On the Separation of Stress-Induced and Texture-Induced Birefringence in Acoustoelasticity

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ABSTRACT. In this paper we develop a simple micromechanical model of a prestressed polycrystalline aggregate, in which the texture-induced and stress-induced anisotropies of the aggregate are precisely defined; here the word "texture" always refers to the texture of the aggregate at the given prestressed configuration, not to that of a perhaps fictitious natural state of the aggregate. We use this model to derive, for a prestressed orthotropic aggregate of cubic crystallites, a birefringence formula which shows explicitly the effects of the orthotropic texture on the acoustoelastic coefficients. From this formula we observe that, generally speaking, we cannot separate the total birefringence into two distinct parts, one reflecting purely the influence of stress on the birefringence, and the other encompassing all the effects of texture. The same formula, on the other hand, provides for each material specific quantitative criteria under which the "separation of stress-induced and texture-induced birefringence" would become meaningful in an approximate sense.

Mathematics Subject Classification (1991): 73B40, 73C99, 73S10

1. INTRODUCTION

The two elastic shear (or quasishear) waves that propagate in the same direction in a stressed solid generally have different speeds. This difference is affected by the stress present, and its measurement conversely delivers information about the stress. In practice, the acoustoelastic birefringence (i.e., the difference of the two quasishear wave speeds divided by their average) is measured. Since the work of Crecraft [1] in the sixties, the method of acoustoelastic birefringence has been among the most intensively studied for ultrasonic measurement of residual stress (cf. the reviews of Pao et al. [2], Thompson et al. [3] and the references therein).

To illustrate how information on residual stress is extracted from the measurement of acoustoelastic birefringence, let us consider a simple example. Let a Cartesian coordinate system be chosen. For the special case of a hyperelastic material point with an isotropic natural state and subjected to an initial stress $\overset{\circ}{\boldsymbol{T}} = \mathrm{diag}(\overset{\circ}{T}_{11}, \overset{\circ}{T}_{22}, \overset{\circ}{T}_{33})$, one of the acoustoelastic birefringences is given by

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = K_1\overset{\circ}{T}_{11} + K_2\overset{\circ}{T}_{22} + K_3\overset{\circ}{T}_{33}, \tag{1}$$

$$K_1 = -K_2 = \frac{1}{2\mu}(1 + \frac{\nu_3}{\mu}), \qquad K_3 = 0, \tag{2}$$

where V_{IJ} is the phase velocity of shear waves propagating in the I-direction with polarization in the J-direction, $\bar{V}_T = (V_{31}+V_{32})/2$, K_i $(i = 1, 2, 3)$ are acoustoelastic coefficients, μ is the shear modulus and ν_3 a "third-order Lamé constant" (as defined by Toupin and Bernstein [4]) pertaining to the isotropic natural state (cf. Tokuoka and Saito [5]; see also Tokuoka and Iwashimizu [6]). By writing

$$C_a = \frac{1}{2\mu}(1 + \frac{\nu_3}{\mu}), \tag{3}$$

we may recast Eq. (1) as

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = C_a(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}). \tag{4}$$

Eq. (4) is not valid if the given material point is anisotropic in its natural state. For instance, if the material point is weakly orthotropic in the natural state and has its planes of symmetry parallel to the coordinate planes, if $\overset{\circ}{\boldsymbol{T}} = \mathrm{diag}(\overset{\circ}{T}_{11}, \overset{\circ}{T}_{22}, \overset{\circ}{T}_{33})$, and if we *ignore* the effect of the original anisotropy on the acoustoelastic coefficients K_i, then Eq. (4) (cf. Tokuoka and Saito [5], Iwashimizu and Kubomura [7]) is replaced by

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = B_o + C_a(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}), \tag{5}$$

where B_o is a parameter depending on the anisotropy in the natural state, and C_a is again given by Eq. (3). While we are primarily interested in measurement of residual stress, which usually does not arise from elastic deformations, we speak of hyperelastic material point, natural state, and elastic deformation when we present Eqs.

(4) and (5) above, for it was within such contexts that these equations were derived in the quoted references. Indeed, until the early eighties, all studies in acoustoelasticity were based on the same theoretical approach, namely the continuum theory of small elastic motions (waves) superimposed on a large deformation of a hyperelastic body, where the large deformation in question takes the body from an unstressed natural state to the initial configuration.

In Eq. (5), the left-hand side is called the total birefringence, B_o is called the texture-induced birefringence, and the term led by C_a the stress-induced birefringence. When Eq. (5) was used in the interpretation of experimental data on a variety of weakly anisotropic aluminum and steel sheets, it was found that C_a varied only slightly among the samples of the same material (i.e., aluminum or steel) in those experiments, but B_o could differ considerably from sample to sample [8, 9, 10]. In a typical ultrasonic measurement, only the total birefringence is determined. In order to use Eq. (5) for the evaluation of stress, the stress-induced birefringence must be separated from the texture-induced birefringence. In practical applications (cf. Thompson et al. [3], Hirao et al. [10]), where a specific material is considered, C_a is often given a previously measured value for that material and is taken as a known constant, and B_o is estimated from some nominally unstressed calibration specimen which is believed to have the same texture as the test sample.

The method of acoustoelastic birefringence and its underlying theory, as presented above, are open to several objections:

1. Eq. (5) is derived only for ultrasonic waves passing through a hyperelastic material point which is elastically deformed from a natural state. As commented by Pao [11], Eq. (5) "might be in large error for measuring residual stresses", because "residual stresses are developed in a body as a result of inhomogeneous plastic deformation". Indeed, experimental investigations on several materials indicate that prior plastic deformations would affect the acoustoelastic response of samples. If B_o is a constant determined by the inherent anisotropy of the material at its natural state, Eq. (5) clearly cannot be valid after the material point has undergone plastic deformations which have induced changes in its anisotropy.

2. When we are given a sample and are asked to evaluate the residual stress in it, we typically do not know the (possibly complicated) thermomechanical history which leads to the given initial state of the sample. Nor do we care about this history. We are really interested only in the residual stress currently residing in the sample. In ultrasonic measurements, what we determine are the instantaneous responses of the sample, which reflect only the character of its given initial state. Making reference to a "natural state" of the sample is, philosophically speaking, unnatural in acoustoelasticity.

3. In Eq. (5), any influence of texture on the acoustoelastic coefficients K_i is ignored. As cautioned by Thompson et al. [3], "care must be taken when [this] assumption is made since the influence of texture on acoustoelastic constants is stronger than its influence on elastic moduli or velocities."

When the influence of texture on the acoustoelastic coefficients cannot be ignored, there is no justification to call B_o the texture-induced birefringence and $C_a(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22})$ the stress-induced birefringence. Certainly we can still speak of the influences of texture and stress on acoustoelastic birefringence, but the separation of stress-induced and texture-induced birefringence will become theoretically impossible. Before we can devise methods to separate stress-induced and texture-induced birefringence in practice, we must make sure that there indeed are well-defined quantities which can be called "stress-induced birefringence" and "texture-induced birefringence". We should quantitatively estimate the influence of texture on the acoustoelastic coefficients K_i before we can safely apply Eq. (5) in cases where it is applicable and discuss how to effect the separation of stress-induced and texture-induced birefringence in practice.

Until now we have been using the word "texture" in a loose physical sense to mean the preferred orientations of crystallites constituting a polycrystalline aggregate. In purely macroscopic continuum mechanics (and thence in all the acoustoelastic theories based upon it), there is no mathematical expression which directly describes the texture of a material point. Although we refer to B_o in Eq. (5) as the texture-induced birefringence, only the intrinsic anisotropy of the natural state, i.e., its slight orthotropy, appears explicitly in the original derivation of the equation; the influence of texture on acoustoelastic response is represented only indirectly through this anisotropy. It is common belief that the presence of texture in a polycrystalline aggregate gives rise to its anisotropic mechanical behavior. In continuum mechanics, however, it is the material anisotropy, not the texture, which has a precise description. In material science, on the other hand, a subject called quantitative texture analysis (cf. Bunge [12]) began to take shape in the sixties. There the texture of a polycrystalline aggregate is mathematically described by an orientation distribution function. The methods of quantitative texture analysis were introduced into acoustoelasticity by Sayers [13, 14] and Johnson [15, 16] in the eighties.

Sayers [13] and Johnson [16] computed the second-order (SOEC) and third-order elastic constants (TOEC), respectively, for an orthotropic aggregate of hyperelastic cubic crystallites, which are unstressed in a given natural state of the aggregate. Both their computations are based on the Voigt assumption, namely: if a macroscopic material point undergoes a deformation with gradient $\boldsymbol{F}$, then all the crystallites pertaining to this material point undergo homogeneous deformations with the same gradient $\boldsymbol{F}$. The effective elastic stiffnesses of the aggregate are then obtained by averaging the corresponding stiffnesses of the crystallites with the orientation distribution function as weight. This model was originally introduced for an isotropic aggregate of hyperelastic cubic crystallites by Voigt [17], who computed its two effective SOEC. Evaluations of the TOEC for the isotropic aggregate were carried out in the sixties [18, 19] by using the same model, which Barsch [19] called "the Voigt approximation of the first kind". Once the effective SOEC and TOEC of the orthotropic aggregate have been evaluated by Sayers and Johnson, we can in principle determine all its acoustoelastic properties under the Voigt model (cf. Remark 5.3). For instance, the texture-induced birefringence in Eq. (5) is found to be [13]

$$B_0 = \frac{16\sqrt{5}\pi^2 c}{35\mu} W_{420}; \tag{6}$$

here $c = c_{11} - c_{12} - 2c_{44}$, $\mu = c_{44} + c/5$, where c_{ij} are the single-crystal second-order elastic stiffnesseses (the reference configuration of the crystallites has been chosen so that the symmetry axes coincide with the coordinate axes), and W_{420} is a texture coefficient which appears with other W_{lmn} as expansion coefficients when the orientation distribution function is expressed as an infinite series of the generalized spherical functions (cf. Roe [20]).

