Math 21-880 Final: SOLUTIONS

December 8, 2014

This is a closed book, closed notes exam. No calculators or smart phones are allowed. You have 3 hours to complete the exam. Please mark your answers clearly and put your name on each piece of paper you submit. There are five questions on the exam.

(1) **20 Points.** Let W be a standard one-dimensional Brownian motion with $W_0 = 0$. Set $\sigma = \inf\{t \geq 0 \mid W_t = 1\}$ and $\tau = \inf\{t \geq \sigma \mid W_t = -1\}$. Note that by definition $\tau > \sigma$. Compute $\mathbb{P}[\tau < t]$.

(2) **20 Points.** Consider the SDE

$$dX_t = b(X_t)dt + dW_t; \qquad X_0 = x \in \mathbb{R}^d.$$

Here, W is a standard d-dimensional Brownian motion and b is a bounded, Lipschitz function. Show that for any Borel set $A \subseteq \mathbb{R}^d$ with positive (Lebesgue) volume that $\mathbb{P}[X_t^x \in A] > 0$ for all $x \in \mathbb{R}^d$ and $t \geq 0$.

(3) **20 Points.** Let X solve the SDE

$$dX_t = \sigma(X_t)dW_t; \qquad X_0 = x \in \mathbb{R}^d.$$

Here, W is again a d-dimensional Brownian motion and we assume σ is Lipschitz, symmetric and point-wise (in x) positive definite. Now, think of X as the share price of some traded asset: i.e. $X_t(\omega)$ is the price at (t, ω). Let $\pi = \{\pi_t\}_{t\geq 0}$ denote a trading strategy: i.e. $\pi_t(\omega) \in \mathbb{R}^d$ is the number of shares of X we hold at (t, ω). For a given initial wealth $w_0 \in \mathbb{R}$, the wealth processes associated to π is denoted by $\mathcal{W}^\pi$ and satisfies the formula

$$\mathcal{W}_t^\pi = w_0 + \int_0^t \pi_u^{\mathrm{T}} dX_u = w_0 + \int_0^t \pi_u^{\mathrm{T}} \sigma(X_u) dW_u,$$

provided the integrals are well-defined. Note that in particular we require π to be adapted to the (augmented) filtration generated by W.

i) **6 Points.** We say the (X, W) market is complete on $[0, T]$ if for any bounded $\mathcal{F}_T$ measurable random variable H, there is some initial capital w_0 and trading strategy π such that $H = \mathcal{W}_T^\pi$ with probability one. $\mathcal{W}^\pi$ is called the "replicating" process for H. Show that the (X, W) market is complete.

ii) **8 Points.** Now assume that $H = g(X_T)$ for a bounded continuous function g. Identify a partial differential equation such that if $u \in C^{1,2}((0,T)\times\mathbb{R}^d)$ solves the PDE then u "should" take the form $u(t,y) = E\left[g(X_T)\,|X_t = y\right]$. Show that if $u \in C^{1,2}((0,T)\times\mathbb{R}^d)$ is a bounded solution of this PDE then u does admit the representation $u(t,y) = E\left[g(X_T) \mid X_t = y\right] = E\left[g(X^y_{T-t})\right]$ where X^y is the solution of the above SDE starting at y.

iii) **6 Points.** In the setting of b), assume u solves the PDE and admits the stochastic representation. Identify the initial capital w_0 and trading strategy π explicitly for the replicating wealth process of $g(X_T)$.

(4) **20 Points.** Let W be a standard Brownian motion starting at 0. Recall that the local time of W near a on $[0,t]$ is given by

$$L_t(a) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\text{Leb}\left[s \le t \mid |W_s - a| \le \varepsilon\right],$$

in that we showed the limit exists almost surely and satisfies Tanaka's formula.

i) **10 Points.** Sketch the proof of Tanaka's formula using the approximating functions g^ε from lecture (or some other approximating functions if you wish). Provide as much detail as you can, but don't spend the whole exam time filling in every step if you are stuck!

