On some constrained variational problems

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Abstract

In part I, it is shown that for integrals of the type

$$I(u,v) := \int_\Omega f(x,u(x),v(x))\,dx,$$

with $\Omega \subset \mathbb{R}^N$ open, bounded, and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ Carathéodory satisfying a growth condition $0 \le f(x,u,v) \le C(1+|v|^p)$, for some $p \in (1,+\infty)$, a sufficient condition for lower semicontinuity along sequences $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^p, $\mathcal{A}v_n \to 0$ in $W^{-1,p}$ is the $\mathcal{A}_x$-quasiconvexity of $f(x,u,.)$. Here $\mathcal{A}$ is a variable coefficients operator of the form

$$\mathcal{A} := \sum_{i=1}^N A^{(i)}(x)\frac{\partial}{\partial x_i},$$

with $A^{(i)} \in C^\infty(\Omega;\mathcal{M}^{l\times d}) \cap W^{1,\infty}$, $i = 1,..N$, satisfying the condition

$$\operatorname{rank}\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right) = \text{const} \quad \text{for } x \in \Omega \text{ and } \omega \in \mathbb{R}^N \setminus \{0\},$$

and $\mathcal{A}_x$ denotes the constant coefficients operator one obtains by freezing x. Under additional regularity conditions on f it is proved that the condition above is also necessary. A characterization of the Young measures generated by bounded sequences $\{v_n\}$ in L^p satisfying the condition $\mathcal{A}v_n \to 0$ in $W^{-1,p}$ is obtained.

In part II, an integral representation for the functional

$$\mathcal{F}(m,M) := \inf\left\{\liminf_{k\to+\infty}\int_\Omega f(x,m_k(x),\nabla m_k(x))\,dx + \int_{\Omega\cap S(m_k)} |[m_k](x)|d\mathcal{H}^{N-1} : \right.$$
$$m_k \in SBV(\Omega;\mathbb{R}^N), \quad |m_k(x)| = 1 \quad \text{a.e. in } \Omega,$$
$$\left. m_k \to m \quad \text{in} \quad L^1(\Omega;\mathbb{R}^N), \quad \nabla m_k \rightharpoonup M \quad \text{in} \quad L^2(\Omega;\mathbb{R}^N)\right\}$$

is obtained. This problem is motivated by equilibrium issues in micromagnetics.

In part III, the effective behavior of second order strain energy densities is obtained using relaxation and Γ convergence techniques. The Cosserat theory is recovered within a dimension reduction analysis for $3D$ thin domains with varying profiles. Homogeneous and inhomogeneous $2D$ models with periodic profiles are treated.

Contents

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1 Introduction

An important problem in the Calculus of Variations, motivated by issues in Mathematics, Physics, Materials Science, is to prove existence of

$$\min\{I(u) : u \in Adm\},$$

where Adm is a subset of some Banach space X and I is a functional defined by an integral.

In order to apply the Direct Method of the Calculus of Variations, which is the most commonly used method to prove existence of minimizers, it is important to establish conditions that ensure lower semicontinuity of I for a given appropriate topology. A well known result in this direction is that for a functional I of the type

$$I(u,v) = \int_\Omega f(x,u(x),v(x))\,dx,$$

with appropriate regularity conditions on f, we have

$$u_n \to u \quad \text{in measure,} \quad v_n \rightharpoonup v \quad \text{in } L^p \Rightarrow \liminf I(u_n,v_n) \geq I(u,v),$$

if and only if $f(x,u,.)$ is convex.

In many problems we don't need to have lower semicontinuity for all weakly convergent sequences, but just for some of them, and then it may happen that convexity is not necessary any more. A typical example is when we deal with gradients, $I : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$,

$$I(u) = \int_\Omega f(x,u(x),Du(x))\,dx.$$

Under appropriate regularity and growth conditions on f, I is weakly lower semicontinuous in $W^{1,p}$ if and only if $f(x,u,.)$ is quasiconvex ([56], [3], [31]). As introduced by Morrey ([56]), a continuous function $f : \mathbb{R}^{m.N} \to \mathbb{R}$ is said to be *quasiconvex* if and only if

$$f(A) \leq \int_Q f(A + D\varphi(x))\,dx,$$

for every $A \in \mathbb{R}^{m.N}$ and every $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$, where $Q := (0,1)^N$, or equivalently,

$$f(A) \leq \int_Q f(A + \omega(x))\,dx,$$

for every $A \in \mathbb{R}^{N.m}$ and for every $\omega \in C^\infty(Q;\mathbb{R}^m)$, $Q-$periodic, $\int_Q \omega(x)\,dx = 0$, with $\operatorname{curl}\omega = 0$. The notion of quasiconvexity lies between convexity and convexity in directions of rank-1, i.e., a quasiconvex function may not be convex in all directions of the space, but at least it is convex in the rank-1 directions. Thus, for $N = 1$ or $m = 1$, quasiconvexity is the same as convexity.

Variational problems involving gradients, as well as some other differential constraints, such as higher order derivatives, divergence free fields, Maxwell Equations, etc, can be treated in a unified way under the notion of $\mathcal{A}$-quasiconvexity, as Fonseca and Müller proved in a recent paper([40]). They studied lower semicontinuity problems when the sequences are constrained by a first order system of PDEs of the form

$$\mathcal{A}u := \sum_{i=1}^N A^i \frac{\partial u}{\partial x_i},$$

under a technical condition called the *constant rank condition*, precisely

$$\operatorname{rank}\left(\sum_{i=1}^N A^{(i)}\omega_i\right) = const,$$

for every $\omega \in \mathbb{R}^N \setminus \{0\}$. They considered integrals of the form

$$I(u,v) := \int_\Omega f(x,u(x),v(x))\,dx,$$

and proved that, under appropriate regularity and growth conditions on f, one has

$$u_n \to u \quad \text{in measure}, \quad v_n \rightharpoonup v \quad \text{in } L^p \quad \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,p} \Rightarrow \liminf I(u_n, v_n) \geq I(u, v),$$

if and only if $f(x, u, .)$ is $\mathcal{A}$-quasiconvex. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(v) \leq \int_Q f(v + \omega(x))\, dx,$$

for all $v \in \mathbb{R}^d$ and all $\omega \in C^\infty(\mathbb{R}^N; \mathbb{R}^d)$, Q-periodic, $\int_Q \omega(x)\, dx = 0$, $\mathcal{A}\omega = 0$. We remark that in case $\mathcal{A} = \text{curl}$, $\mathcal{A}$-quasiconvexity reduces to the usual quasiconvexity.

In part I we study lower semicontinuity when the sequences are constrained by a system of PDEs with variable coeficients, precisely, we deal with operators $\mathcal{A}$ of the type,

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)}(x) \frac{\partial v}{\partial x_i},$$

with $A^{(i)} \in C^\infty \cap W^{1,\infty}$, and rank $\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right) = \text{const}$, for every $x \in \Omega$ and every $\omega \in \mathbb{R}^N \setminus \{0\}$. For integrals of the type

$$I(u) := \int_\Omega f(x, u(x))\, dx,$$

under appropriate regularity and growth conditions on f, we were able to prove that I is lower semicontinuity along sequences $u_n \rightharpoonup u$ in L^p, $\mathcal{A}u_n \to 0$ in $W^{-1,p}$, if and only if $f(x, .)$ is $\mathcal{A}_x$-quasiconvex, where $\mathcal{A}_x$ is the constant coefficients operators we obtain by freezing x. We also characterize the Young measures generating by such sequences. This work is CNA Report 03-2003.

In many problems we have also to deal with surface energies and consider functionals in SBV of the type

$$I(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx + \int_{\Omega \cap S(u)} g(u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1}.$$

The study of the lower semicontinuity for these integrals brings new difficulties since it may happen that there is interaction between the volume and the surface energies. The Compactness Theorem (2.27) in SBV due to Ambrosio ([9], [14]) identifies a class of functionals for which we can study lower semicontinuity considering the volume and the surface integrals separately, namely, if $f(x, u, v) \geq c|v|^r$ for some $r > 1$ and $g(u, v, p) \geq c|p|$, $f(x, u, .)$ convex and g jointly convex (see [14]), plus appropriate regularity conditions, then I is lower semicontinuous in SBV for sequences $u_n \to u$ in L^1, $\sup_n ||u_n||_\infty < +\infty$. If, in addition, we have $\sup_n \mathcal{H}^{N-1}(S(u_n) \cap \Omega) < +\infty$, then it is enough to assume quasiconvexity of $f(x, u, .)$ (see [10], [14]).

When lower semicontinuity fails, the effective energy is given by the relaxation of I,

$$\mathcal{F}(u) := \inf\{\liminf I(u_n) : u_n \to u\}.$$

Then a question arises: can we find a 'good' representation for $\mathcal{F}$, i.e., can we represent $\mathcal{F}$ as an integral of some new densities? In part II we deal with a relaxation problem in SBV, where the conditions of Ambrosio's Compactness Theorem fail. We find an integral representation for the functional

$$\mathcal{F}(m, M) := \inf \Bigg\{ \liminf \int_\Omega f(x, m_n(x), \nabla m_n(x))\, dx + \int_{\Omega \cap S(m_n)} |[m_n](x)|\, d\mathcal{H}^{N-1} : m_n \in SBV(\Omega; \mathbb{R}^N),$$
$$m_n \to m \quad \text{in } L^1, \quad |m_n(x)| = 1 \quad \text{a.e. in } \Omega, \quad \nabla m_k \rightharpoonup M \quad \text{in } L^2 \Big\},$$

with $c(|v|^2 - 1) \leq f(x, u, v) \leq C(1 + |v|^2)$, satisfying appropriate regularity conditions.