While the word "texture" has acquired a precise meaning in the work of Sayers and of Johnson, their acoustoelastic theory follows essentially the old paradigm. Their starting point is the polycrystalline aggregate in a natural state. The word "texture" refers to the texture of the aggregate at this natural state. The stress in the initial configuration arises as a result of an elastic deformation of the aggregate from the natural state. Thus most of the objections against the old theory remain in force.

In this paper we shall develop a simple micromechanical model of a prestressed polycrystalline aggregate, in which the texture-induced and stress-induced anisotropies of the aggregate are defined precisely by appropriate subgroups of the rotation group SO(3). In our model the word "texture" always refers to the texture of the aggregate at the given initial configuration. Indeed, only the initial configuration will ever be used as the reference configuration of the aggregate, which need not have an unstressed natural state. Using this model, we shall derive, for a prestressed orthotropic aggregate of cubic crystallites, a birefringence formula which shows explicitly the effects of the orthotropic texture on the acoustoelastic coefficients. It will be clear from this formula that, generally speaking, we cannot separate the total birefringence into two distinct parts, one reflecting purely the influence of stress on the birefringence, and the other encompassing all the effects of texture. The same formula, on the other hand, will provide quantitative criteria under which the ad hoc approximation (5) would become acceptable (cf. Remark 5.4 below).

2. PRELIMINARIES

We shall develop a simple micromechanical model by which the acoustoelastic coefficients of a polycrystalline aggregate can be expressed in terms of its texture and the (second-order and third-order) elastic constants of its constituent crystallites. To this end, we begin by casting the constitutive equation of a single crystallite in a suitable form (see Eq. (25) below).

Consider a single crystallite $\mathcal{B}_\alpha$, which has an unstressed natural state $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$. We assume that $\mathcal{B}_\alpha$ obeys the kinematics and constitutive laws of continuum mechanics. Let $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ be the initial configuration of $\mathcal{B}_\alpha$, which can be obtained from $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ by a homogeneous transplacement $\boldsymbol{\Phi}$. For convenience, we shall refer to the transplacement $\boldsymbol{\Phi}$ as the pre-deformation of $\mathcal{B}_\alpha$, although the crystallite $\mathcal{B}_\alpha$ may have been prepared at its initial configuration and the transplacement $\mathbf{\Phi}$ need not have taken place physically. Let $\boldsymbol{P} = \nabla\mathbf{\Phi}$, $\boldsymbol{I}$ be the identity tensor, and

$$\overset{\circ}{\boldsymbol{E}} = \frac{1}{2}(\boldsymbol{P}^T\boldsymbol{P} - \boldsymbol{I}) \tag{7}$$

be the Lagrangian prestrain. We assume that $\mathbf{\Phi}$ is a transplacement with possibly large rotation but small prestrain $\overset{\circ}{\boldsymbol{E}}$, and that $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ carries a homogeneous initial (Cauchy) stress $\overset{\circ}{\boldsymbol{T}}$. Strictly speaking, we should have attached a subscript α to each physical quantity pertaining to the crystallite $\mathcal{B}_\alpha$. We have suppressed the subscript for simplicity and will continue to do so in the remaining part of this section. Later in this paper, we will restore the subscript α whenever confusion might arise.

We consider a homogeneous deformation $\boldsymbol{\chi}$ superimposed on $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$. Let $\boldsymbol{F} = \nabla\boldsymbol{\chi}$ and $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$ be the deformation gradient and displacement gradient pertaining to $\boldsymbol{\chi}$, respectively. We assume that $\|\boldsymbol{H}\|$ be small. The crystallite $\mathcal{B}_\alpha$ is strained by the incremental deformation $\boldsymbol{\chi}$. Let

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{8}$$

$$\boldsymbol{E}_{\boldsymbol{\lambda}} = \frac{1}{2}((\boldsymbol{F}\boldsymbol{P})^T\boldsymbol{F}\boldsymbol{P} - \boldsymbol{I}) \tag{9}$$

be the resulting Lagrangian strains of $\mathcal{B}_\alpha$ with the configurations $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ and $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ as reference, respectively. It follows immediately from Eqs. (7)–(9) that

$$\boldsymbol{E}_{\boldsymbol{\lambda}} = \overset{\circ}{\boldsymbol{E}} + \boldsymbol{P}^T\boldsymbol{E}\boldsymbol{P}. \tag{10}$$

Let $\boldsymbol{T}$ be the Cauchy stress at the current configuration $\boldsymbol{\chi}(\boldsymbol{\kappa}(\mathcal{B}_\alpha))$ that results from the transplacement $\boldsymbol{\chi}\circ\mathbf{\Phi}$ (i.e., $\mathbf{\Phi}$ followed by $\boldsymbol{\chi}$). The corresponding second Piola-Kirchhoff stresses with $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ and $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ as reference are

$$\boldsymbol{\sigma} = (\det\boldsymbol{F})\,\boldsymbol{F}^{-1}\boldsymbol{T}\,\boldsymbol{F}^{-T}, \tag{11}$$

$$\boldsymbol{\sigma}_{\boldsymbol{\lambda}} = (\det\boldsymbol{F}\boldsymbol{P})(\boldsymbol{F}\boldsymbol{P})^{-1}\boldsymbol{T}(\boldsymbol{F}\boldsymbol{P})^{-T}, \tag{12}$$

respectively. In this paper we restrict our attention to incremental deformation $\boldsymbol{\chi}$ and pre-deformation $\mathbf{\Phi}$ which are both elastic. Hence we can express the constitutive equations in question as

$$\boldsymbol{\sigma} = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{E}), \tag{13}$$

$$\boldsymbol{\sigma}_{\boldsymbol{\lambda}} = \mathfrak{F}_{\boldsymbol{\lambda}}(\boldsymbol{E}_{\boldsymbol{\lambda}}), \tag{14}$$

respectively, where $\mathfrak{F}_{\boldsymbol{\kappa}}(\cdot)$ and $\mathfrak{F}_{\boldsymbol{\lambda}}(\cdot)$ are the constitutive functions. By eliminating $\boldsymbol{T}$ between Eqs. (11) and (12), we obtain the relation

$$\mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{E}) = (\det\boldsymbol{P})^{-1}\boldsymbol{P}\,\mathfrak{F}_{\boldsymbol{\lambda}}(\boldsymbol{E}_{\boldsymbol{\lambda}})\,\boldsymbol{P}^T. \tag{15}$$

We assume that $\mathfrak{F}_{\boldsymbol{\lambda}}(\cdot)$ (and thus also $\mathfrak{F}_{\boldsymbol{\kappa}}(\cdot)$) be of class C^2. The initial Cauchy stress at the configuration $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ is given by

$$\overset{\circ}{\boldsymbol{T}} = \mathfrak{F}_{\boldsymbol{\kappa}}(\mathbf{0}) = (\det\boldsymbol{P})^{-1}\boldsymbol{P}\,\mathfrak{F}_{\boldsymbol{\lambda}}(\overset{\circ}{\boldsymbol{E}})\,\boldsymbol{P}^T. \tag{16}$$

Let $\boldsymbol{e} = (\boldsymbol{H} + \boldsymbol{H}^T)/2$ be the infinitesimal strain pertaining to the incremental deformation $\boldsymbol{\chi}$. From Eqs. (10), (13), (15) and the fact that $\boldsymbol{E} = \boldsymbol{e} + o(\|\boldsymbol{H}\|)$, we appeal to Taylor's theorem to obtain the formula (cf. Haupt et al. [21], equation (3.10))

$$\boldsymbol{\sigma} = \overset{\circ}{\boldsymbol{T}} + (\det \boldsymbol{P})^{-1} \boldsymbol{P}\, D\mathfrak{F}_{\boldsymbol{\lambda}}(\overset{\circ}{\boldsymbol{E}})[\boldsymbol{P}^T \boldsymbol{e} \boldsymbol{P}] \boldsymbol{P}^T + o(\|\boldsymbol{H}\|), \tag{17}$$

where $D\mathfrak{F}_{\boldsymbol{\lambda}}(\cdot)$ denotes the Frechét derivative of $\mathfrak{F}_{\boldsymbol{\lambda}}(\cdot)$. We are interested only in situations where $\|\boldsymbol{H}\|$ is small and will henceforth discard all terms which are of order $o(\|\boldsymbol{H}\|)$.

Let

$$\boldsymbol{P} = \overset{\circ}{\boldsymbol{R}}\overset{\circ}{\boldsymbol{U}} = \overset{\circ}{\boldsymbol{V}}\overset{\circ}{\boldsymbol{R}} \tag{18}$$

be the polar decomposition of $\boldsymbol{P}$. From the identity $\overset{\circ}{\boldsymbol{U}}{}^2 = \boldsymbol{I} + 2\overset{\circ}{\boldsymbol{E}}$, we deduce that

$$\overset{\circ}{\boldsymbol{U}} = \boldsymbol{I} + \overset{\circ}{\boldsymbol{E}} + o(\|\overset{\circ}{\boldsymbol{E}}\|). \tag{19}$$

By Taylor's theorem, we have

$$D\mathfrak{F}_{\boldsymbol{\lambda}}(\overset{\circ}{\boldsymbol{E}})[\cdot] = D\mathfrak{F}_{\boldsymbol{\lambda}}(\boldsymbol{0})[\cdot] + D(D\mathfrak{F}_{\boldsymbol{\lambda}})(\boldsymbol{0})[\overset{\circ}{\boldsymbol{E}}, \cdot] + o(\|\overset{\circ}{\boldsymbol{E}}\|). \tag{20}$$

Let $\mathbb{C}^\circ = D\mathfrak{F}_{\boldsymbol{\lambda}}(\boldsymbol{0})$, $\mathbb{K}^\circ = (\mathbb{C}^\circ)^{-1}$, and $\mathbb{D}^\circ = D(D\mathfrak{F}_{\boldsymbol{\lambda}})(\boldsymbol{0})$ be the stiffness tensor, the compliance tensor, and the "third-order" stiffness tensor, respectively. As Cartesian tensors, $\mathbb{C}^\circ$ and $\mathbb{K}^\circ$ are fourth-order tensors, and $\mathbb{D}^\circ$ is of sixth-order.