ii) **10 Points.** Recall that we also showed for all Borel measurable non-negative functions h on $\mathbb{R}$ the almost sure identity

$$\int_0^t h(W_s)ds = \int_{\mathbb{R}} h(a)L_t(a)da \tag{1}$$

Now, let $f \in C^2(\mathbb{R})$ be strictly increasing with $f(\pm\infty) = \pm\infty$. Define the process $Y_t = f(W_t)$. Motivated by (1) above, we define the local time $L^Y_t(a)$ as the two parameter random field such that (amongst other properties) for all non-negative Borel measurable functions h on $\mathbb{R}$ we have almost surely that

$$\int_0^t h(Y_s)d\langle Y\rangle_s = \int_{\mathbb{R}} h(a)L^Y_t(a)da; \qquad t \ge 0.$$

Assuming it exists, identify $L^Y_t(a)$ explicitly.

(5) **20 Points.** For a fixed $T > 0$, identify the Laplace transform of $\int_0^T W_t^2 dt$ where W is a standard one dimensional Brownian motion starting at 0, i.e. compute

$$E\left[e^{-\lambda\int_0^T W_t^2 dt}\right]; \qquad \lambda > 0. \tag{2}$$

Complete this answer through the following steps:

i) **5 Points.** If $\phi \in L^2[0,T]$ is deterministic show that $\int_0^T \phi_u dB_u \sim N(0, \int_0^T \phi_u^2 du)$ where B is a standard one-dimensional Brownian motion.

ii) **5 Points.** Define Y as the solution of the SDE $dY_t = -\gamma Y_t dt + dB_t, Y_0 = 0$ where B is a standard one-dimensional Brownian motion and $\gamma > 0$. Compute the distribution of Y_T and use this to evaluate $E\left[e^{\beta Y_T^2}\right]$ for $\beta \in \mathbb{R}$. Are there any restrictions upon β?

iii) **10 Points.** Use your answer above to compute (2) for any $\lambda > 0$.

Solutions

(1) There are two ways to answer this question. The slickest way is to define $B_t = W_{t\wedge\sigma} + (W_{t\wedge\sigma} - W_t)$ and note that by Levy it follows that B is a Brownian motion. Then, we clearly have τ as the first hitting time of B to 3 and thus

$$\mathbb{P}\left[\tau < t\right] = 2\mathbb{P}\left[B_t > 3\right] = \frac{2}{\sqrt{2\pi t}} \int_3^\infty e^{-z^2/(2t)} dz$$

A more straightforward way to solve this problem is to note by the strong Markov property that $B_t = W_{t+\sigma} - W_\sigma$ is a Brownian motion in it's own filtration (this follows since $\sigma < \infty$ almost surely). Furthermore, B is independent of $\mathcal{F}_\sigma$. We thus see that $\mathbb{P}\left[\tau < t\right] = \mathbb{P}\left[\sigma + \tau_{-2} < t\right]$ where τ_{-2} is the first hitting time to -2 of B and σ, τ_{-2} are independent. This latter fact and the symmetry of Brownian motion allow us to conclude that $\mathbb{P}\left[\tau < t\right] = \mathbb{P}\left[\sigma + \tau_2 < t\right]$ and hence using convolution

$$\begin{aligned} \mathbb{P}\left[\tau < t\right] &= \int_0^t \mathbb{P}\left[\tau_2 < t - s\right] \mathbb{P}\left[\sigma \in ds\right] \\ &= 2\int_0^t \mathbb{P}\left[W_{t-s} > 2\right] \mathbb{P}\left[\sigma \in ds\right] \\ &= 2\mathbb{P}\left[W_t > 3\right]. \end{aligned}$$

Here, we have set W as a "generic" Brownian motion since we are only interested in the laws. Thus, the two formulas coincide.