This problem is motivated by equilibrium issues in micromagnetics. In considering the relaxed functional defined for pairs (m, M), instead of only m, we look for a better description of defects (situations where a given magnetization can be attained via a diffusion of discontinuities), in the same spirit as the *structured deformations* introduced by Owen and Del Piero (see [36]) to understand defects in crystals and then studied by Chosky and Fonseca in [27]. This work is CNA report 029-2002 and it was submitted to the *Journal of Nonlinear Differential Equations and Applications.*

In part III we consider a sequence of functionals

$$F_\varepsilon(u) := \int_{\Omega_\varepsilon} W_\varepsilon(x, D^2u(x))\,dx$$

where $\beta'|H|^p \leq W_\varepsilon(x,H) \leq \beta(1+|H|^p)$ for some $p \in (1,+\infty)$ and $0 \leq \beta' \leq \beta < \infty$, which represents a second order strain energy of a 3D thin structure, possibly with variable thickness between $\gamma\varepsilon$ and ε for a fixed $\gamma > 0$, i.e., $\Omega_\varepsilon := \{x \in \mathbb{R}^3 : (x_1,x_2) \in \omega, \quad 0 < x_3 < \varepsilon f_\varepsilon(x_1,x_2)\}$, with $\omega \subset \mathbb{R}^2$ and $\gamma \leq f_\varepsilon(x_1,x_2) \leq 1$. This is a generalization of the energies considered in [17], where to a multiple well elastic energy is added a quadratic term of the second derivative to account for interfacial energy, to the case where the dependence in the second order derivate is not necessarily quadratic, and as there is strong convergence in the lower order terms we focus simply on a term depending only on the second derivatives. By considering dependence on x and possibility of variable thickness, we allow for material heterogeneity and periodic profiles, as Braides, Fonseca and Francfort did in [20] for plate models.

For each sequence $\{\varepsilon\}$, we prove the existence of a subsequence $\{\varepsilon^R\}$, such that $\{F_{\varepsilon^R}\}$ converges to a limit functional

$$F_{\{\varepsilon^R\}}(u,b) := \int_\omega W_{\{\varepsilon^R\}}(x_1,x_2,D^2u,Db)\,dx.$$

Although we could not prove that this convergence is Γ-convergence, it has similar properties and the techniques used in the proofs are similar to the framework of Γ convergence, namely sequences $\{u_{\varepsilon^R}\}$ of minimizers (or almost minimizers) of F_{ε^R} converge in a appropriate sense to minimizers of the 2D limit problem $F_{\{\varepsilon^R\}}$. We derive applications to homogeneous, inhomogeneous and periodic models. This is a joint work with Elvira Zappale, it is CNA report 021-2002 and it was accepted in the *Journal of Nonlinear Analysis.*

2 Preliminaries

In this section we present some notation and results that will be useful for the next chapters.

In the sequel Ω is an open, bounded subset of $\mathbb{R}^N$, $\mathcal{L}^N$ is the N-dimensional Lebesgue measure, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, $Q := (0,1)^N$ and $Q(x_0, r) := x_0 + r(-\frac{1}{2}, \frac{1}{2})^N$. Let $\nu \in S^{N-1}$, and denote by $Q_\nu(x_0, r) := x_0 + rQ_\nu$, where

$$Q_\nu := \left\{ x \in \mathbb{R}^N : |x.\nu_i| < \frac{1}{2}, |x.\nu| < \frac{1}{2}, i = 1, .., N-1 \right\},$$

for some orthonormal basis $\{\nu_1, .., \nu_{N-1}, \nu\}$ of $\mathbb{R}^N$. We denote by $E^s_t(\mathbb{R}^m)$ the s-tuple of completely symmetric t-linear forms on $\mathbb{R}^m$. Given a set A the function χ_A is

$$\chi_A := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

$\mathcal{A}(A)$ denotes the set of all open subsets of A, and $\mathcal{A}_\infty(A)$ those which have Lipschitz boundary. The set of the finite Radon measures on $\mathbb{R}^d$ is denoted by $\mathcal{M}(\mathbb{R}^d)$.

2.1 Young measures

In this section we present some results on Young measures. For more details and proofs we refer the reader to [61], [15], [58].

Given a bounded sequence $\{u_n\}$ in $L^p(\Omega)$, for some $p \in [1, +\infty]$, it is often useful to have a characterization of the weak limit of $f(u_n)$, whenever exits, where f is a nonlinear function. The Young measure associated to the sequence (or to a subsequence of) $\{u_n\}$ provides such a characterization. The precise result is stated below.

Theorem 2.1 (Fundamental Theorem on Young Measures). *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{z_n\}$ be a sequence of measurable functions, $z_n : E \to \mathbb{R}^d$. Then there exists a subsequence $\{z_{n_k}\}$ and a weak* measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following hold:*

i) *$v_x \geq 0$, $||\nu_x||_{\mathcal{M}} \leq 1$ for a.e. $x \in E$;*

ii) *One has i') $||\nu_x||_{\mathcal{M}} = 1$ for a.e. $x \in E$ if and only if*

$$\lim_{M \to +\infty} \sup_k \mathcal{L}^N \left(\{|z_{n_k}| \geq M\} \right) = 0; \tag{2.1}$$

iii) *if $K \subset \mathbb{R}^d$ is a compact subset and* $\operatorname{dist}(z_{n_k}, K) \to 0$ *in measure then*

$$\operatorname{supp}\nu_x \subset K \quad \text{for a.e. } x \in E;$$

iv) *if i') holds then in iii) one may replace 'if' by 'if and only if';*

v) *if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory integrand, bounded from below, then*

$$\liminf_{k \to +\infty} \int_\Omega f(x, z_{n_k}(x))\, dx \geq \int_\Omega \bar{f}(x)\, dx$$

where

$$\bar{f}(x) := \langle \nu_x, f(x, .) \rangle = \int_{\mathbb{R}^d} f(x, y) d\nu_x(y);$$

vi) *if i') holds and if f is as in v), then*

$$\liminf_{k\to+\infty} \int_\Omega f(x, z_{n_k}(x))\, dx = \int_\Omega \bar{f}(x)\, dx < +\infty$$

if and only if $\{f(., z_{n_k}(.))\}$ is equi-integrable. In this case

$$f(., z_{n_k}(.)) \rightharpoonup \bar{f} \quad \text{in } L^1(\Omega).$$

The map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ is called the *Young measure generated by the sequence* $\{z_{n_k}\}$. The Young measure ν is said to be *homogeneous* if there is $\nu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_x = \nu_0$ for a.e. $x \in E$.

Remark 2.2. Condition *(2.1)* holds if for some $p > 0$

$$\sup_{n\in\mathbb{N}} \int_E |z_n|^p\, dx < +\infty$$

Proposition 2.3. *If $\{v_n\}$ generates a Young measure ν and if $\omega_n \to \omega$ in measure then $\{v_n + \omega_n\}$ generates the 'translated' Young measure*

$$\tilde{\nu}_x := \Gamma_{\omega(x)}\nu_x$$

where

$$\langle \Gamma_a \mu, \varphi\rangle := \langle \mu, \varphi(. + a)\rangle$$

for $a \in \mathbb{R}^d$, $\varphi \in C_0(\mathbb{R}^d)$. In particular, if $\omega_n \to 0$ in measure then $\{v_n + \omega_n\}$ generates the Young measure ν.

Proposition 2.4. *If $\{v_n\}$ generates a Young measure ν and $u_n \to u$ a.e. in Ω then the sequence $\{(u_n, v_n)\}$ generates the Young measure μ defined by*

$$\mu_x := \delta_{u(x)} \otimes \nu_x, \quad \text{a.e. } x \in \Omega$$

2.2 A-quasiconvexity

Here we present some results on $\mathcal{A}$-quasiconvexity, which are due to Fonseca and Müller. For more details and proofs we refer the reader to [40].