Substituting Eqs. $(18)_1$, (19) and (20) into Eq. (17) (with the $o(\|\boldsymbol{H}\|)$ term discarded), we obtain the relation

$$\begin{aligned}\boldsymbol{\sigma} \;=\; & \overset{\circ}{\boldsymbol{T}} + \overset{\circ}{\boldsymbol{R}}\mathbb{C}^\circ[\overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{R}}{}^T + \overset{\circ}{\boldsymbol{R}}\mathbb{D}^\circ[\overset{\circ}{\boldsymbol{E}}, \overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{R}}{}^T \\ & - (\operatorname{tr}\overset{\circ}{\boldsymbol{E}})\overset{\circ}{\boldsymbol{R}}\mathbb{C}^\circ[\overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{R}}{}^T + \overset{\circ}{\boldsymbol{R}}(\overset{\circ}{\boldsymbol{E}}\mathbb{C}^\circ[\overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}] + \mathbb{C}^\circ[\overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{E}})\overset{\circ}{\boldsymbol{R}}{}^T \\ & + \overset{\circ}{\boldsymbol{R}}\mathbb{C}^\circ[\overset{\circ}{\boldsymbol{E}}\overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}} + \overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}\overset{\circ}{\boldsymbol{E}}]\overset{\circ}{\boldsymbol{R}}{}^T + o(\|\overset{\circ}{\boldsymbol{E}}\|).\end{aligned} \tag{21}$$

As mentioned above, we assume that $\|\overset{\circ}{\boldsymbol{E}}\|$ be small and will henceforth drop the $o(\|\overset{\circ}{\boldsymbol{E}}\|)$ term in Eq. (21).

We are now ready to recast Eq. (21) into the constitutive equation we want. Since $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ is an unstressed natural state, we have $\mathfrak{F}_{\boldsymbol{\lambda}}(\boldsymbol{0}) = \boldsymbol{0}$ and thence

$$\mathfrak{F}_{\boldsymbol{\lambda}}(\overset{\circ}{\boldsymbol{E}}) = \mathbb{C}^\circ[\overset{\circ}{\boldsymbol{E}}] + o(\|\overset{\circ}{\boldsymbol{E}}\|). \tag{22}$$

From Eqs. (16), $(18)_1$, (19) and (22), we observe that

$$\overset{\circ}{\boldsymbol{T}} = \overset{\circ}{\boldsymbol{R}}\mathbb{C}^\circ[\overset{\circ}{\boldsymbol{E}}]\overset{\circ}{\boldsymbol{R}}{}^T + o(\|\overset{\circ}{\boldsymbol{E}}\|). \tag{23}$$

By dropping the $o(\|\overset{\circ}{\boldsymbol{E}}\|)$ term, we obtain the expression

$$\overset{\circ}{\boldsymbol{E}} = \mathbb{K}^\circ[\overset{\circ}{\boldsymbol{R}}{}^T\overset{\circ}{\boldsymbol{T}}\overset{\circ}{\boldsymbol{R}}]. \tag{24}$$

Substituting Eq. (24) into Eq. (21) and discarding the $o(\|\mathring{\boldsymbol{E}}\|)$ term, we arrive at the formula

$$
\begin{aligned}
\boldsymbol{\sigma} &= \boldsymbol{\sigma}(\boldsymbol{e};\mathring{\boldsymbol{T}},\mathring{\boldsymbol{R}}) \\
&= \mathring{\boldsymbol{T}} + \mathring{\boldsymbol{R}}\mathbb{C}^{\circ}[\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}]\mathring{\boldsymbol{R}}^{T} + \mathring{\boldsymbol{R}}\mathbb{D}^{\circ}[\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}],\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}]\mathring{\boldsymbol{R}}^{T} \\
&\quad - (\operatorname{tr}\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}])\mathring{\boldsymbol{R}}\mathbb{C}^{\circ}[\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}]\mathring{\boldsymbol{R}}^{T} \\
&\quad + \mathring{\boldsymbol{R}}(\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}]\,\mathbb{C}^{\circ}[\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}] + \mathbb{C}^{\circ}[\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}]\,\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}])\mathring{\boldsymbol{R}}^{T} \\
&\quad + \mathring{\boldsymbol{R}}\mathbb{C}^{\circ}[\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}]\mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}} + \mathring{\boldsymbol{R}}^{T}\boldsymbol{e}\mathring{\boldsymbol{R}}\mathbb{K}^{\circ}[\mathring{\boldsymbol{R}}^{T}\mathring{\boldsymbol{T}}\mathring{\boldsymbol{R}}]]\mathring{\boldsymbol{R}}^{T},
\end{aligned}
\tag{25}
$$

which is the constitutive equation that we will use in developing our model. For a fixed $\boldsymbol{e}$ and $\mathring{\boldsymbol{T}}$, the function $\boldsymbol{\sigma}(\boldsymbol{e};\mathring{\boldsymbol{T}},\cdot)$ is clearly continuous on the rotation group SO(3).

3. A SIMPLE MICROMECHANICAL MODEL

Consider a polycrystalline aggregate which has a macroscopic material point X given in an initial configuration $\boldsymbol{\kappa}(X)$ with prestress $\mathring{\boldsymbol{T}}(X)$. Henceforth we shall simply refer to X as the given aggregate point and shall only use $\boldsymbol{\kappa}(X)$ as the reference configuration of X. Whether X has an unstressed natural state is irrelevant to our discussion below.

We assume that X consists of numerous crystallites $\mathcal{B}_{\alpha}$. At the initial configuration $\boldsymbol{\kappa}(X)$ of the aggregate point X, each crystallite $\mathcal{B}_{\alpha}$ pertaining to X is itself given in some initial configuration. By abuse of language, we simply denote the initial configuration of $\mathcal{B}_{\alpha}$ by $\boldsymbol{\kappa}(\mathcal{B}_{\alpha})$. As we are considering a polycrystalline aggregate, different crystallites need not have the same initial configuration, even if they pertain to the same aggregate point X. In our micromechanical model of the polycrystalline aggregate, we make two basic assumptions:

1. Every crystallite $\mathcal{B}_{\alpha}$ pertaining to the macroscopic material point X carries the same homogeneous initial stress $\mathring{\boldsymbol{T}} = \mathring{\boldsymbol{T}}(X)$ at the initial configuration $\boldsymbol{\kappa}(\mathcal{B}_{\alpha})$.

2. When a transplacement with deformation gradient $\boldsymbol{F}(X)$ is superimposed on the initial configuration $\boldsymbol{\kappa}(X)$, every crystallite $\mathcal{B}_{\alpha}$ pertaining to X undergoes a homogeneous transplacement with deformation gradient $\boldsymbol{F}$ with respect to the reference configuration $\boldsymbol{\kappa}(\mathcal{B}_{\alpha})$.

Remark 3.1. If $\mathring{\boldsymbol{T}}(X) = \boldsymbol{0}$, our model will reduce to that of Voigt for textured aggregates. Should $\mathring{\boldsymbol{T}}(X)$ arise as a result of an elastic deformation from an unstressed configuration of the aggregate, Assumption 1 would be nothing but the familiar assumption of Reuss as applied to the pre-deformation. Here we are really interested in the general situation that the given configuration of the polycrystalline aggregate is prestressed, the aggregate need not have an unstressed configuration, and the initial stress $\overset{\circ}{\boldsymbol{T}} \neq \boldsymbol{0}$ does not arise from an elastic pre-deformation of the aggregate. □

Let $\mathcal{B}$ be a perfect single crystal of the material in question. Let $\boldsymbol{\lambda}(\mathcal{B})$ be a fixed (unstressed) natural configuration of $\mathcal{B}$; $\boldsymbol{\lambda}(\mathcal{B})$ completely occupies the three-dimensional Euclidean space E^3. For each crystallite $\mathcal{B}_\alpha$, we assume that the initial configuration $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ may be identified as the image of a homogeneous transplacement $\boldsymbol{\Phi}_\alpha : \boldsymbol{\lambda}(\mathcal{B}) \to E^3$, as restricted to some appropriate subset of $\boldsymbol{\lambda}(\mathcal{B})$. For simplicity, we denote $\boldsymbol{\Phi}_\alpha^{-1}(\boldsymbol{\kappa}(\mathcal{B}_\alpha))$ by $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$. We take $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ as the reference natural configuration of $\mathcal{B}_\alpha$. All the reference configurations $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$ have the same orientation, namely, that of $\boldsymbol{\lambda}(\mathcal{B})$. While the crystallites are assumed to have unstressed configurations $\boldsymbol{\lambda}(\mathcal{B}_\alpha)$, the polycrystalline aggregate itself need not have a natural configuration.