(2) This will follow via Girsanov's theorem. Namely, fix $x \in \mathbb{R}^d$ and define

$$Z_t^x = \mathcal{E}\left(\int_0^\cdot -b(X_u^x) dW_u\right)_t; \qquad t \geq 0.$$

Since b is bounded we know that Z is a martingale and we may define a measure $\mathbb{Q}^x$ equivalent to $\mathbb{P}$ via $d\mathbb{Q}^x/d\mathbb{P}|_{\mathcal{F}_t} = Z_t^x$ for each $t \geq 0$. By Girsanov's theorem we know that $\tilde{W}^x = W + \int_0 b(X_u^x) du$ is a $\mathbb{Q}^x$ Brownian motion. We thus have under $\mathbb{Q}^x$ that

$$dX_t^x = b(X_t^x) dt + dW_t = d\tilde{W}_t^x$$

Thus, X^x is a Brownian motion starting at x under $\mathbb{Q}^x$ and hence $\mathbb{Q}^x[X_t \in A] > 0$ for any A with positive volume. Thus, by the equivalence of $\mathbb{Q}^x$ and $\mathbb{P}$ on $\mathcal{F}_t$ the result follows.

(3) i) Consider the martingale M given by

$$M_t = E[H \mid \mathcal{F}_t]; \qquad t \leq T.$$

Since H is bounded, M is indeed a martingale. By the martingale representation theorem there exists an adapted process ϕ such that $\int_0^t \phi_t^{\mathrm{T}} dW_t$ is well defined and such that

$$M_t = M_0 + \int_0^t \phi_u^{\mathrm{T}} dW_u.$$

In fact, $M_0 = E[H]$. Evaluating this at $t = T$ we obtain

$$M_T = H = E[H] + \int_0^T \phi_u^{\mathrm{T}} dW_u.$$

Now, we know for any initial capital and trading strategy we have

$$\mathcal{W}_T^\pi = w_0 + \int_0^T \pi_u^{\mathrm{T}} \sigma(X_u) dW_u$$

Thus, we can replicate H with initial capital $w_0 = E[H]$ and trading strategy π_t provided that

$$\sigma(X_t)\pi_t = \phi_t \Longrightarrow \pi_t = \sigma(X_t)^{-1}\phi_t.$$

Here, we have used that σ is symmetric and invertible. Since $\int_0^t \phi_u^{\mathrm{T}} dW_u$ is well defined, we know that $\mathcal{W}^\pi$ is well defined.

ii) Define the differential operator L by

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} \Sigma^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

for C^2 functions f. Consider the PDE

$$\begin{aligned} u_t(t,y) + Lu(t,y) &= 0; \qquad t \in (0,T), y \in \mathbb{R}^d \\ u(T,y) &= g(y); \qquad y \in \mathbb{R}^d. \end{aligned}$$

Assume $u \in C^{1,2}((0,T) \times \mathbb{R}^d)$ is a bounded solution to the PDE. Write $X^{t,x}$ for the solution of the SDE starting at x. Define $\tau_n = \inf\{s \geq t \mid \|X_s^{t,x}\| \geq n\}$. Assume $\|x\| < n$ we have

$$u(T \wedge \tau_n, X_{T\wedge\tau_n}^{t,x}) = u(t,x) + \int_t^{T\wedge\tau_n} \nabla u(u, X_u^{t,x})^{\mathrm{T}} \sigma(X_u^{t,x}) dW_u$$

Taking conditional expectations with respect to $\mathcal{F}_t$ and noting that ∇u and σ are bounded on $\{x \,|\|x\| \leq n\}$ we have

$$E\left[u(T \wedge \tau_n, X^{t,x}_{T\wedge\tau_n}) \mid \mathcal{F}_t\right] = u(t,x)$$

Since $\tau_n \to \infty$ almost surely we have since u is bounded and continuous that

$$E\left[u(T, X^{x,t}_T) \mid \mathcal{F}_t\right] = E\left[g(X^{x,t}_T) \mid \mathcal{F}_t\right] = u(t,x)$$

By uniqueness of solutions we have we have for $\mathbb{P}\left[X\right]^{-1}_t$ almost every y that

$$E\left[g(X_T) \mid X_t = y\right] = u(t,y)$$

Lastly, by the Markovian property of X we may re-write this as

$$u(t,y) = E\left[g(X^y_{T-t})\right]$$

iii) With $w_0 = u(0,x)$ and $\pi_t = \nabla u(t, X_t)$ we have

$$\begin{aligned}\mathcal{W}^\pi_T &= w_0 + \int_0^T \pi_t^{\mathrm{T}} \sigma(X_t) dW_t = u(0,x) + \int_0^T \nabla u(t,X_t)^{\mathrm{T}} \sigma(X_t) dW_t \\ &= u(T, X_T) = g(X_T).\end{aligned}$$

So that $\mathcal{W}^\pi$ replicates. Note that $u(0,x) = E\left[g(X_T)\right]$ so this is the initial capital w_0.