Consider a first order linear partial differential operator of the form

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i},$$

where $v : \mathbb{R}^N \to \mathbb{R}^d$, $A^{(i)} \in \mathcal{M}^{l\times d}$ and

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^{(i)}\omega_i\right) = \text{const}, \tag{2.2}$$

for all $\omega \in \mathbb{R}^N \setminus \{0\}$. The *constant rank hypothesis* (2.2) plays a pivot role in the construction of a continuous projection onto the kernel of $\mathcal{A}$, $\mathbb{T} : L^q(T_N; \mathbb{R}^d) \to L^q(T_N; \mathbb{R}^d)$, where $1 < q < +\infty$ and $L^q(T_N; \mathbb{R}^d)$ denotes the space of functions $v : \mathbb{R}^N \to \mathbb{R}^d$, $Q-$periodic, $v \in L^q(Q)$, with the following properties

Lemma 2.5. **i)** $\mathcal{A}(\mathbb{T}v) = 0$*;*

ii) $\mathbb{T}^2 = \mathbb{T}$

iii) $||v - \mathbb{T}v||_{L^q} \le C_q ||\mathcal{A}v||_{W^{-1,q}}$ *for every $v \in L^q(T_N; \mathbb{R}^d)$ such that $\int_Q v\, dx = 0$;*

iv) *if $\{v_n\}$ is q-equi-integrable then $\{\mathbb{T}v_n\}$ is also q-equi-integrable.*

A Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(v) \leq \int_Q f(v + \omega(x))\, dx,$$

for all $v \in \mathbb{R}^d$ and all $Q-$periodic $\omega \in C^\infty(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}\omega = 0$ and $\int_Q \omega(x)\, dx = 0$. The $\mathcal{A}$-quasiconvexity lies between convexity and convexity in certain directions of the space, defined by the *characteristic cone*

$$\Lambda := \cup_{\omega \in \mathbb{R}^N \setminus \{0\}} \ker \left(\sum_{i=1}^N A^{(i)} \omega_i \right),$$

introduced by Murat and Tartar (see [61], [57]). Precisely,

Proposition 2.6. *If $f : \mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous and $\mathcal{A}$-quasiconvex, then*

$$f(\theta y + (1-\theta)z) \leq \theta f(y) + (1-\theta) f(z)$$

for all $\theta \in (0,1)$, $y, z \in \mathbb{R}^d$ such that $y - z \in \Lambda$.

If the directions in Λ are enough to generate all the space $\mathbb{R}^d$ then the $\mathcal{A}$-quasiconvex functions are locally Lipschitz continuous (see Theorem 2.3, pag. 29, in [31]). In the curl-free case (the usual quasiconvexity) the characteristic cone is $\Lambda = \{a \otimes b : a \in \mathbb{R}^m, b \in \mathbb{R}^N\}$, $m.N = d$, i.e., all the rank-one directions, which are enough to generate all $\mathbb{R}^d$, thus, as it is well known, quasiconvex functions are continuous.

The notion of $k-$quasiconvexity, which was introduced by Meyers in [54] to treat higher order variational problems, can be included also in the framework of $\mathcal{A}$-quasiconvexity, for an appropriate operator $\mathcal{A}$ (see [40], [19]). We recall that a Borel function $f : E_k^m(\mathbb{R}^N) \to \mathbb{R}$ is said to be $k-$quasiconvex if

$$f(v) \leq \int_Q f(v + D^k \varphi(x))\, dx,$$

for all $v \in E_k^m(\mathbb{R}^N)$ and all $\varphi \in C_c^\infty(Q; \mathbb{R}^m)$. For such a functions the characteristic cone is given by (see for instance [16])

$$\Lambda_k := \cup_{b \in \mathbb{R}^N \setminus \{0\}} \{a \otimes b^{\otimes k} : a \in \mathbb{R}^m\}.$$

We now prove that Λ_k generates the space $E_k^m(\mathbb{R}^N)$, thus, $k-$quasiconvex functions are continuous, for any $k \in \mathbb{N}$. This is a consequence of a Theorem of Tensor Algebra we present below. We first need to introduce some notation: let V be a real vector space of dimension m and $V^{(k)}$ the set of completely symmetric $k-$linear forms on V. Let S_m denote the set of all permutations of $\{1, 2, .., m\}$. Given $v^1, v^2, ..., v^m \in V$ we define

$$v^1 \bullet v^2 \bullet .... \bullet v^m := \frac{1}{|S_m|} \sum_{\sigma \in S_m} v^{\sigma^{-1}(1)} \otimes v^{\sigma^{-1}(2)} \otimes ... \otimes v^{\sigma^{-1}(m)},$$

i.e., the symmetrized tensor product (see Theorem 3.1, pag. 89 in [53]). Let $G_{k,m}$ denote the collection of nondecreasing sequences of positive integers of lenght k chosen from $1, .., m$, ω denote the generic element of $G_{k,m}$ and $k_t(\omega)$ the number of times the integer t appears in the range of ω. We then have the following result (Theorem 1.7, pag 191 in [53]).

Theorem 2.7. *Let $\{e_1, .., e_m\}$ be a basis for V. Let $e(\omega) := \sum_{t=1}^m k_t(\omega) e_t$, $\omega \in G_{k,m}$, and set*

$$e^\bullet(\omega) := e(\omega) \bullet ... \bullet e(\omega) \in V^{(k)}.$$

Then $\{e^\bullet(\omega) : \omega \in G_{k,m}\}$ is a basis for $V^{(k)}$. Thus $V^{(k)}$ is spanned by elements of the form $x \bullet ... \bullet x$ (k times), $x \in V$.

Theorem 2.8. *Let $k \in \mathbb{N}$. If $f : E_k^m(\mathbb{R}^N) \to \mathbb{R}$ is k-quasiconvex, then f is locally Lipschitz.*

Moreover, if

$$|f(v)| \leq \alpha\, (1 + |v|^p) \tag{2.3}$$

then

$$|f(v_1) - f(v_2)| \leq \beta \left(1 + |v_1|^{p-1} + |v_2|^{p-1}\right) |v_1 - v_2|,$$

for every v_1, v_2 in $E_k^m(\mathbb{R}^N)$.

Proof. Applying Theorem 2.7 with $V = \mathbb{R}^N$ and $V^{(k)} = E_k(\mathbb{R}^N)$, we get that the set $\{\omega^{\otimes k} : \omega \in \mathbb{R}^N\}$ generates $E_k(\mathbb{R}^N)$. Thus $\Lambda_k = \{a \otimes \omega^{\otimes k} : a \in \mathbb{R}^m, \omega \in \mathbb{R}^N\}$ generates all space $E_k^m(\mathbb{R}^N)$. As f is convex in each direction of Λ_k, by Theorem 2.3, pag. 29, in [31], f is locally Lipschitz continuous.

If, in addition, f satisfies 2.3, then by Lemma 2.2, pag. 156, in [31], f is $p-$Lipschitz continuous. □

This result is well known for $k = 1$, (cf. [31], [52]), and it was recently proved in the case $k = 2$ in [46].

The Theorem below is a characterization of the Young measures generated by bounded sequences $\{v_n\}$ in L^q, satisfying $\mathcal{A}v_n = 0$. This result generalizes that of Kinderleher and Pedregal about Young measures generated by sequences of gradients bounded in L^q, that correspond to the particular case $\mathcal{A} = \text{curl}$ (see [48], [49]).

Theorem 2.9. *Let $1 \le q < +\infty$, and let $\nu = \{\nu\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. There exists a q-equi-integrable sequence $\{v_n\}$ in $L^q(\Omega;\mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}v_n = 0$ in Ω if and only if the following three conditions hold:*

i) *there exists $v \in L^q(\Omega;\mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and*

$$v(x) = \langle \nu_x, Id\rangle \quad a.e.\ x \in \Omega$$

ii)

$$\int_\Omega \langle \nu_x, |z|^q\rangle \, dx < +\infty$$

iii) *for a.e. $x \in \Omega$ and all continuous functions g that satisfy $|g(v)| \le C\,(1 + |v|^q)$ for some $C > 0$ and all $v \in \mathbb{R}^d$ one has*

$$\langle \nu_x, g\rangle \ge Q_{\mathcal{A}}g(\langle \nu_x, Id\rangle),$$

where for $v \in \mathbb{R}^d$

$$Q_{\mathcal{A}}g(v) := \inf\left\{\int_Q f(v + \omega(x))\, dx : \omega \in L^q(T_N;\mathbb{R}^d) \cap \text{Ker}\mathcal{A}\right\}.$$

2.3 Pseudodifferential operators

We present some results on *Pseudodifferential Operators*. For more details and proofs we refer the reader to [59], [6], [62].