Let $\boldsymbol{P}_\alpha = \nabla\boldsymbol{\Phi}_\alpha$, and let

$$\boldsymbol{P}_\alpha = \overset{\circ}{\boldsymbol{R}}_\alpha \overset{\circ}{\boldsymbol{U}}_\alpha = \overset{\circ}{\boldsymbol{V}}_\alpha \overset{\circ}{\boldsymbol{R}}_\alpha \tag{26}$$

be the polar decomposition of $\boldsymbol{P}_\alpha$. The rotation tensor $\overset{\circ}{\boldsymbol{R}}_\alpha$ defines the orientation of the initial configuration $\boldsymbol{\kappa}(\mathcal{B}_\alpha)$ with respect to that of $\boldsymbol{\lambda}(\mathcal{B})$. Thus the orientations of the crystallites at X are represented by appropriate elements of the rotation group SO(3).

In what follows we shall appeal to some standard notions and theorems in measure theory, as applied to probability measures and the Haar measure defined on the rotation group. All that we shall need can be found easily in texts on analysis [22, 23].

We describe the texture at the configuration $\boldsymbol{\kappa}(X)$ of the aggregate point X (i.e., the totality of orientations of crystallites at X) by a probability (Borel) measure $\wp$ defined on the rotation group SO(3). We call $\wp$ the orientation measure of the aggregate point X. For a Borel subset A of SO(3), $\wp(A)$ gives the probability that we find a specific crystallite having an orientation represented by an element in A. For brevity, henceforth we write $\mathcal{G}$ for the rotation group SO(3). Clearly, we have

$$\wp(\mathcal{G}) = 1. \tag{27}$$

The natural configuration $\boldsymbol{\lambda}(\mathcal{B})$ of the perfect single crystal $\mathcal{B}$ usually possesses certain rotational symmetry defined by a point group $\mathcal{G}_{\rm cr} \subset \mathcal{G}$. We call $\mathcal{G}_{\rm cr}$ the group of crystal symmetry. By the definition of the orientation measure $\wp$, it is clear that $\wp$ should be right invariant under the action of $\mathcal{G}_{\rm cr}$, i.e.,

$$\wp(A\boldsymbol{Q}) = \wp(A) \tag{28}$$

for each rotation $\boldsymbol{Q}$ in $\mathcal{G}_{\rm cr}$ and for each Borel subset A of $\mathcal{G}$.

The collection of rotations $\boldsymbol{Q}$ which satisfy

$$\wp(\boldsymbol{Q}A) = \wp(A) \tag{29}$$

for each Borel subset A of $\mathcal{G}$ constitutes a subgroup $\mathcal{G}_{\rm tex}$ of $\mathcal{G}$. This subgroup $\mathcal{G}_{\rm tex}$ describes the symmetry of the texture at X; we call it the group of texture symmetry.

If the crystallites pertaining to X are randomly oriented, then $\wp$ satisfies Eq. (29) for any $\boldsymbol{Q}$ in $\mathcal{G}$—i.e., it is left invariant under the action of the rotation group $\mathcal{G}$. In that case $\wp$ will simply be the Haar measure $\wp_H$, with $\wp_H(\mathcal{G}) = 1$.

Remark 3.2. In the literature [12, 20] on texture analysis, the texture of a macroscopic material point in a polycrystalline aggregate is usually described by the orientation distribution function (ODF) w. Strictly speaking, w is well defined only if the orientation measure $\wp$ is absolutely continuous with respect to the Haar measure $\wp_H$. With this proviso, we can define w in terms of $\wp$ and $\wp_H$ by

$$w = \frac{1}{8\pi^2}\frac{d\wp}{d\wp_H}\,, \tag{30}$$

where $d\wp/d\wp_H$ denotes the Radon-Nikodym derivative of $\wp$ with respect to $\wp_H$. Using $\wp$ instead of w will not only lead to a slightly more general theory but will also facilitate our theoretical analysis. On the other hand, we shall follow common practice and use w in specific computations.

To describe rotations, the literature [12, 20] on texture analysis usually adopts the passive viewpoint (i.e., rotation of coordinate system, fixed objects). For our present discussion, we find it more convenient to use active rotations (i.e., rotation of objects, fixed coordinate system) throughout this paper. Readers who are accustomed to using passive rotations should keep in mind this difference in convention lest confusion might arise. □

Remark 3.3. The groups $\mathcal{G}_{\rm cr}$ and $\mathcal{G}_{\rm tex}$ can be defined in terms of the orientation distribution function w (cf. Bunge [24]). Here we have simply recast the usual definitions in terms of $\wp$. The group $\mathcal{G}_{\rm tex}$ is often called the group of sample symmetry in the literature [24] on texture analysis because there the crystallites are taken as unstressed. In our present context, however, the presence of the initial stress $\overset{\circ}{\boldsymbol{T}}$ will also influence the mechanical anisotropy of the aggregate point in question. Hence we call $\mathcal{G}_{\rm tex}$ the group of texture symmetry to distinguish it from the group that describes the mechanical isotropy or anisotropy of the aggregate point. □

By Assumptions 1 and 2, all the crystallites at X have the same infinitesimal strain $\boldsymbol{e}$ and the same prestress $\overset{\circ}{\boldsymbol{T}}$ for small elastic motions superimposed on $\boldsymbol{\kappa}(X)$. Let

$$\mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\overset{\circ}{\boldsymbol{R}}) = \boldsymbol{\sigma}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\overset{\circ}{\boldsymbol{R}}) - \overset{\circ}{\boldsymbol{T}}, \tag{31}$$

where $\boldsymbol{\sigma}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\overset{\circ}{\boldsymbol{R}})$ is given in Eq. (25), be the incremental elasticity tensor (cf. Man and Carlson [25]) of a typical crystallite $\mathcal{B}_\alpha$ at X. Since $\boldsymbol{\sigma}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\cdot)$ is continuous on the rotation group $\mathcal{G}$, the function $\mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\cdot)$ is Lebesgue integrable with respect to any probability (Borel) measure $\wp$ on $\mathcal{G}$.

We *postulate* that the incremental elasticity tensor of the polycrystalline aggregate at the macroscopic material point X is, in effect, given by the average of $\mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\overset{\circ}{\boldsymbol{R}})$ over the orientation measure $\wp$ (see also Remark 4.1 below), i.e.,

$$\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}] = \int_{\mathcal{G}} \mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\overset{\circ}{\boldsymbol{R}})\,d\wp(\overset{\circ}{\boldsymbol{R}}). \tag{32}$$

Under Assumptions 1 and 2, the preceding postulate is equivalent to the assumption that the Cauchy stress $\boldsymbol{T}(X)$ at the aggregate point X in the current configuration is equal to the average over $\wp$ of the Cauchy stresses carried by the crystallites pertaining to X. Indeed, because the Cauchy stress carried by $\mathcal{B}_\alpha$ is given by

$$\boldsymbol{T}_\alpha = (\det \boldsymbol{F})^{-1} \boldsymbol{F} \boldsymbol{\sigma}(\boldsymbol{e}; \overset{\circ}{\boldsymbol{T}}, \overset{\circ}{\boldsymbol{R}}) \boldsymbol{F}^T, \tag{33}$$

and because by assumption $\boldsymbol{F}$, $\boldsymbol{e}$ and $\overset{\circ}{\boldsymbol{T}}$ do not vary over the crystallites in question, we have

$$\begin{aligned} \boldsymbol{T}(X) &= (\det \boldsymbol{F})^{-1} \boldsymbol{F}(\overset{\circ}{\boldsymbol{T}} + \mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]) \boldsymbol{F}^T \\ &= \int_{\mathcal{G}} (\det \boldsymbol{F})^{-1} \boldsymbol{F}(\overset{\circ}{\boldsymbol{T}} + \mathcal{L}(\boldsymbol{e}; \overset{\circ}{\boldsymbol{T}}, \overset{\circ}{\boldsymbol{R}})) \boldsymbol{F}^T d\wp(\overset{\circ}{\boldsymbol{R}}) \\ &= \int_{\mathcal{G}} \boldsymbol{T}_\alpha(\boldsymbol{e}(X); \overset{\circ}{\boldsymbol{T}}(X), \overset{\circ}{\boldsymbol{R}}) d\wp(\overset{\circ}{\boldsymbol{R}}) \end{aligned} \tag{34}$$

if and only if Eq. (32) holds.

A glance at Eqs. (25), (31) and (32) reveals that $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$, as defined by Eq. (32), may be written as the sum of two terms:

$$\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}] = \mathbb{C}[\boldsymbol{e}] + \mathbb{D}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}], \tag{35}$$

where

$$\mathbb{C}[\boldsymbol{e}] = \int_{\mathcal{G}} \overset{\circ}{\boldsymbol{R}} \mathbb{C}^\circ [\overset{\circ}{\boldsymbol{R}}{}^T \boldsymbol{e} \overset{\circ}{\boldsymbol{R}}] \overset{\circ}{\boldsymbol{R}}{}^T \, d\wp(\overset{\circ}{\boldsymbol{R}}) \tag{36}$$

is independent of $\overset{\circ}{\boldsymbol{T}}$ and $\mathbb{D}(\overset{\circ}{\boldsymbol{T}})[e]$, as defined by $\mathbb{D}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}\,] = \mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}] - \mathbb{C}[\boldsymbol{e}]$, is linear in $\overset{\circ}{\boldsymbol{T}}$.