(4) i) Recall the functions g_ε from lecture:

$$g_\varepsilon(x) = \begin{cases} |x-a| & |x-a| > \varepsilon \\ \frac{1}{2}\left(\varepsilon + \frac{(x-a)^2}{\varepsilon}\right) & |x-a| \leq \varepsilon \end{cases}.$$

We showed that $g_\varepsilon \in C^1(\mathbb{R})$ and that $\ddot{g}_\varepsilon$ is piecewise continuous taking the value 0 for $|x-a| > \varepsilon$ and $1/\varepsilon$ for $|x-a| < \varepsilon$. We are allowed to use Ito's formula for such functions and hence

$$\begin{aligned}g_\varepsilon(W_t) &= g_\varepsilon(z) + \int_0^t \dot{g}_\varepsilon(W_s) dW_s + \frac{1}{2}\int_0^t \ddot{g}_\varepsilon(W_s) ds \\ &= g_\varepsilon(z) + \int_0^t \dot{g}_\varepsilon(W_s) dW_s + \frac{1}{2\varepsilon}\mathrm{Leb}\left[s \leq t \mid |W_s - a| \leq \varepsilon\right]\end{aligned}$$

In the last equality we substituted in the $\leq$ since the level set of $|W_s - a| = \varepsilon$ has 0 Lebesgue measure. Now clearly we have that $g_\varepsilon(W_t) \to |W_t - a|$ almost surely. Furthermore, since

$$|\dot{g}_\varepsilon(x)| \leq 1; \qquad \dot{g}_\varepsilon(x) \to \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

we have that for any $T \geq 0$ that

$$\lim_{\varepsilon\downarrow 0} E\left[\int_0^T (\dot{g}_\varepsilon(W_s) - \text{sign}(W_s))^2\, ds\right] = \lim_{\varepsilon\downarrow 0} E\left[\int_0^T 1_{|W_s-a|\leq\varepsilon} (\dot{g}_\varepsilon(W_s) - \text{sign}(W_s))^2\, ds\right]$$

$$\leq \lim_{\varepsilon\downarrow 0} 4E\left[\int_0^T 1_{|W_s-a|\leq\varepsilon} ds\right] = 0$$

Thus, for any $\varepsilon_k \to 0$ we can find a further sub-sequence $\varepsilon_{k_j} \to 0$ such that almost surely

$$\lim_{\varepsilon_{k_j}\to 0} \frac{1}{2\varepsilon_{k_j}} \text{leb}\left[s \leq t \mid |W_s - a| \leq \varepsilon_{k_j}\right] = |W_t - a| - |z-a| - \int_0^t \text{sign}(W_s) dW_s$$

which is the desired result.

ii) We have

$$\begin{aligned}
\int_0^t h(Y_s) d\langle Y\rangle_s &= \int_0^t h(Y_s)\dot{f}(W_s)^2 ds \\
&= \int_0^t h(f(W_s))\dot{f}(W_s)^2 ds \\
&= \int_{\mathbb{R}} h(f(a))\dot{f}(a)^2 L_t(a) da; \qquad z = f(a), dz = \dot{f}(a)da \\
&= \int_{\mathbb{R}} h(z)\dot{f}(f^{-1}(z)) L_t(f^{-1}(z)) dz
\end{aligned}$$

where we have used (1) and the fact that f is strictly increasing and hence invertible. Thus, we see that

$$L_t^Y(z) = \dot{f}(f^{-1}(z)) L_t(f^{-1}(z))$$

(5) i) Let $\alpha > 0$. From Novikov we know that $M_t = e^{\alpha \int_0^t \phi_u dB_u - (1/2)\alpha^2 \int_0^t \phi_u^2 du}$ is a martingale. We thus have

$$E\left[e^{\alpha \int_0^T \phi_u dB_u}\right] = e^{\frac{1}{2}\alpha^2 \int_0^T \phi_u^2 du}$$

which yields the desired result.