We start by introducing some notation. Given a function $u : \mathbb{R}^N \to \mathbb{C}$, we denote by ∂_j the partial derivative with respect to x_j, and by $D_j := -\underline{i}\partial_j$, where $\underline{i}$ is the imaginary unit. Given two functions u and v in $L^2(\mathbb{R}^N)$ we denote by

$$(u, v) := \int_{\mathbb{R}^N} u(x)\overline{v(x)}\, dx.$$

We denote by $\mathcal{S}$ the space of $C^\infty(\mathbb{R}^N)$ functions that are rapidly decreasing at infinity, i.e., a function φ belongs to $\mathcal{S}$ if $x^\alpha\partial^\beta\varphi$ are bounded in $\mathbb{R}^N$ for all pairs α, β of multiindices. The topology on $\mathcal{S}$ is defined by the norms ($k \in \mathbb{Z}_0^+$)

$$||\varphi||_k = \sup_{|\alpha+\beta|\le k} ||x^\alpha\partial^\beta\varphi||_\infty.$$

We denote by $\mathcal{S}'$ the set of semilinear forms u on $\mathcal{S}$ (i.e. $(u, \alpha\varphi + \beta\psi) = \bar{\alpha}(u,\varphi) + \bar{\beta}(u,\psi)$) such that there exits $C \in \mathbb{R}$ and $M \in \mathbb{Z}_0^+$ verifying

$$|(u,\varphi)| \le C||\varphi||_M \qquad \text{for } \varphi \in \mathcal{S}.$$

For a function $u \in \mathcal{S}$, the Fourier transform $\hat{u}$ (or $\mathcal{F}u$) of u, is defined by the formula

$$\hat{u}(\lambda) := \int_{\mathbb{R}^N} u(x)e^{-\underline{i}x.\lambda}\, dx.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}u(\lambda) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} u(x)e^{ix.\lambda}\,dx.$$

Given $s \in \mathbb{R}$ we denote by $L^{s,p}(\mathbb{R}^N)$ the image of $L^p(\mathbb{R}^N)$ under the linear mapping

$$J^s u = \mathcal{F}^{-1}\left((1+|\lambda|^2)^{-\frac{s}{2}}\mathcal{F}u\right).$$

If $u \in L^{s,p}(\mathbb{R}^N)$ then there exits a unique $\tilde{u} \in L^p(\mathbb{R}^N)$ with $u = J^s\tilde{u}$. The space $L^{s,p}(\mathbb{R}^N)$ is a Banach space with norm

$$||u||_{L^{s,p}} := ||\tilde{u}||_{L^p}.$$

The spaces $L^{s,p}$, with $p = 2$, coincide with $H^s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$, and for $p \in (1, +\infty)$ and $s \in \mathbb{Z}$ they coincide with $W^{s,p}(\mathbb{R}^N)$. We have the duality relation

$$[L^{s,p}(\mathbb{R}^N)]' = L^{-s,p'}(\mathbb{R}^N),$$

where $p' = \frac{p}{p-1}$.
For more details about the spaces $L^{s,p}$ we refer the reader to [4].

Let $q \in \mathbb{R}$ and $b(x, \lambda)$ be a C^∞ complex-valued function on $\mathbb{R}^N \times \mathbb{R}^N$. We say that b is a *symbol of order-q*, and we write $b \in S^q$, if there exists constants $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_\lambda^\beta b(x,\lambda)| \leq C_{\alpha\beta}\left(1+|\lambda|^2\right)^{\frac{q-|\beta|}{2}}, \tag{2.4}$$

for $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N$, α, $\beta \in \mathbb{Z}_+^N$. We have $S^q \subset S^l$ for $q \leq l$, and define $S^\infty := \cup_q S^q$.
Given a symbol $b \in S^q$ we say that $b \sim \sum_j b_j$, with $j \in \mathbb{Z}_0^+$, if $b_j \in S^{q-j}$ and

$$b - \sum_{j<k} b_j \in S^{q-k}.$$

We define below two operations on symbols, the compound, $b\#c$, and the adjoint, $b^\star$.

Theorem 2.10. *Let $b \in S^q$ and $c \in S^l$. Then the oscillatory integrals*

$$b^\star(x,\lambda) = \frac{1}{(2\pi)^N}\int \bar{b}(x-y,\lambda-\eta)e^{-iy\cdot\eta}\,dy\,d\eta$$

$$b\#c(x,\lambda) = \frac{1}{(2\pi)^N}\int b(x,\lambda-\eta)c(x-y,\lambda)e^{-iy\cdot\eta}\,dy\,d\eta$$

define symbols $b^\star \in S^q$ and $b\#c \in S^{q+l}$ with the following asymptotic expansions

$$b^\star \sim \sum_\alpha \frac{1}{\alpha!}\partial_\lambda^\alpha D_x^\alpha \bar{b} \qquad b\#c \sim \sum_\alpha \frac{1}{\alpha!}\partial_\lambda^\alpha b D_x^\alpha c.$$

Remark 2.11. For any $b \in S^\infty$ we have

$$(b^\star)^\star = b.$$

For $t \in \mathbb{R}$, denote by τ^t the symbol $\tau^t(\lambda) = \left(1+|\lambda|^2\right)^{\frac{t}{2}}$, we then have

i) $(\tau^t)^\star = \tau^t$

ii) $\tau^{t_1}\#\tau^{t_2} = \tau^{t_1+t_2}$

We associate a pseudo-differential operator B (or $b(x,D)$) to the symbol $b(x,\lambda)\in S^q$, by the formula

$$B\varphi(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} b(x,\lambda)\hat{\varphi}(\lambda)e^{ix.\lambda}, \qquad \varphi\in\mathcal{S}(\mathbb{R}^N).$$

The function $B\varphi\in\mathcal{S}(\mathbb{R}^N)$ and the application is continuous from $\mathcal{S}$ to $\mathcal{S}$ (see Theorem 3.1. in [59]).

The adjoint symbol is associated with the adjoint operator, it is the tool to extend the domain of a pseudodifferential operator to $S^{'}$, and the compound symbol is associated with composition, as the theorem below shows.

Theorem 2.12. *For any b and $c\in S^\infty$ and φ and $\psi\in\mathcal{S}$ one has*

i) $(b^\star(x,D)\varphi,\psi)=(\varphi,b(x,D)\psi)$

ii) $(b\#c(x,D)\varphi,\psi)=(b(x,D)c(x,D)\varphi,\psi)$

Remark 2.13. Given $b=b(x,\lambda)\in S^q$ and $c=c(\lambda)\in S^l$, the symbol correspondent to the composition $b(x,D)c(D)$ is the multiplication of the symbols, i.e.,

$$b\#c(x,\lambda)=b(x,\lambda)c(\lambda).$$

However, the general case where the symbol c also depend on x, is more complicated. In that case, according to Theorems 2.10 and 2.12, all one can say is that

$$b\#c(x,\lambda)=b(x,\lambda)c(x,\lambda)+\textit{symbol of order } q+l-1$$

The domain of a pseudodifferential can be extended to $\mathcal{S}^{'}$, as it is illustrated below

Definition 2.14. *Given a $b\in S^\infty$, we call* pseudodifferential operator of symbol b, *the operator $b(x,D)$: $\mathcal{S}^{'}\to\mathcal{S}^{'}$ defined by*

$$(b(x,D)u,\varphi)=(u,b^\star(x,D)\varphi), \qquad \textit{for} \quad u\in\mathcal{S}^{'},\varphi\in\mathcal{S}$$

If $b\in S^q$, $b(x,D)$ is said to have order q.

In particular, we can define the action of a pseudodifferential operator on Sobolev spaces, and the continuity result below holds.

Theorem 2.15. *Let $b\in S^q$. Then for every $s\in\mathbb{R}$ there exists a constant C_s such that $b(x,D)u\in H^{s-q}$ for all $u\in H^s$, with*

$$||b(x,D)u||_{H^{s-q}}\le C_s||u||_{H^s}.$$

For $p\neq 2$ a similar result holds if we replace the Sobolev spaces by the spaces $L^{s,p}$. In order to prove that we need the following result, due to Coifman and Meyer ([30]).

Theorem 2.16. *Let $b\in S^0$ and $p\in(1,+\infty)$. Then $b(x,D)u\in L^p(\mathbb{R}^N)$ for all $u\in L^p(\mathbb{R}^N)$, and*

$$||b(x,D)\varphi||_{L^p}\le C||\varphi||_{L^p}, \qquad \forall\varphi\in L^p(\mathbb{R}^N),$$

Theorem 2.17. *Let $b\in S^q$. Then for every $s\in\mathbb{R}$ there exists a constant C_s such that $b(x,D)u\in L^{s-q,p}$ for all $u\in L^{s,p}$, with*

$$||b(x,D)u||_{L^{s-q,p}}\le C_s||u||_{L^{s,p}}.$$

Proof. The proof is similar to the proof of Theorem 2.15 presented in [59].

Let $b\in S^q$. We first prove that

$$||b^\star(x,D)\varphi||_{L^{-s,p}}\le C||\varphi||_{L^{q-s,p}}, \qquad \text{for} \quad \varphi\in\mathcal{S}.$$

Note that $\mathcal{S}\subset L^{s,p}$ and

$$||\varphi||_{L^{s,p}}=||\tau^s(D)\varphi||_{L^p}.$$

We have

$$\begin{aligned}||b^\star(x,D)\varphi||_{L^{-s,p}} &= ||\tau^{-s} b^\star(x,D)\varphi||_{L^p}\\ &= ||\tau^{-s}(D)b^\star(x,D)\tau^{-q+s}(D)\tau^{q-s}(D)\varphi||_{L^p}\\ &\leq C||\tau^{q-s}(D)\varphi||_{L^p} = C||\varphi||_{L^{q-s,p}}.\end{aligned}$$

Let $u \in L^{s,p}$, we now prove that

$$|(b(x,D)u,\varphi)| \leq C||u||_{L^{s,p}}||\varphi||_{L^{m-s,p'}}, \qquad \forall \varphi \in \mathcal{S}.$$

We have

$$\begin{aligned}|(b(x,D)u,\varphi)| &= |(u, b^\star(x,D)\varphi)|\\ &= |(u, \tau^s(D)\tau^{-s}(D)b^\star(x,D)\varphi)|\\ &= |(\tau^s(D)u, \tau^{-s}(D)b^\star(x,D)\varphi)|\\ &\leq ||u||_{L^{s,p}}||b^\star(x,D)\varphi||_{L^{-s,p'}}\\ &\leq C||u||_{L^{s,p}}||\varphi||_{L^{q-s,p'}},\end{aligned}$$

thus $b(x,D)u \in L^{s-q,p}$ and

$$||b(x,D)u||_{L^{s-q,p}} \leq C||u||_{L^{s,p}}.$$

□

In what follows we are interested in pseudodifferential operators associated with matrix-valued symbols. Given a matrix $B(x,\lambda) := [b_{jk}(x,\lambda)]_{j,k=1}^{s,t}$, we say that $B(x,\lambda) \in (S^q)^{s\times t}$ if $b_{jk}(x,\lambda) \in S^q$ for $j = 1,..,s$, $k = 1,..,t$. Given $u \in \mathcal{S}'(\mathbb{R}^N;\mathbb{R}^t)$ we define $Bu \in \mathcal{S}'(\mathbb{R}^N;\mathbb{R}^s)$ by

$$(Bu)_j := \sum_{k=1}^t b_{jk}(x,D)u_k, \qquad j = 1,..,s.$$

It is easy to check that all the results we presented above for scalar-valued symbols still hold for matrix-valued symbols.