If the crystallites are hyperelastic, the fourth-order tensors $\mathbb{C}^\circ$ and $\mathbb{K}^\circ$ will enjoy the major symmetry (i.e., $\boldsymbol{A}_1 \cdot \mathbb{C}^\circ[\boldsymbol{A}_2] = \mathbb{C}^\circ[\boldsymbol{A}_1] \cdot \boldsymbol{A}_2$ for any symmetric tensors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$), and the sixth-order tensor $\mathbb{D}^\circ$ will obey the following symmetry: $\boldsymbol{A}_1 \cdot \mathbb{D}^\circ[\boldsymbol{A}_2, \boldsymbol{A}_3] = \boldsymbol{A}_{\sigma(1)} \cdot \mathbb{D}^\circ[\boldsymbol{A}_{\sigma(2)}, \boldsymbol{A}_{\sigma(3)}]$ for any symmetric tensors $\boldsymbol{A}_i$ $(i = 1, 2, 3)$ and for each permutation σ of the indices $1, 2, 3$. For the aggregate, it follows immediately from Eqs. (25), (31), (32), (35) and (36) that the tensor $\mathbb{C}[\cdot]$ and, for each $\overset{\circ}{\boldsymbol{T}}$, the fourth-order tensors $\mathbb{D}(\overset{\circ}{\boldsymbol{T}})[\cdot]$ and $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\cdot]$ will also enjoy the major symmetry. ΔΔΔΔ Received: from t1.mscf.uky.edu by t2.ms.uky.edu id aa27314; 18

Jul 96 19:25 EDT From: mclxyh@ms.uky.edu Subject: sec4.tex To: Roberto Paroni ¡roberto@ms.uky.edu¿ Date: Thu, 18 Jul 1996 19:25:50 -0400 (EDT) X-Mailer: ELM [version 2.4 PL23] MIME-Version: 1.0 Content-Type: text/plain; charset=US-ASCII Content-Transfer-Encoding: 7bit Message-ID: ¡9607181925.aa27314@t2.t2.mscf.uky.edu¿ Status: OR

4. MATERIAL SYMMETRY. TEXTURE- AND STRESS-INDUCED ANISOTROPIES

Let $L^1(\mathcal{G}, \wp)$ be the set of real-valued functions defined on the rotation group $\mathcal{G}$ which are Lebesgue integrable with respect to the orientation measure $\wp$. The following lemma will be instrumental in our derivation of the symmetry properties of the incremental elasticity tensor $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$.

Lemma 4.1. For each $f \in L^1(\mathcal{G}, \wp)$ and each $\boldsymbol{Q} \in \mathcal{G}_{\text{tex}}$,

$$\int_{\mathcal{G}} f(\boldsymbol{Q}^T \boldsymbol{R}) d\wp(\boldsymbol{R}) = \int_{\mathcal{G}} f(\boldsymbol{R}) d\wp(\boldsymbol{R}). \tag{37}$$

Proof: For $f = \chi_A$, where $A \subset \mathcal{G}$ is a Borel set and χ_A is the characteristic function of A, the given formula follows from the left invariance of $\wp$ under the action of $\mathcal{G}_{\text{tex}}$. Hence this formula is valid for any simple function f and, by the monotone convergence theorem, it remains valid for any nonnegative function f measurable with respect to $\wp$. For $f \in L^1(\mathcal{G}, \wp)$, we write $f = f^+ - f^-$, where $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$ are nonnegative. □

The following theorem gives the key symmetry property of $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$ as a function of the pair $(\overset{\circ}{\boldsymbol{T}}, \boldsymbol{e})$.

Theorem 4.2. For each $\boldsymbol{Q}$ in $\mathcal{G}_{\text{tex}}$, we have

$$\mathcal{L}(\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T)[\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]\boldsymbol{Q}^T. \tag{38}$$

Proof: In Eqs. (25) and (31), $\mathcal{L}(\boldsymbol{e}; \overset{\circ}{\boldsymbol{T}}, \overset{\circ}{\boldsymbol{R}})$ is given as a sum of five terms. To prove this theorem, it suffices to show that the corresponding assertion is valid for the average over $\wp$ of each of these terms. This demonstration is easily done by appealing to Lemma 4.1. For instance, we have for each $\boldsymbol{Q} \in \mathcal{G}_{\text{tex}}$

$$\begin{aligned}
&\int_{\mathcal{G}} \overset{\circ}{\boldsymbol{R}}\, \mathbb{D}^{\circ}[\mathbb{K}^{\circ}[\overset{\circ}{\boldsymbol{R}}{}^T(\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T)\overset{\circ}{\boldsymbol{R}}], \overset{\circ}{\boldsymbol{R}}{}^T(\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T)\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{R}}{}^T\, d\wp(\overset{\circ}{\boldsymbol{R}}) \\
=\ & \boldsymbol{Q}\left(\int_{\mathcal{G}} \boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}}\, \mathbb{D}^{\circ}[\mathbb{K}^{\circ}[(\boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}})^T\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}}], (\boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}})^T\boldsymbol{e}\boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}}](\boldsymbol{Q}^T\overset{\circ}{\boldsymbol{R}})^T\, d\wp(\overset{\circ}{\boldsymbol{R}})\right)\boldsymbol{Q}^T \\
=\ & \boldsymbol{Q}\left(\int_{\mathcal{G}} \overset{\circ}{\boldsymbol{R}}\, \mathbb{D}^{\circ}[\mathbb{K}^{\circ}[\overset{\circ}{\boldsymbol{R}}{}^T\overset{\circ}{\boldsymbol{T}}\overset{\circ}{\boldsymbol{R}}], \overset{\circ}{\boldsymbol{R}}{}^T\boldsymbol{e}\overset{\circ}{\boldsymbol{R}}]\overset{\circ}{\boldsymbol{R}}{}^T\, d\wp(\overset{\circ}{\boldsymbol{R}})\right)\boldsymbol{Q}^T.
\end{aligned}$$

The other four terms can be treated similarly. □

Let $\mathcal{G}_{\text{str}}$ be the subgroup of rotations $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}(X)\boldsymbol{Q}^T = \overset{\circ}{\boldsymbol{T}}(X). \tag{39}$$

We call $\mathcal{G}_{\text{str}}$ the group of stress symmetry and call

$$\mathcal{G}_{\text{agg}} = \mathcal{G}_{\text{tex}} \cap \mathcal{G}_{\text{str}} \tag{40}$$

the group of aggregate symmetry for our present model. The groups $\mathcal{G}_{\text{tex}}$, $\mathcal{G}_{\text{str}}$ and $\mathcal{G}_{\text{agg}}$ all refer to the aggregate point X at its initial configuration $\boldsymbol{\kappa}(X)$. From Theorem 4.2 and Eq. (39) we immediately obtain the following corollary.

Corollary 4.3. For each $\boldsymbol{Q}$ in $\mathcal{G}_{\text{agg}}$, we have

$$\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]\boldsymbol{Q}^T. \tag{41}$$

Remark 4.1. Let Lin and Sym be the set of second-order tensors and of second-order symmetric tensors, respectively. In the (macroscopic) theory of linear elasticity with initial stress, we may take the constitutive equation of a material point X (with respect to the initial configuration $\boldsymbol{\kappa}(X)$) as (cf. Man and Carlson [25])

$$\boldsymbol{S}(\boldsymbol{H}) = \overset{\circ}{\boldsymbol{T}} + \boldsymbol{H}\overset{\circ}{\boldsymbol{T}} + \mathsf{L}[\boldsymbol{H}]; \tag{42}$$

here $\boldsymbol{S}$ is the first Piola-Kirchhoff stress; $\overset{\circ}{\boldsymbol{T}} = \boldsymbol{S}(\mathbf{0})$ is the initial (Cauchy) stress; $\boldsymbol{H}$ is the displacement gradient; $\mathsf{L} : \mathrm{Lin} \to \mathrm{Sym}$ is the incremental elasticity tensor, which satisfies $\mathsf{L}[\boldsymbol{W}] = \mathbf{0}$ for all skew-symmetric tensor $\boldsymbol{W}$. Transplanting a definition sometimes adopted in finite elasticity (cf. Gurtin [26], Ch. 5; Hoger, [27, 28]; see also Coleman and Noll [29]), Man and Lu [30] define the material symmetry group $\mathcal{G}_m$ of X at the configuration $\boldsymbol{\kappa}(X)$, in the context of the linearized theory, as the subgroup of rotations $\boldsymbol{Q}$ which satisfy

$$\boldsymbol{S}(\boldsymbol{Q}\boldsymbol{H}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{S}(\boldsymbol{H})\boldsymbol{Q}^T \tag{43}$$

for each $\boldsymbol{H} \in \mathrm{Lin}$; here $\boldsymbol{S}(\cdot)$ is given by Eq. (42). It follows [28, 30] immediately from Eqs. (42) and (43) that $\boldsymbol{Q} \in \mathcal{G}_m$ if and only if $\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T = \overset{\circ}{\boldsymbol{T}}$ and $\mathsf{L}[\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathsf{L}[\boldsymbol{e}]\boldsymbol{Q}^T$ for each $\boldsymbol{e} \in \mathrm{Sym}$. In our present model, $\mathsf{L}[\boldsymbol{e}]$ is represented by $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$. Let us identify $\mathsf{L}[\boldsymbol{e}]$ with $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$ for each $\boldsymbol{e} \in \mathrm{Sym}$. As every $\boldsymbol{Q} \in \mathcal{G}_{\mathrm{agg}}$ satisfies Eqs. (39) and (41), clearly we have $\mathcal{G}_{\mathrm{agg}} \subset \mathcal{G}_m$. Whether the stronger condition $\mathcal{G}_{\mathrm{agg}} = \mathcal{G}_m$ in fact prevails remains to be investigated. □

The groups $\mathcal{G}_{\mathrm{tex}}$ and $\mathcal{G}_{\mathrm{str}}$ define the texture- and stress-induced anisotropies of the aggregate point X, respectively. ΔΔΔΔ

5. COMPUTATION OF THE ACOUSTOELASTIC COEFFICIENTS. EXAMPLES

Given the orientation measure $\wp$ of an aggregate point X, in principle we can evaluate, through Eq. (32), the incremental elasticity tensor $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}]$ of X. Then all the linearized elastic properties of X, including its acoustoelastic coefficients, are determined. In this paper we restrict our attention to the case that the aggregate point X is orthotropic (as defined by the group $\mathcal{G}_{\mathrm{agg}}$). Moreover, we assume that the crystallites in question are hyperelastic. As a result, the fourth-order tensors $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\cdot]$, $\mathbb{C}[\cdot]$, and $\mathbb{D}(\overset{\circ}{\boldsymbol{T}})[\cdot]$ will enjoy the major symmetry (cf. Section 3 above).

