\le 2 \sum_{i=1}^{n} E\left[(X_{t_i} - X_{t_{i-1}})^4\right] + 2 \sum_{i=1}^{n} (\langle X \rangle_{t_i} - \langle X \rangle_{t_{i-1}})^2$$

(17)

$$\leq 2E\left[V_t^{(4)}(\Pi)\right] + 2E\left[\langle X\rangle_t \cdot m_t(\langle X\rangle;\Pi)\right]$$

pull out max, telescoping sum with $\langle X\rangle_0 = 0$

$\to 0$

b/c $\langle X\rangle_t \leq K$ + previous lemma.

Step 4: Un-localize....

$$T_n = \inf\left[t \geq 0 \;\middle|\; |X|_t \geq n \text{ or } \langle X\rangle_t \geq n\right]$$

$\Rightarrow \{X_{t\wedge T_n}\}_{t\geq 0}$; $\{X^2_{t\wedge T_n} - \langle X\rangle_{t\wedge T_n}\}_{t\geq 0}$ bdd martingales

Also, $\langle X_{\cdot\wedge T_n}\rangle_t = \langle X\rangle_{t\wedge T_n}$ by uniqueness in Doob-decomposition.

By step 3:

$$\lim_{\|\Pi\|\to 0} E\left[\left(\sum_{k=1}^{m}(X_{t_k\wedge T_n} - X_{t_{k-1}\wedge T_n})^2 - \left(\langle X\rangle_{t_k\wedge T_n} - \langle X\rangle_{t_{k-1}\wedge T_n}\right)\right)^2\right]$$

$= 0$

(18)

Since $T_n \to \infty$ a.s. it follows that

$V_t^{(2)}(\pi) \to \langle X \rangle_t$ in Prob.

- make sure you can show this! ∎

Covariations

Let $X, Y \in \mathcal{M}_2$. We define the covariation, or cross-variation, of X and Y as

$$\langle X, Y \rangle_t = \frac{1}{4}\left[\langle X+Y \rangle_t - \langle X-Y \rangle_t\right] \quad t \geq 0$$

- $X+Y,\ X-Y \in \mathcal{M}_2$

Note

$$ab = \frac{1}{4}\left((a+b)^2 - (a-b)^2\right) \quad a, b \in \mathbb{R}.$$

$$\Rightarrow \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) = \frac{1}{4}\sum_{i=1}^{n}\left[(X_{t_i} + Y_{t_i} - (X_{t_{i-1}} + Y_{t_{i-1}}))^2 - (X_{t_i} - Y_{t_i} - (X_{t_{i-1}} - Y_{t_{i-1}}))^2\right]$$

so if $X, Y \in \mathcal{M}_2^c$ then

$$\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) \xrightarrow{\text{prob.}} \langle X, Y \rangle_t$$

as $\|\Pi\| \to 0$ for $t \geq 0$

- why we call it covariation.

<u>Also</u>

1) $X_t Y_t - \langle X, Y \rangle_t$

$$= \frac{1}{4}\Big[(X_t + Y_t)^2 - \langle X+Y \rangle_t - \big((X_t - Y_t)^2 - \langle X-Y \rangle_t\big)\Big]$$

so $XY - \langle X, Y \rangle$ is a martingale.

2) We say X, Y are orthogonal if $\langle X, Y \rangle_t = 0 \ \forall \, t \geq 0$ with probability one.

so $\langle X, Y \rangle$ orthogonal implies XY is a martingale.

3) $\langle X \rangle_t = \langle X, X \rangle_t$ for $X \in \mathcal{M}_2$

4) Another way to think about $\langle X, Y \rangle$ for $X, Y \in \mathcal{M}_2^c$:

$\langle X, Y \rangle$: unique continuous process A of bounded variation with $A_0 = 0$ s.t. $\{X_t Y_t - A_t\}_{t \geq 0}$ is a martingale.

pf. If $\exists$ A, B s.t. statement holds then $B - A$ is a continuous martingale of finite variation

$\Rightarrow B - A \equiv 0$. (Read Thm 5.13)

21

5) $\langle X, Y\rangle$ well defined for X, Y in $M^{c,loc}$

$\exists!$ adapted, continuous process A, $A_0 = 0$ s.t. $XY - A \in M^{c,loc}$.

write $A = \langle X, Y\rangle$

Idea:
$X^n_t = X_{t\wedge T_n}$; $Y^n_t = Y_{t\wedge T_n}$ $\in M^c_2$ so A^n s.t. $X^nY^n - A^n \in M^2_c$. By uniqueness $A^n = A^m$ on $T_n \wedge T_m$

since $T_n \nearrow \infty$ we define $A = A^n$ on $[0, T_n]$ and A is consistently defined $\forall\ t \geq 0$ and $XY - A \in M^{c,loc}$ with $\{T_n\}$