We now derive some estimates that will be useful in Chapter 3. Consider the operator defined in (3.1) and denote by $A(x,\lambda)$ its symbol, i.e.

$$A(x,\lambda) := \sum_{i=1}^N A^{(i)}(x)\lambda_i,$$

and by $P(x,\lambda)$ the projection onto $\mathrm{Ker}(A(x,\lambda))$. Define $Q(x,\lambda)$ by the implicit equation

$$Q(x,\lambda)A(x,\lambda) := I_m - P(x;\lambda). \tag{2.5}$$

The function $Q(x,\lambda)$ is positively homogeneous of degree -1 in λ and using (3.2) we get that $Q(x,\lambda) \in C^\infty(\Omega\times\mathbb{R}^N\setminus\{0\};\mathcal{M}^{m\times d})$. Define

$$Q_\eta(x,\lambda) := \eta(x)Q(x,\lambda)\chi(|\lambda|),$$

where $\chi : [0,+\infty) \to \mathbb{R}$ is a C^∞-function for which we can find numbers r, R, $0 < r < R < +\infty$, such that $\chi(|\lambda|) = 0$ for $|\lambda| < r$ and $\chi(|\lambda|) = 1$ for $|\lambda| > R$, and $\eta \in C_c^\infty(\Omega;[0,1])$, $\eta = 1$ on $\tilde\Omega$, for some open set $\tilde\Omega \subset\subset \Omega$. It is easy to check that

$$|\partial_x^\alpha\partial_\lambda^\beta Q_\eta(x,\lambda)| \leq C_{\alpha,\beta}\Big(1+|\lambda|^2\Big)^{\frac{-1-|\beta|}{2}},$$

for $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^N$. Thus $Q_\eta(x,\lambda)$ is a symbol of order -1 and we denote by Q_η the corresponding pseudo-differential operator.

We denote by $A_\eta(x,\lambda)$ the symbol

$$A_\eta(x,\lambda) := \sum_{i=1}^{N} \eta(x) A^{(i)}(x)\lambda_i,$$

and by $\mathcal{A}_\eta$ the corresponding operator.

By Remark 2.13, the compound operator $Q_\eta \mathcal{A}_\eta$ has order 0 and symbol

$$\eta(x)Q(x,\lambda)\chi(|\lambda|)A_\eta(x,\lambda) + \quad \text{symbol of order -1},$$

or, using (2.5),

$$\eta^2(x)I_m - \eta^2(x)P(x,\lambda)\chi(|\lambda|) + \quad \text{symbol of order -1}.$$

We denote by P_η the operator correspondent to the order 0 symbol $\eta^2(x)P(x,\lambda)\chi(|\lambda|)$, thus

$$u - P_\eta u = Q_\eta \mathcal{A}u + Ku,$$

for $u \in L^p(\Omega)$ with compact support in $\tilde{\Omega}$, where K is a pseudo-differential operator of order -1. Using Theorem 2.17, we get the estimates

$$||u - P_\eta u||_{L^p} \le C||\mathcal{A}u||_{W^{-1,p}} + C||u||_{W^{-1,p}}, \tag{2.6}$$

and

$$||\mathcal{A}P_\eta u||_{W^{-1,p}} \le C||u||_{W^{-1,p}}, \tag{2.7}$$

where we have used the fact that $\mathcal{A}P_\eta$ is an operator of order 0, because of the relation

$$A(x,\lambda)P(x,\lambda) = 0.$$

2.4 Spaces BV, SBV, and sets of finite perimeter

We recall some basic definitions and properties of the space BV of *functions of bounded variation*, of the space SBV of *functions of special bounded variation*, and also of *sets of finite perimeter*. For more details and proofs we refer the reader to [14].

Definition 2.18. *A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of* bounded variation*, $u \in BV(\Omega;\mathbb{R}^d)$, if for all $i \in \{1,..,d\}$, $j \in 1,..,N$, there exists a bounded Radon measure μ_{ij} such that*

$$\int_\Omega u_i(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx = -\int_\Omega \varphi(x)\,d\mu_{ij}$$

for every $\varphi \in C^1_c(\Omega)$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}. The total variation of the gradient measure, $|Du|(\Omega)$, is given by $|Du|(\Omega) = \sum_1^d |Du_i|(\Omega)$, where

$$|Du_i|(\Omega) := \sup_\varphi \left\{ \int_\Omega u_i \,\mathrm{div}\varphi\,dx : \varphi \in C^1_c(\Omega;\mathbb{R}^N), \quad ||\varphi||_\infty \le 1 \right\}.$$

The space BV is a Banach space equipped with the norm

$$||u||_{BV(\Omega;\mathbb{R}^d)} := ||u||_{L^1(\Omega;\mathbb{R}^d)} + |Du|(\Omega).$$

Definition 2.19. *A set A is said to be of* finite perimeter in Ω if $\chi_A \in BV(\Omega)$, where χ_A denotes the characteristic function of A. The perimeter of A is defined by

$$Per_\Omega(A) := |D\chi_A|(\Omega)$$

There is an important connexion between sets of finite perimeter and level sets of BV functions which we state next.

Theorem 2.20 (Co-area formula for BV functions). *If $u \in BV(\Omega)$, then $E_t := \{x \in \Omega : u(x) > t\}$ has finite perimeter for a.e. $t \in \mathbb{R}$, and*

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{E_t}|(\Omega)\, dt$$

Let $E \subset \mathbb{R}^N$ be a $\mathcal{L}^N$-measurable set. We define

$$E^1 := \left\{ x \in \mathbb{R}^N : \lim_{\varepsilon \to 0} \frac{\mathcal{L}^N(E \cap Q(x,\varepsilon))}{\varepsilon^N} = 1 \right\}$$

and

$$E^0 := \left\{ x \in \mathbb{R}^N : \lim_{\varepsilon \to 0} \frac{\mathcal{L}^N\left(E \cap Q(x,\varepsilon)\right)}{\varepsilon^N} = 0 \right\},$$

the *measure theoretic interior* (the set of points of density 1 in E) and the *measure theoretic exterior* (the set of points of density 0 in E) of E, respectively. Also, $\partial^* E$ is the *essential boundary* of E, i.e.

$$\partial^* E := \mathbb{R}^N \setminus (E^0 \cup E^1).$$

We note that $\mathcal{L}^N(E \triangle E^1) = 0$ and $\mathcal{L}^N((\mathbb{R}^N \setminus E) \triangle E^0) = 0$.

For sets of finite perimeter it is possible to define a normal on part of the boundary, the *reduced boundary* $\mathcal{F}E$.

Definition 2.21. *Let $E \subset \Omega$ be a set of finite perimeter in Ω. We define reduced boundary $\mathcal{F}E$ to be the set of points x such that*

i) $|D\chi_E|(Q(x,\varepsilon)) > 0$ *for every $\varepsilon > 0$ such that $Q(x,\varepsilon) \subset \Omega$,*

ii) $\nu_E(x) := \lim_{\varepsilon \to 0} \frac{D\chi_E(Q(x,\varepsilon))}{|D\chi_E|(Q(x,\varepsilon)}$ *exits in $\mathbb{R}^N$,*

iii) $|\nu_E(x)| = 1$.

The function $\nu_E : \mathcal{F}E \to S^{N-1}$ is called the generalized inner normal to E.

It can be shown that $\mathcal{F}E \subset \partial^* E$ and $\mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0$ (see [14]). The set $\mathcal{F}E$ is $(N-1)$-countable rectifiable, i.e.,

$$\mathcal{F}E = \cup_{n=1}^{\infty} K_n \cup E,$$

and $\mathcal{H}^{N-1}(E) = 0$, K_n is a compact subset of a C^1 hypersurface S_n for each n, and $\nu_E|S_n$ is normal to S_n.