We choose a Cartesian coordinate system in which the coordinate planes are the planes of orthotropic symmetry of X. We denote the basis vectors of this coordinate system by $\boldsymbol{i}_k$ $(k = 1, 2, 3)$. Since $\mathcal{G}_{\mathrm{agg}} \subset \mathcal{G}_{\mathrm{str}}$, $\overset{\circ}{\boldsymbol{T}}$ must be diagonal under the chosen coordinate system, i.e., $\overset{\circ}{\boldsymbol{T}} = \mathrm{diag}\,(\overset{\circ}{T}_{11}, \overset{\circ}{T}_{22}, \overset{\circ}{T}_{33})$. Let $L_{ij}(\overset{\circ}{\boldsymbol{T}})$, C_{ij}, and $D_{ij}(\overset{\circ}{\boldsymbol{T}})$ be the components of $\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\cdot]$, $\mathbb{C}[\cdot]$, and $\mathbb{D}(\overset{\circ}{\boldsymbol{T}})[\cdot]$ (see Eq. (35)) in the "abbreviated subscripts" notation (cf. Auld [31], Sections 1.F, 2.D, and 3.C). Then we have (cf. Man and Lu [30])

$$\rho V_{31}^2 = \overset{\circ}{T}_{33} + L_{55} = \overset{\circ}{T}_{33} + C_{55} + D_{551}\overset{\circ}{T}_{11} + D_{552}\overset{\circ}{T}_{22} + D_{553}\overset{\circ}{T}_{33}, \tag{44}$$

$$\rho V_{32}^2 = \overset{\circ}{T}_{33} + L_{44} = \overset{\circ}{T}_{33} + C_{44} + D_{441}\overset{\circ}{T}_{11} + D_{442}\overset{\circ}{T}_{22} + D_{443}\overset{\circ}{T}_{33}, \tag{45}$$

where ρ is the mass density at the given initial configuration $\boldsymbol{\kappa}(X)$ of the aggregate point X, V_{IJ} is the phase velocity of shear waves propagating in the I-direction with polarization in the J-direction, and

$$D_{ijk} = (\mathbb{D}(\boldsymbol{i}_k \otimes \boldsymbol{i}_k)[\cdot])_{ij} \qquad \text{for} \quad k = 1, 2, 3. \tag{46}$$

It follows immediately from Eqs. (44) and (45) that

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = B_0 + K_1\overset{\circ}{T}_{11} + K_2\overset{\circ}{T}_{22} + K_3\overset{\circ}{T}_{33}, \tag{47}$$

where

$$B_0 = \frac{1}{2\rho\bar{V}_T^2}(C_{55} - C_{44}), \qquad \bar{V}_T = (V_{31} + V_{32})/2, \tag{48}$$

$$K_i = \frac{1}{2\rho\bar{V}_T^2}(D_{55i} - D_{44i}), \quad \text{for } i = 1, 2, 3. \tag{49}$$

In most applications, replacing $\rho\bar{V}_T^2$ in Eqs. (48) and (49) by μ, the shear modulus pertaining to an unstressed isotropic aggregate of the crystallites in question, would introduce an error of at most a few percent in B_0 and K_i ($i = 1, 2, 3$). With the understanding that an approximation has been made, we shall henceforth replace $\rho\bar{V}_T^2$ by μ in all the birefringence formulae below.

In what follows we shall evaluate K_i for the case where the crystallites have *cubic* symmetry. We choose a reference configuration $\boldsymbol{\lambda}(\mathcal{B})$ of the perfect single crystal $\mathcal{B}$ (see Section 3) so that the coordinates axes are the axes of cubic symmetry of $\mathcal{B}$. Since B_0 for the present instance has already been evaluated in the work of Sayers [13] (see Eq. (6) above), Eq. (47) will become an explicit formula for the acoustoelastic birefringence when the coefficients K_i are ascertained. In our computations of K_i, we shall assume that the orientation measure $\wp$ of the aggregate point X is absolutely continuous with respect to the Haar measure $\wp_H$ so that the orientation distribution function w is well defined (cf. Eq. (30)); we shall follow common practice and use the orientation distribution function w instead of the orientation measure $\wp$.

To begin with, we parametrize the rotation group $\mathcal{G}$ with the Euler angles (ψ, θ, ϕ), the definition of which we follow the convention adopted by Roe [20]. A rotation $\boldsymbol{R}$ in $\mathcal{G}$ is then represented by a 3×3 orthogonal matrix with its entries given by the following equations (cf. Roe [20]; note that we use active rotations in this paper while Roe's are passive—see Remark 3.2 above):

$$R_{11} = \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi,$$

$$
\begin{aligned}
R_{12} &= -\cos\psi\cos\theta\sin\phi - \sin\psi\cos\phi, \\
R_{13} &= \cos\psi\sin\theta, \\
R_{21} &= \sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi, \\
R_{22} &= -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi, \\
R_{23} &= \sin\psi\sin\theta, \qquad R_{31} = -\sin\theta\cos\phi, \\
R_{32} &= \sin\theta\sin\phi, \qquad R_{33} = \cos\theta.
\end{aligned} \tag{50}
$$

In terms of w and the Euler angles, an integral over $\mathcal{G}$ with respect to $\wp$ is given by

$$
\int_{\mathcal{G}} \cdots \, d\wp = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \cdots \, w(\psi,\theta,\phi)\sin\theta d\psi d\theta d\phi. \tag{51}
$$

In particular, Eq. (32) becomes

$$
\begin{aligned}
\mathcal{L}(\overset{\circ}{\boldsymbol{T}})[\boldsymbol{e}] &= \int_{\mathcal{G}} \mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\boldsymbol{R})d\wp(\boldsymbol{R}) \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \mathcal{L}(\boldsymbol{e};\overset{\circ}{\boldsymbol{T}},\boldsymbol{R}(\psi,\theta,\phi))w(\psi,\theta,\phi)\sin\theta d\psi d\theta d\phi,
\end{aligned} \tag{52}
$$

where we have written the dummy variable in Eq. (32) as $\boldsymbol{R}$ rather than $\overset{\circ}{\boldsymbol{R}}$.

Henceforth we assume that w is square-integrable on $\mathcal{G}$ with respect to the Haar measure $\wp_H$. We may then expand w as an infinite series of the generalized spherical functions:

$$
w(\psi,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l} W_{lmn}Z_{lmn}(\cos\theta)e^{-im\psi}e^{-in\phi}, \tag{53}
$$

where Z_{lmn} are the normalized Jacobi polynomials. We call the expansion coefficients W_{lmn} the texture coefficients. From the normalization condition $\wp(\mathcal{G}) = 1$, we always have [12, 20]

$$
W_{000} = \frac{1}{4\sqrt{2}\pi^2} \tag{54}
$$

for any orientation distribution function w.

To illustrate the evaluation of K_i, we consider first as examples the simpler subcases of random and transversely-isotropic textures.

Example 5.1. The case with cubic crystallites, $\overset{\circ}{\boldsymbol{T}} = \mathrm{diag}(\overset{\circ}{T}_{11}, \overset{\circ}{T}_{22}, \overset{\circ}{T}_{33})$, and random texture. Here w is simply a constant function, namely $w \equiv 1/8\pi^2$. By using Maple V to carry out the integrations and summations symbolically, we find from Eqs. (25), (31), (44)–(50), and (52) that*

$$
\frac{V_{31}-V_{32}}{\bar{V}_T} = C_{\text{iso}}(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}), \tag{55}
$$

$$
\begin{aligned}
C_{\text{iso}} = \frac{1}{70\mu}\Big(&(c_{11} - c_{12})(8\,s_{11} - 8\,s_{12} + 3\,s_{44}) \\
&+ 3\,c_{44}(4\,s_{11} - 4\,s_{12} + 5\,s_{44}) + 2(c_{111} - 3\,c_{112} + 2\,c_{123})(s_{11} - s_{12}) \\
&+ 9\,c_{456}\,s_{44} + 6(c_{155} - c_{144})(s_{11} - s_{12} + s_{44})\Big);
\end{aligned} \tag{56}
$$

*The formula for C_{iso} reported in reference [32] is erroneous. Eq. (56) is the correct formula.

here the c_{ij}, c_{ijk}, and s_{ij} are the second-order stiffnesses, third-order stiffnesses, and second-order compliances of the cubic crystallites, respectively, and we have made the approximation $\rho \bar{V}_T^2 \approx \mu$, where $\mu = (c_{11} - c_{12} + 3c_{44})/5$. Thus we have obtained a formula which expresses the acoustoelastic coefficient C_{iso} of the polycrystalline aggregate point X explicitly in terms of the elastic parameters of the crystallites. A glance at Eqs. (4) and (55) reveals that the coefficient C_{iso} in our present model corresponds to the coefficient C_a in the acoustoelastic theory of Tokuoka and Saito [5].