ii) Using Ito's formula on $e^{\gamma t} Y_t$ it follows that

$$e^{\gamma t} Y_T = \int_0^T e^{\gamma s} dB_s \Longrightarrow Y_T \sim N\left(0, e^{-2\gamma T}\int_0^T e^{2\gamma s} ds\right) \sim \sqrt{\frac{1-e^{-2\gamma T}}{2\gamma}} N(0,1)$$

Set $C_T = \sqrt{(1-e^{-2\gamma T})(2\gamma)}$. We thus have for any $\beta > 0$ that

$$\begin{aligned}
E\left[e^{\beta Y_T^2}\right] &= E\left[e^{\beta C_T^2 N(0,1)^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\beta C_T^2 z^2 - z^2/2} dz \\
&= \frac{1}{\sqrt{1-2\beta C_T^2}}; \qquad -\infty < \beta < \frac{1}{2C_T^2}.
\end{aligned}$$

iii) Now, let $\lambda > 0$. We have

$$
\begin{aligned}
E\left[e^{-\lambda\int_0^T W_t^2dt}\right] &= E\left[e^{\pm\sqrt{2\lambda}\int_0^T W_tdW_t-\lambda\int_0^T W_t^2dt}\right] \\
&= E\left[e^{\sqrt{2\lambda}\int_0^T W_tdW_t}\mathcal{E}\left(-\sqrt{2\lambda}\int_0^\cdot W_tdW_t\right)_T\right] \\
&= E\left[e^{\sqrt{\frac{\lambda}{2}}(W_T^2-T)}\mathcal{E}\left(-\sqrt{2\lambda}\int_0^\cdot W_tdW_t\right)_T\right]
\end{aligned}
$$

Now, W clearly does not explode to ∞ under $\mathbb{P}$. If we could change the measure through the stochastic exponential above then $W^\lambda = W + \sqrt{2\lambda}\int W_tdt$ is a Brownian motion under the new measure $\mathbb{P}^\lambda$. This gives that

$$dW_t = dW_t = -\sqrt{2\lambda}W_tdt + dW_t^\lambda$$

Since W has the dynamics of the process in part b) we know that W does not explode under $\mathbb{P}^\lambda$ and hence the stochastic exponential above is a martingale. Thus, we have

$$E\left[e^{-\lambda\int_0^T W_t^2dt}\right] = e^{-\sqrt{\frac{\lambda}{2}}T}E^{\mathbb{P}^\lambda}\left[e^{\sqrt{\frac{\lambda}{2}}W_T^2}\right]$$

We now use the result from part b) with $\gamma = -\sqrt{2\lambda}$. First, note that

$$\sqrt{\frac{\lambda}{2}} \le \frac{1}{2}\left(\frac{2\sqrt{2\lambda}}{1-e^{-2\sqrt{2\lambda}T}}\right) \Leftrightarrow 1-e^{-2\sqrt{2\lambda}T} \le 2$$

so that the restriction in part b) always holds. Thus,

$$
\begin{aligned}
E\left[e^{-\lambda\int_0^T W_t^2dt}\right] &= e^{-\sqrt{\frac{\lambda}{2}}T}\frac{1}{\sqrt{1-\frac{2\sqrt{\frac{\lambda}{2}}\left(1-e^{-2\sqrt{2\lambda}T}\right)}{2\sqrt{2\lambda}}}} \\
&= \sqrt{\frac{e^{-\sqrt{2\lambda}T}}{1-\frac{1}{2}\left(1-e^{-2\sqrt{2\lambda}T}\right)}} \\
&= \sqrt{\frac{1}{\frac{1}{2}\left(e^{\sqrt{2\lambda}T}+e^{-\sqrt{2\lambda}T}\right)}} \\
&= \left(\cosh\left(\sqrt{2\lambda}T\right)\right)^{-1/2}
\end{aligned}
$$

It is interesting to note that the above formula implies

$$\lim_{T\to\infty}\frac{1}{T}\log\left(E\left[e^{-\lambda\int_0^T W_t^2dt}\right]\right) = -\sqrt{\frac{\lambda}{2}}$$