Given $u \in BV(\Omega; \mathbb{R}^d)$, the *approximate upper* and *lower limit* of each component u_i, $i \in \{1, .., d\}$, are given by

$$u_i^+(x) := \inf \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}^N(\{u_i > t\} \cap Q(x,\varepsilon))}{\varepsilon^N} = 0 \right\}$$

$$u_i^-(x) := \sup \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}^N(\{u_i < t\} \cap Q(x,\varepsilon))}{\varepsilon^N} = 0 \right\}.$$

The set

$$S(u) := \cup_{i=1}^d \left\{ x \in \Omega : u_i^-(x) < u_i^+(x) \right\}$$

is called *jump set* of u, and the value $\tilde{u}(x) := \frac{1}{2}(u^+(x) + u^-(x))$ is defined for every $x \in \Omega$. It is well known that $S(u)$ is $(N-1)$-countable rectifiable. The result below is about some fine properties that BV functions enjoy.

Theorem 2.22. *If $u \in BV(\Omega; \mathbb{R}^d)$, then*

(i) *for $\mathcal{L}^N$-a.e. $x \in \Omega$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N+1}} \left\{ \int_{Q(x_0,\varepsilon)} |u(y) - u(x) - \nabla u(x).(y-x)|^{\frac{N}{N-1}}\, dy \right\}^{\frac{N-1}{N}} = 0;$$

(ii) *for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, there exists a unit vector $\nu(x) \in S^{N-1}$, normal to $S(u)$ at x, and there exist vectors $u_-(x), u_+(x) \in \mathbb{R}^d$ such that*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon):(y-x).\nu(x)>0\}} |u(y) - u_+(x)|^{\frac{N}{N-1}} \, dy = 0;$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon):(y-x).\nu(x)<0\}} |u(y) - u_-(x)|^{\frac{N}{N-1}} \, dy = 0;$$

(iii) *for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \Omega \setminus S(u)$*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_Q |u(y) - \tilde{u}(x_0)| \, dx = 0.$$

We remark that, in general, $(u_i)^\pm \neq (u_\pm)_i$. We denote by $[u](x)$ the jump of u at x, defined by

$$[u](x) := u_+(x) - u_-(x).$$

If $u \in BV(\Omega;\mathbb{R}^d)$, then the measure Du may be represented as

$$Du = \nabla u \mathcal{L}^N + (u_+ - u_-) \otimes \nu \mathcal{H}^{N-1} \lfloor S(u) + C(u), \tag{2.8}$$

where ∇u is the density of the absolutely continuous part of Du with respect to $\mathcal{L}^N$, and $C(u)$ is the Cantor part. The three measures in 2.8 are mutually singular.

It is possible to define the trace of a function $u \in BV(\Omega;\mathbb{R}^d)$ on the reduced boundary of a set E of finite perimeter in Ω (see Theorem 3.77, pag. 171 of [14]).

Theorem 2.23. *Let $u \in BV(\Omega;\mathbb{R}^d)$ and let $E \subset \Omega$ be a set of finite perimeter in Ω. For $\mathcal{H}^{N-1}$-almost every $x \in \mathcal{F}E$ there exist $u_{+,\mathcal{F}E}(x)$, $u_{-,\mathcal{F}E}(x)$ in $\mathbb{R}^d$ such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{Q^+_{\nu_E}(x,\varepsilon)} |u(y) - u_{+,\mathcal{F}E}(x)| \, dy = 0$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{Q^-_{\nu_E}(x,\varepsilon)} |u(y) - u_{-,\mathcal{F}E}(x)| \, dy = 0$$

Moreover $Du \lfloor \mathcal{F}E = (u_{+,\mathcal{F}E} - u_{-,\mathcal{F}E}) \otimes \nu_E \mathcal{H}^{N-1} \lfloor \mathcal{F}E$.

Theorem 2.24. *Let $u, v \in BV(\Omega;\mathbb{R}^d) \cap L^\infty$ and let E be a set of finite perimeter in Ω. Then $w := u\chi_E + v\chi_{\Omega \setminus E} \in BV(\Omega;\mathbb{R}^d)$ and*

$$Dw = Du \lfloor E^1 + (u_{+,\mathcal{F}E} - v_{-,\mathcal{F}E}) \otimes \nu_E \mathcal{H}^{N-1} \lfloor (\mathcal{F}E) + Dv \lfloor E^0$$

The next theorem is a generalization of the Besicovitch Differentiation Theorem (see Ambrosio and Dal Maso [11], Proposition 2.2).

Theorem 2.25. *If λ and μ are Radon measures in Ω, $\mu \geq 0$, then there exists a Borel set $E \subset \Omega$ such that $\mu(E) = 0$, and for every $x \in \mathrm{supp}\mu \setminus \mathrm{E}$*

$$\frac{d\lambda}{d\mu}(x) := \lim_{\varepsilon \to 0} \frac{\lambda(x + \varepsilon C)}{\mu(x + \varepsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

The following subspace of BV was introduced and studied by De Giorgi and Ambrosio [13].

Definition 2.26. *A function $u \in BV(\Omega;\mathbb{R}^d)$ is said to be of* special bounded variation *if $C(u) = 0$. We write $u \in SBV(\Omega;\mathbb{R}^d)$.*

The following SBV compactness theorem is due to Ambrosio [9].

Theorem 2.27. *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\theta : (0, +\infty) \to \mathbb{R}$ be nondecreasing lower semicontinuous functions satisfying*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \qquad \lim_{t\to 0^+} \frac{\theta(t)}{t} = \infty.$$

Let $\{u_n\}$ be a sequence of functions in $SBV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ such that $\sup_n ||u_n||_\infty < \infty$ and

$$\sup_n \left\{ \int_\Omega \varphi(|\nabla u_n|)\, dx + \int_{\Omega\cap S(u_n)} \theta([u_n])\mathcal{H}^{N-1} \right\} < \infty.$$

Then there exists a subsequence $\{u_{n_i}\}$ and a function $u \in SBV(\Omega;\mathbb{R}^d)$ such that

$$u_{n_i} \to u \quad in \quad L^1, \qquad \nabla u_{n_i} \rightharpoonup \nabla u \quad in \quad L^1.$$

The next theorem was obtained by Alberti [5].

Theorem 2.28. *Let $f \in L^1(\Omega;\mathbb{R}^{d\times N})$. There exists $u \in SBV(\Omega;\mathbb{R}^d)$ and a Borel function $g : \Omega \to \mathbb{R}^{d\times N}$ such that*

$$Du = f\mathcal{L}^N + g\mathcal{H}^{N-1}\lfloor S(u), \qquad \int_{\Omega\cap S(u)} |g| d\mathcal{H}^{N-1} \le C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})},$$

where C depends only on N.

The Lemma below is a simple corollary of the co-area formula for BV functions, and it is an improvement to the Lemma 2.9 in [27].

Lemma 2.29. *Let $u \in BV(\Omega;\mathbb{R}^d)$. There exist functions $u_n \in SBV$, with $\nabla u_n = 0$, such that $u_n - u \to 0$ in L^∞ and*

$$\lim_{n\to+\infty} |Du_n|(\Omega) = \lim_{n\to+\infty} \int_{\Omega\cap S(u_n)} |[u_n](x)| d\mathcal{H}^{N-1}(x) = |Du|(\Omega).$$

Proof. Without loss of generality we may assume that $d = 1$. We also assume u to be non-negative; the general case follows by considering the positive and negative parts of u. Let $E_t := \{x \in \Omega : u(x) > t\}$ and define

$$u_n(x) := \sum_{i=1}^{\infty} \frac{1}{n} \chi_{E_{(a_{in})}}(x),$$

where, using Theorem 2.20, $a_{in} \in (\frac{i-1}{n}, \frac{i}{n})$ is such that $E_{a_{in}}$ has finite perimeter and

$$\frac{|D\chi_{E_{a_{in}}}|(\Omega)}{n} \le \int_{\frac{i-1}{n}}^{\frac{i}{n}} |D\chi_{E_t}|(\Omega)\, dt.$$

Clearly $u_n - u \to 0$ in $L^\infty(\Omega)$, and

$$|Du_n|(\Omega) \le \sum_{i=0}^{\infty} \frac{1}{n} |D\chi_{E_{a_{in}}}|(\Omega) \le \int_0^\infty |D\chi_{E_t}|(\Omega)\, dt = |Du|(\Omega).$$

This yelds

$$\limsup |Du_n|(\Omega) \le |Du|(\Omega)$$

The converse inequality follows from the lower semicontinuity of the total variation. $\square$

2.5 Global method for relaxation

We present an integral representation result for a class of functionals. The following result was established in [24].