Using the experimental data of Thomas [33] on the second-order and third-order elastic constants of (99.95% to 99.99% pure) single-crystal aluminum at 25 °C, we find from formula (56) that $C_{\text{iso}} = -4.62 \times 10^{-5}$ MPa^{-1} at 25 °C. If we replace the values of the third-order stiffnesses of the crystallites by those reported by Sarma and Reddy [34] (for single-crystal aluminum of 99.999% purity, at 25 °C; they did not report the second-order elastic constants of their samples), we obtain $C_{\text{iso}} = -4.32 \times 10^{-5}$ MPa^{-1}.

Fukuoka and Toda [35] conducted acoustoelastic measurements on (slightly anisotropic) polycrystalline samples of 99.5% pure aluminum. From their experimental data, they obtained an average value of $C_a = -4.42 \times 10^{-5}$ MPa^{-1} through Eq. (4). They did not report the temperature at which their experiments were performed, although they mentioned that "the environmental temperature is kept within 0.1 °C of fluctuation". Kobori and Iwashimizu [36] obtained the experimental value of $C_a = -4.62 \times 10^{-5}$ MPa^{-1} for an aluminum alloy (with 97.5% aluminum) at 20 °C. □

Example 5.2. The case with cubic crystallites, $\mathring{\boldsymbol{T}} = \text{diag}(\mathring{T}_{11}, \mathring{T}_{22}, \mathring{T}_{33})$, and transversely-isotropic texture (with the 3-axis as the symmetry axis). Every entry R_{ij} of the rotation matrix (cf. Eq. (50)) can be expressed as a linear combination of the generalized spherical functions with $l = 1$. It follows then from the orthogonality and from the multiplication formula of the generalized spherical functions (cf. Bunge [12], section 13.1.3) that in our evaluation of K_i, we may truncate all terms with $l > 6$ in the series expansion (53) of w. Moreover, from the cubic symmetry of the crystallites and from the transverse isotropy of the texture, many W_{lmn} coefficients are null. It turns out that in our evaluation of K_i, we may in effect put (cf. Roe [20, 37], Morris and Heckler [38], Johnson [16])

$$\begin{aligned} w \;=\; & \frac{1}{8\pi^2} + \frac{\sqrt{2}}{128}\Big(27 + 60\cos 2\theta + 105\cos 4\theta + 45\cos 4\phi \\ & - 60\cos 2\theta\cos 4\phi + 15\cos 4\theta\cos 4\phi\Big) W_{400} \\ & + \frac{21\sqrt{26}}{1024}\Big(\frac{50}{21} + 5\cos 2\theta + 6\cos 4\theta + 11\cos 6\theta - 10\cos 4\phi \\ & - 5\cos 2\theta\cos 4\phi + 26\cos 4\theta\cos 4\phi - 11\cos 6\theta\cos 4\phi\Big) W_{600}. \qquad (57) \end{aligned}$$

By using Maple V to complete the symbolic computations, we obtain from Eqs.

(25), (31), (44)–(50), (52), and (57) that

$$\frac{V_{31}-V_{32}}{\bar{V}_T} = K_1\overset{\circ}{T}_{11} + K_2\overset{\circ}{T}_{22}, \tag{58}$$

$$K_1 = -K_2 = C_{\text{iso}} + \beta_1 W_{400} + \beta_4 W_{600}, \tag{59}$$

where C_{iso} is given by Eq. (56), and

$$\begin{aligned}\beta_1 = {} & -\frac{4\sqrt{2}\pi^2}{385\mu}\Big(4(c_{11}-c_{12})(7s_{11}-7s_{12}+2s_{44}) \\ & -c_{44}(78s_{11}-78s_{12}+5s_{44})+7(c_{111}-3c_{112}+2c_{123})(s_{11}-s_{12}) \\ & -21c_{456}s_{44}-(c_{155}-c_{144})(39s_{11}-39s_{12}-16s_{44})\Big),\end{aligned} \tag{60}$$

$$\begin{aligned}\beta_4 = {} & \frac{8\sqrt{26}\pi^2}{1001\mu}\Big(2(c_{11}-c_{12}-2c_{44})(2s_{11}-2s_{12}-s_{44}) \\ & +(c_{111}-3c_{112}+2c_{123})(s_{11}-s_{12}) \\ & +8c_{456}s_{44}-4(c_{155}-c_{144})(s_{11}-s_{12}+s_{44})\Big).\end{aligned} \tag{61}$$

In the preceding formulae for β_1 and β_4, we have put $\mu = (c_{11}-c_{12}+3c_{44})/5$ in place of $\rho\bar{V}_T^2$. □

We now consider the case where X is generally orthotropic. The procedure to evaluate K_i is essentially the same as in Example 5.2, although the orientation distribution function w, which now involves the independent texture coefficients $W_{400}, W_{420}, W_{440}, W_{600}, W_{620}, W_{640}$ and W_{660}, is much more complicated than Eq. (57). Here we shall be content to outline the results of the computations and to examine a concrete example. Again, by using Maple V, we obtain

$$\frac{V_{31}-V_{32}}{\bar{V}_T} = B_0 + K_1\overset{\circ}{T}_{11} + K_2\overset{\circ}{T}_{22} + K_3\overset{\circ}{T}_{33}, \tag{62}$$

$$\begin{aligned}K_1 = {} & C_{\text{iso}} + \beta_1 W_{400} + \beta_2 W_{420} + \beta_3 W_{440}, \\ & +\beta_4 W_{600} + \beta_5 W_{620} + \beta_6 W_{640}\end{aligned} \tag{63}$$

$$\begin{aligned}K_2 = {} & -C_{\text{iso}} - \beta_1 W_{400} + \beta_2 W_{420} - \beta_3 W_{440} \\ & -\beta_4 W_{600} + \beta_5 W_{620} - \beta_6 W_{640},\end{aligned} \tag{64}$$

$$K_3 = \gamma_2 W_{420} + \gamma_5 W_{620}, \tag{65}$$

where B_0, C_{iso}, β_1, and β_4 are given by Eqs. (6), (56), (60), and (61), respectively, and the coefficients $\beta_2, \beta_3, \beta_5, \beta_6, \gamma_2, \gamma_5$ are given by the following formulae:

$$\begin{aligned}\beta_2 = {} & \frac{-8\sqrt{5}\pi^2}{385\mu}\Big((c_{11}-c_{12})(2s_{11}-24s_{12}-s_{44}) \\ & +c_{44}(18s_{11}+26s_{12}-9s_{44})-c_{111}(5s_{11}+17s_{12})-7c_{112}(s_{11}-s_{12}) \\ & +2c_{123}(6s_{11}+5s_{12})-7c_{456}s_{44}+c_{144}(13s_{11}+31s_{12}+2s_{44}) \\ & +c_{155}(31s_{11}+57s_{12}-2s_{44})\Big),\end{aligned} \tag{66}$$

$$\begin{aligned}
\beta_3 &= \frac{8\sqrt{35}\pi^2}{385\mu}\Big((2\,c_{11}-2\,c_{12}+7\,c_{44})(2\,s_{11}-2\,s_{12}-\,s_{44})\\
&\quad +(\,c_{111}-3\,c_{112}+2\,c_{123})(\,s_{11}-\,s_{12})\\
&\quad -3\,c_{456}\,s_{44}+(\,c_{155}-\,c_{144})(7\,s_{11}-7\,s_{12}-4\,s_{44})\Big), \qquad (67)\\
\beta_5 &= \frac{-4\sqrt{105}}{15}\,\beta_4, \qquad (68)\\
\beta_6 &= \sqrt{14}\,\beta_4, \qquad (69)\\
\gamma_2 &= \frac{-16\sqrt{5}\pi^2}{385\mu}\Big(-(\,c_{11}-\,c_{12})(13\,s_{11}-2\,s_{12}-\,s_{44})\\
&\quad +c_{44}(4\,s_{11}+18\,s_{12}+9\,s_{44})-\,c_{111}(6\,s_{11}+5\,s_{12})+7\,c_{112}(\,s_{11}-\,s_{12})\\
&\quad -c_{123}(\,s_{11}-12\,s_{12})+7\,c_{456}\,s_{44}+\,c_{144}(9\,s_{11}+13\,s_{12}-2\,s_{44})\\
&\quad +c_{155}(13\,s_{11}+31\,s_{12}+2\,s_{44})\Big), \qquad (70)\\
\gamma_5 &= \frac{8\sqrt{105}}{15}\,\beta_4, \qquad (71)
\end{aligned}$$

where $\mu = (c_{11} - c_{12} + 3c_{44})/5$. For later discussion, it will be convenient to recast Eq. (62)–(65) in the form

$$\begin{aligned}
\frac{V_{31}-V_{32}}{\bar{V}_T} &= B_0 + (C_{\rm iso} + \beta_1 W_{400} + \beta_3 W_{440} + \beta_4 W_{600} + \beta_6 W_{640})(\mathring{T}_{11} - \mathring{T}_{22})\\
&\quad + (\beta_2 W_{420} + \beta_5 W_{620})(\mathring{T}_{11} + \mathring{T}_{22}) + (\gamma_2 W_{420} + \gamma_5 W_{620})\mathring{T}_{33}. \qquad (72)
\end{aligned}$$

Note that while w involves the texture coefficient W_{660}, the acoustoelastic coefficients in question are independent of W_{660}.