Theorem 2.30. *Let $\mathcal{F}_1 : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty)$ be a functional satisfying the following assumptions*

$$\begin{array}{ll} (C1) & \mathcal{F}_1(u,\cdot) \text{ is the restriction to } \mathcal{A}(\Omega) \text{ of a Radon measure,} \\ (C2) & \mathcal{F}_1(u,A) = \mathcal{F}_1(v,A) \text{ whenever } u = v \quad \mathcal{L}^N \text{ a.e. on } A \in \mathcal{A}(\Omega), \\ (C3) & \mathcal{F}_1(\cdot A) \text{ is } L^1(\Omega;\mathbb{R}^m) - \text{ lower semicontinuous,} \\ (C4) & \text{there exists } C > 0 \text{ such that } \frac{1}{C}\int_A |Du|^p dx \le \mathcal{F}_1(u,A) \le C\int_A (1+|Du|^p)dx. \end{array}$$

Then, for every $u \in W^{1,p}(\Omega,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}_1(u,A) = \int_A f_1(x,u,Du)dx, \ \ \text{where } f_1(x_0,u_0,\xi) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_1(u_0+\xi(\cdot - x_0);Q(x_0,\varepsilon))}{\varepsilon^N}$$

for all $x_0 \in \omega, u_0 \in \mathbb{R}^m, \xi \in \mathbb{R}^{dN}$, and where, for $(v,A) \in W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega)$,

$$\mathbf{m}_1(v,A) := \inf\left\{\mathcal{F}_1(w,A) : w \in W^{1,p}(A;\mathbb{R}^m) \text{ with } u = v \text{ in a neighborhood of } \partial A\right\}.$$

With a proof entirely analogous to that of Theorem 2.30, one can obtain a similar result valid for functionals defined in $W^{2,p}(\Omega;\mathbb{R}^m)\times$ $W^{1,p}(\Omega,\mathbb{R}^m)$. Let $\mathcal{F} : W^{2,p}(\Omega,\mathbb{R}^m)\times W^{1,p}(\Omega,\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty)$ be an energy functional such that

$$\begin{array}{ll} (A1) & \mathcal{F}(u,b,\cdot) \text{ is the restriction of a Radon measure to } \mathcal{A}(\Omega), \\ (A2) & \mathcal{F}(\cdot,\cdot A) \text{ is } W^{2,p}\times W^{1,p} - \text{sequentially weakly lower semicontinuous, that is} \\ & u_n \rightharpoonup u \text{ in } W^{2,p}(\Omega,\mathbb{R}^m) \text{ and } b_n \rightharpoonup b \in W^{1,p}(\Omega,\mathbb{R}^m) \Rightarrow \mathcal{F}(u,b,A) \le \liminf_{n\to+\infty} \mathcal{F}(u_n,b_n A) \\ & \text{for all } A \in \mathcal{A}(\Omega); \\ (A3) & \mathcal{F}(\cdot,\cdot,A) \text{ is local, i.e. if } u = v \text{ and } b = d \text{ a.e. in } A \in \mathcal{A}(\Omega) \text{ then } \mathcal{F}(u,b,A) = \mathcal{F}(v,d,A); \\ (A4) & \text{there exists } C > 0 \text{ such that} \\ & \frac{1}{C}\int_A \left(\left|D^2 u\right|^p + |Db|^p\right) dx \le \mathcal{F}(u,b,A) \le C\int_A \left(1 + \left|D^2 u\right|^p + |Db|^p\right) dx \end{array}$$

for $u \in W^{2,p}(\Omega,\mathbb{R}^m)$, $b \in W^{1,p}(\Omega,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Theorem 2.31. *Let $\mathcal{F} : W^{2,p}(\Omega,\mathbb{R}^m)\times W^{1,p}(\Omega,\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty)$ be a functional satisfying the assumptions $(A1)-(A4)$. For every $(u,b) \in W^{2,p}(\Omega,\mathbb{R}^m)\times W^{1,p}(\Omega,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u,b,A) = \int_A f(x,u,Du,D^2u,b,Db)dx$$

where

$$f(x_0,u_0,\xi,H,b_0,\beta) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}(u_0+\xi(\cdot - x_0)+\frac{1}{2}(\cdot - x_0)^T H(\cdot - x_0), b_0+\beta(\cdot - x_0), Q(x_0,\varepsilon))}{\varepsilon^N}, \tag{2.9}$$

and

$$\mathbf{m}(u_0,b_0,A) := \inf\Big\{\mathcal{F}(u,b,A) : (u,b) \in W^{2,p}(A,\mathbb{R}^m)\times W^{1,p}(A,\mathbb{R}^m), \ \ u = u_0 \text{ and } b = b_0 \text{ on a neighborhood of } \partial A\Big\}$$

for $u_0 \in W^{2,p}(\Omega,\mathbb{R}^m), b_0 \in W^{1,p}(\Omega,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

The proof of Theorem 2.31 is hinged on several lemmata. Most of them are analogous to those related to the proof of Theorem 2.30. We present just those which differ from their counterpart in [24]. To this end, and just as in [22] and [24], we start by introducing the following family of functions. For $\delta > 0$, $u \in W^{2,p}(\Omega, \mathbb{R}^m)$,$b \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$,

$$\mathbf{m}^\delta(u_0, b_0, A) := \inf \Big\{ \sum_{i=1}^{\infty} \mathbf{m}(u_0, b_0, Q_i) : Q_i \text{ cubes },\ \ Q_i \cap Q_j = \emptyset \text{ for } j \neq i,$$
$$Q_i \subset A,\ \ \mathrm{diam}(Q_i) < \delta,\ \mathcal{L}^N(A \setminus \cup Q_i) = 0 \Big\}.$$

For every $A \in \mathcal{A}(\Omega)$ and $(u_0, b_0) \in W^{2,p}(\Omega; \mathbb{R}^m) \times W^{1,p}(\Omega, \mathbb{R}^m)$ we set

$$\mathbf{m}^*(u_0, b_0, A) := \lim_{\delta \to 0} \mathbf{m}^\delta(u_0, b_0, A). \tag{2.10}$$

Note that (2.10) is meaningful since the $\{\mathbf{m}_\delta(u_0, b_0, A)\}$ is a decreasing sequence in δ.

Lemma 2.32. *If $A' \subset\subset A$, with $A', A \in \mathcal{A}(\Omega)$, then*

$$\mathbf{m}(u, b, A) \le \mathbf{m}(u, b, A') + C \int_{A \setminus A'} \left(1 + \left|D^2 u\right|^p + |Db|^p\right) dx,$$

and

$$\mathcal{F}(u, b, A) \le \mathcal{F}(u, b, A') + C \int_{A \setminus A'} \left(1 + \left|D^2 u\right|^p + |Db|^p\right) dx.$$

Proof. Given $\eta > 0$ choose $(v, d) \in W^{2,p}(A', \mathbb{R}^m) \times W^{1,p}(A', \mathbb{R}^m)$ such that $v = u$ and $d = b$ on a neighborhood of $\partial A'$ and

$$\mathbf{m}(u, b, A') \ge \mathcal{F}(v, d, A') - \eta.$$

Define

$$w := \begin{cases} v \text{ in } A' \\ u \text{ in } \omega \setminus \overline{A'} \end{cases} \quad \text{and } q := \begin{cases} d \text{ in } A' \\ b \text{ in } \omega \setminus \overline{A'} \end{cases}$$

Let $\varepsilon > 0$, set $\tilde{A}'_\varepsilon := \{x \in A', \mathrm{dist}(x, \partial A') > \varepsilon\}$. By $(A4)$ we have

$$\begin{aligned} \mathbf{m}(u, b, A) &\le \mathcal{F}(w, q, A) \\ &\le \mathcal{F}(v, d, A') + C \int_{A \setminus \overline{\tilde{A}'_\varepsilon}} \left(1 + \left|D^2 u\right|^p + |Db|^p\right) \\ &\le \mathbf{m}(u, b, A') + \eta + C \int_{A \setminus \overline{\tilde{A}'_\varepsilon}} \left(1 + \left|D^2 u\right|^p + |Db|^p\right) dx. \end{aligned}$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$ we get the desired inequality. A similar proof can be performed for $\mathcal{F}$. $\square$

Lemma 2.33. *Let $(u, b) \in W^{2,p}(\Omega, \mathbb{R}^m) \times W^{1,p}(\Omega, \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$. Under assumptions $(A1) - (A4)$*

$$\mathbf{m}^*(u, b, A) = \mathcal{F}(u, b, A).$$

Proof. The proof of Lemma 2.33 is entirely similar to the proof of Lemma 3.3 in [22].

$\square$

Lemma 2.34. *If $\mathcal{F}$ satisfies $(A1) - (A4)$, then*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}((u, b), Q(x_0, \varepsilon))}{\varepsilon^N} = \lim_{\varepsilon \to 0} \frac{\mathbf{m}((u, b), Q(x_0, \varepsilon))}{\varepsilon^N} \tag{2.11}$$

for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$.

Proof. The proof is entirely similar to the proof of Lemma 3.5 in [22]. See also Lemma 2.2.2 in [24]. $\square$

The proofs of the two following lemmas are proposed for the convenience of the reader, since there are slight modifications with respect to the proofs of Lemmas 2.3.1 and 2.3.2 in [24], due to the presence of second order derivatives.

Lemma 2.35. *For a.e.* $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u,b,\cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), Du(x_0), D^2u(x_0), b(x_0), Db(x_0)) \tag{2.12}$$

where f is the function defined in (2.9).

Proof. In the proof the constant C may vary from line to line.