Remark 5.1. Equation (72), of course, is consistent with the corresponding formula in the more general, purely macroscopic theory (cf. King and Fortunko [39], equation (5), for the case $\mathring{T}_{33} = 0$). But Eq. (72) contains more information than its more general counterpart. Once the second- and third-order elastic constants of the crystallites are ascertained, B_0 and all the acoustoelastic coefficients in Eq. (72) become explicit linear functions of the texture coefficients. Even when we know nothing about the second- and third-order elastic constants of the crystallites, Eq. (72) gives us more insight than the corresponding formula in the purely macroscopic theory. For instance, King and Fortunko ([39], Appendix) surmised that the $\mathring{T}_{11} + \mathring{T}_{22}$ term could be ignored for "weak anisotropy", but they suggested using $2\mu B_0$ to characterize the degree of anisotropy. From Eqs. (6) and (72), however, we observe that even if $B_0 = 0$, i.e., $W_{420} = 0$, the coefficient of the $\mathring{T}_{11} + \mathring{T}_{22}$ term, which is now $\beta_5 W_{620}$, could remain substantial. □

Remark 5.2. Both the birefringence fomulae (59) and (72) for the cases of transversely-isotropic and orthotropic texture contains a term $C_{\rm iso}(\mathring{T}_{11} - \mathring{T}_{22})$ identical to the birefringence for random texture, because the normalization condition $\wp(\mathcal{G}) = 1$ dictates that the orientation distribution function w always contains the constant term $1/8\pi^2$ equal to the ODF for the case of random texture. Clearly, for the same reason, this $C_{\rm iso}$ term will arise in all cases of anisotropic texture. As a result, a sufficient condition for Eq. (5) to be a good approximation as a formula for the total birefringence is that the texture be sufficiently weak, i.e., that all the W_{lmn} coefficients in question be sufficiently small. □

Remark 5.3. Equation (72) gives, for the first time, an explicit formula which expresses the total birefringence of a prestressed orthotropic aggregate in terms of its initial texture, initial stress, and the single-crystal SOEC and TOEC. As we have mentioned in the introduction, in the Voigt model, once the SOEC and TOEC of an orthotropic aggregate have been evaluated by Sayers and Johnson, all the acoustoelastic properties of the aggregate can be determined in that model. To the best of our knowledge, however, no analog of Eq. (72) has yet been published for the Voigt model. □

Let us now consider a concrete example. Using the data of Thomas [33] on the second- and third-order elastic constants of single-crystal aluminum at 25 °C, we obtain from our computations the following formula for the acoustoelastic birefringence:

$$\begin{aligned}\frac{V_{31}-V_{32}}{\bar{V}_T} = {} & -3.97W_{420} + (-0.046 - 0.49\,W_{400} + 0.05\,W_{440} + 0.95\,W_{600} \\ & + 3.56\,W_{640})(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}) - (0.12\,W_{420} + 2.60\,W_{620})(\overset{\circ}{T}_{11} + \overset{\circ}{T}_{22}) \\ & + (0.72\,W_{420} + 5.20\,W_{620})\overset{\circ}{T}_{33}, \end{aligned} \tag{73}$$

where the stresses are in units of GPa. If we use the third-order stiffnesses of aluminum reported by Sarma and Reddy [34] (but keeping Thomas' values of the second-order elastic constants), then the birefringence formula becomes

$$\begin{aligned}\frac{V_{31}-V_{32}}{\bar{V}_T} = {} & -3.97W_{420} + (-0.043 - 0.47\,W_{400} + 0.04\,W_{440} + 0.92\,W_{600} \\ & + 3.45\,W_{640})(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}) - (0.07\,W_{420} + 2.52\,W_{620})(\overset{\circ}{T}_{11} + \overset{\circ}{T}_{22}) \\ & + (0.72\,W_{420} + 5.03\,W_{620})\overset{\circ}{T}_{33}. \end{aligned} \tag{74}$$

Remark 5.4. Whether the ad hoc formula (5) would furnish an acceptable approximation to Eq. (72) for a polycrystalline aggregate depends on the texture of the aggregate and on the values of $C_{\rm iso}, \beta_1, \beta_2, ..., \gamma_5$ for the material in question. Eq. (73) or (74), for instance, shows that for aluminum the coefficients W_{620} and W_{640} dominate the influence of texture on the acoustoelastic coefficients in the birefringence formula. Indeed, for textures commonly encountered in rolled plates (say, with $|W_{400}| \le 0.005, |W_{420}| \le 0.005, |W_{440}| \le 0.01$), we expect that Eq. (5) would be approximately valid when $|W_{600}| \le 0.003$, $|W_{620}| \le 0.001$, and $|W_{640}| \le 0.001$. Should we have more information on the initial stress (e.g., the case that $\overset{\circ}{T}_{22} = \overset{\circ}{T}_{33} = 0$), less stringent conditions would suffice. This example illustrates the fact that when we have evaluated the coefficients $C_{\rm iso}$, $\beta_1, \beta_2, ..., \gamma_5$ for a specific material, the resulting birefringence formula would then provide quantitative criteria to delineate situations where the ad hoc formula (5) would become acceptable. □

Johnson and Springer [40] reported the following texture coefficients for a sample of 7039-T64 aluminum plate: $W_{400} = -0.0017$, $W_{420} = 0.0043$, $W_{440} = -0.0039$, $W_{600} = -0.0037$, $W_{620} = -0.0055$, $W_{640} = 0.0012$, and $W_{660} = -0.00057$, which pertain to the coordinate system with the 1-, 2-, and 3-axis coinciding with the normal, transverse, and rolling direction of the plate, respectively. Substituting these values into Eq. (73) and (74), we obtain for this sample the birefringence formula

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = -0.0171 - 0.031\,\overset{\circ}{T}_{11} + 0.059\,\overset{\circ}{T}_{22} - 0.026\,\overset{\circ}{T}_{33}, \tag{75}$$

and

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = -0.0171 - 0.028\,\overset{\circ}{T}_{11} + 0.055\,\overset{\circ}{T}_{22} - 0.024\,\overset{\circ}{T}_{33}, \tag{76}$$

respectively; here the stresses are in units of GPa.

6. DISCUSSIONS

Clearly Eq. (5), the formula often used in practical applications of birefringence measurements, will become an acceptable approximation for Eq. (72) if and only if the contributions of the β_i $(i = 1, 2, ..., 6)$ and γ_j $(j = 2, 5)$ terms in Eq. (72) are negligible (cf. Remark 5.4 above). In general, however, this need not be the case. Indeed a particularly acute situation could arise if $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} \neq 0$. There the $C_{\rm iso}$ term drops out of formula (72), but the initial stress will contribute to the birefringence if $W_{420} \neq 0$ or $W_{620} \neq 0$. Take, for instance, the aluminum sample studied by Johnson and Springer [40] (cf. the last paragraph in Section 5 of the present paper). For $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} = 100$ MPa and $\overset{\circ}{T}_{33} = 0$, Eq. (75) gives the total birefringence a value of -1.43%. By formula (5), we should have attributed -1.71% to "texture-induced birefringence" and 0.28% to "stress-induced birefringence". Using the value of $C_a = -4.62 \times 10^{-5}$ MPa^{-1}, we shall then draw the totally erroneous conclusion that $\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22} = -61$ MPa. If $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} = 100$ MPa and $\overset{\circ}{T}_{33} = -100$ MPa, then the estimate by formula (5) will be further worsened to $\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22} = -117$ MPa. Thus application of formula (5) must be exercised with caution.

In this paper we have presented a simple micromechanical model of a prestressed polycrystalline aggregate. We consider a macroscopic material point X of the aggregate given in a prestressed and textured configuration $\boldsymbol{\kappa}(X)$. In our model, only $\boldsymbol{\kappa}(X)$ is used as the reference configuration for the aggregate point X, which need not have an unstressed natural configuration. Both the prestress $\overset{\circ}{\boldsymbol{T}}(X)$ and the orientation measure $\wp(X)$, which defines the texture, refer to the stress and texture at the initial configuration $\boldsymbol{\kappa}(X)$. Material anisotropies of X induced by the initial stress $\overset{\circ}{\boldsymbol{T}}$ and the initial texture $\wp$ are precisely defined. Moreover, explicit formulae which describe the influence of the texture $\wp$ on the acoustoelastic coefficients in birefringence formulae could be derived. Examples of such formulae are given in Section 5. From these examples it is clear that, generally speaking, we cannot separate the total birefringence into two parts, one reflecting purely the influence of the initial stress $\overset{\circ}{\boldsymbol{T}}$, and the other encompassing all the effects of the initial texture $\wp$. On the other hand, such a separation (as embodied in Eq. (5)) would be approximately valid if the texture is sufficiently weak. The criteria for the validity of Eq. (5) depend on the material and on the situation at hand (see Remark 5.4). Under the present model, these criteria can be derived from the analog of Eq. (73) for the polycrytalline aggregate in question.

Of course, all the discussions above are based on the presumption that the formulae derived under our simple model be corroborated by the actual behavior of polycrystalline aggregates. It is unclear how the simplifying assumptions in the present modelling would affect the validity of these formulae. Further theoretical and experimental studies remain to be undertaken to confirm or to contradict these formulae, to delineate their limitations and to explore their possible applications. Since we are interested in measurement of residual stresses in structural metals, a particularly intriguing question arises which concerns polycrystalline aggregates that have undergone plastic deformations. It has been suggested [32] that a formula such as Eq. (73), where the W_{lmn} coefficients pertain to the texture of the given initial configuration, might remain valid even if the aggregate in question has, in the past, undergone plastic deformations. The effects of plastic deformations on the acoustoelastic response of a polycrystalline aggregate would then be reflected only in the change of texture that they could have induced on the aggregate. Whether this suggestion has any merit could be answered only in the laboratory.

Acknowledgment. We thank Daniel McDonald, who willingly answered all the questions we had on using Maple. The research reported here was supported in part by a grant from the National Science Foundation (Grant No. CMS-9522829).

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