By (2.11) and arguing as in [24] we have

$$\frac{d\mathcal{F}(u,b,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\lambda\to 1^-}\lim_{\varepsilon\to 0^+}\frac{\mathbf{m}(u,b,Q(x_0,\lambda\varepsilon))}{\varepsilon^N}. \tag{2.13}$$

In order to get (2.12) it is enough to verify that the right hand side of (2.13) is less than or equal to $f(x_0, u(x_0), Du(x_0), D^2u(x_0), b(x_0), Db(x_0))$. Fix $0 < s < 1$ and let us consider $v_\varepsilon \in W^{2,p}(Q(x_0, s\lambda\varepsilon))$ and $d_\varepsilon \in W^{1,p}(Q(x_0, s\lambda\varepsilon))$ such that $v_\varepsilon(x) = u(x_0) + Du(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T D^2u(x_0)(x - x_0)$ and $d_\varepsilon(x) = b(x_0) + Db(x_0)(x - x_0)$ on a neighborhood of $\partial Q(x_0, s\lambda\varepsilon)$ and such that

$$\begin{aligned}\mathcal{F}(v_\varepsilon, d_\varepsilon, Q(x_0, s\lambda\varepsilon)) \leq \mathbf{m}\Big(&u(x_0) + Du(x_0)(\cdot - x_0) + \\ &\frac{1}{2}(\cdot - x_0)^T D^2u(x_0)(\cdot - x_0), b(x_0) + Db(x_0)(\cdot - x_0), Q(x_0, s\lambda\varepsilon)\Big) + (\lambda\varepsilon)^{N+1}.\end{aligned}$$

Outside $Q(x_0, s\lambda\varepsilon)$ we can consider the layer L of width $(1-s)\lambda\frac{\varepsilon}{2}$ and a smooth cut off function φ, $0 < \varphi < 1$ in L, $\varphi \equiv 1$ on $Q(x_0, s\lambda\varepsilon)$, $\varphi \equiv 0$ on $\partial Q(x_0, \lambda\varepsilon)$, $\|D\varphi\|_\infty \leq \frac{C}{(1-s)\lambda\varepsilon}$, $\left\|D^2\varphi\right\|_\infty \leq \frac{C}{(1-s)^2\lambda^2\varepsilon^2}$. We define

$$w_\varepsilon(x) := \varphi v_\varepsilon(x) + (1-\varphi)u(x) \text{ and } o_\varepsilon(x) := \varphi d_\varepsilon(x) + (1-\varphi)b(x).$$

This functions agree respectively with u and b on a neighborhood of $\partial Q(x_0, \lambda\varepsilon)$. In view of Lemma 2.32 we have

$$\begin{aligned}
&\mathbf{m}\left(u, b, Q(x_0, \lambda\varepsilon)\right) \leq \mathcal{F}\left(w_\varepsilon, o_\varepsilon, Q(x_0, \lambda\varepsilon)\right)\\
&\leq \mathcal{F}\left(v_\varepsilon, d_\varepsilon, Q(x_0, \lambda s\varepsilon)\right) + C\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big(1 + \left|D^2 w_\varepsilon(x)\right|^p + |Do_\varepsilon(x)|^p\Big)\,dx\\
&\leq \mathbf{m}\left(u(x_0) + Du(x_0)(\cdot - x_0) - \tfrac{1}{2}(\cdot - x_0)^T D^2u(x_0)(\cdot - x_0), b(x_0) + Db(x_0)(\cdot - x_0), Q(x_0, s\lambda\varepsilon)\right)\\
&\quad + (\lambda\varepsilon)^{N+1} + C\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big(1 + \left|D^2u(x_0)\right|^p + \left|D^2u(x)\right|^p + |Db(x_0)|^p + |Db(x)|^p\Big)\,dx\\
&\quad + \frac{C}{(1-s)^p(\lambda\varepsilon)^p}\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big(\left|Du(x_0) - D^2u(x_0)(x - x_0) - Du(x)\right|^p\\
&\quad + |b(x_0) - Db(x_0)(x - x_0) - b(x)|^p\Big)\,dx + \frac{C}{(1-s)^{2p}(\lambda\varepsilon)^{2p}}\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big|u(x) - u(x_0)\\
&\quad - Du(x_0)(x - x_0) - \frac{1}{2}(x - x_0)^T D^2u(x_0)(x - x_0)\Big|^p\,dx.
\end{aligned}$$

By dividing by $(\lambda\varepsilon)^N$ we have

$$\begin{aligned}
&\frac{\mathbf{m}\left(u, b, Q(x_0, \lambda\varepsilon)\right)}{(\lambda\varepsilon)^N}\\
&\leq \frac{\mathbf{m}\left(u(x_0) + Du(x_0)(\cdot - x_0) - \frac{1}{2}(\cdot - x_0)D^2u(x_0)(\cdot - x_0), b(x_0) + Db(x_0)(\cdot - x_0); Q(x_0, s\lambda\varepsilon)\right)}{(\lambda\varepsilon)^N}\\
&\quad + \lambda\varepsilon + C(1 - s^N) + \frac{C}{(\lambda\varepsilon)^N}\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big(\left|D^2u\right|^p + |Db|^p\Big)\,dx\\
&\quad + \frac{C}{(1-s)^p(\lambda\varepsilon)^{p+N}}\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big(\left|Du(x_0) - D^2u(x_0)(x - x_0) - Du(x)\right|^p\\
&\quad + |b(x_0) - Db(x_0)(x - x_0) - b(x)|^p\Big)\,dx + \frac{C}{(1-s)^{2p}(\lambda\varepsilon)^{2p+N}}\int_{Q(x_0,\lambda\varepsilon)\setminus Q(x_0,\lambda s\varepsilon)}\Big|u(x) - u(x_0)\\
&\qquad - Du(x_0)(x - x_0) - \frac{1}{2}(x - x_0)^T D^2u(x_0)(x - x_0)\Big|^p\,dx.
\end{aligned} \tag{2.14}$$

Clearly

$$\lim_{s\to 1^-}\lim_{\varepsilon\to 0^+}\left[\fint_{Q(x_0,\lambda\varepsilon)}\left|D^2u\right|^p - s^N\fint_{Q(x_0,s\lambda\varepsilon)}\left|D^2u\right|^p\right] = \lim_{s\to 1^-}\left[\left|D^2u(x_0)\right|^p - s^N\left|D^2u(x_0)\right|^p\right] = 0$$

and

$$\lim_{s\to 1^-}\lim_{\varepsilon\to 0^+}\left[\fint_{Q(x_0,\lambda\varepsilon)}|Db|^p - s^N\fint_{Q(x_0,s\lambda\varepsilon)}|Db|^p\right] = \lim_{s\to 1^-}\left[|Db(x_0)|^p - s^N|Db(x_0)|^p\right] = 0$$

And the last two lines of (2.14) go to 0 as ε goes to 0 by the fine properties of Sobolev functions (see Theorem 3.4.2 page 129, [63]). Finally, we conclude that

$$\begin{aligned}
&\frac{d\mathcal{F}(u,b,\cdot)}{d\mathcal{L}^N}(x_0) \le \liminf_{\lambda\to 1^-}\liminf_{s\to 1^-}\\
&\lim_{\varepsilon\to 0}\frac{\mathbf{m}\left(u(x_0)+Du(x_0)(\cdot-x_0)-\frac{1}{2}(\cdot-x_0)D^2u(x_0)(\cdot-x_0), b(x_0)+Db(x_0)(\cdot-x_0); Q(x_0,s\lambda\varepsilon)\right)}{(s\lambda\varepsilon)^N}\\
&\le \limsup_{\varepsilon\to 0}\frac{\mathbf{m}\left(u(x_0)+Du(x_0)(\cdot-x_0)-\frac{1}{2}(\cdot-x_0)D^2u(x_0)(\cdot-x_0), b(x_0)+Db(x_0)(\cdot-x_0); Q(x_0,\varepsilon)\right)}{\varepsilon^N}\\
&= f(x_0,u(x_0),Du(x_0),D^2u(x_0),b(x_0),Db(x_0)).
\end{aligned}$$

□

Lemma 2.36. *For $\mathcal{L}^N$ a.e. $x_0\in\Omega$*

$$\frac{d\mathcal{F}(u,b,\cdot)}{d\mathcal{L}^N}(x_0) \ge f(x_0,u(x_0),Du(x_0),D^2u(x_0),b(x_0),Db(x_0)). \tag{2.15}$$

Proof. Let $\varepsilon_h\to 0$ be such that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\frac{\mathbf{m}\left(u(x_0)+Du(x_0)(\cdot-x_0)-\frac{1}{2}(\cdot-x_0)^TD^2u(x_0)(\cdot-x_0), b(x_0)+Db(x_0)(\cdot-x_0); Q(x_0,\varepsilon)\right)}{(\varepsilon)^N}\\
&= \lim_{\varepsilon_h\to 0^+}\frac{\mathbf{m}\left(u(x_0)+Du(x_0)(\cdot-x_0)-\frac{1}{2}(\cdot-x_0)D^2u(x_0)(\cdot-x_0), b(x_0)+Db(x_0)(\cdot-x_0); Q(x_0,\varepsilon_h)\right)}{(\varepsilon_h)^N}.
\end{aligned}$$

Fix $0<s<1$ and let $v_{\varepsilon_h}\in W^{2,p}(Q(x_0,s\varepsilon_h))$ and $d_{\varepsilon_h}\in W^{1,p}(Q(x_0,s_{\varepsilon_h}))$ be such that $v_{\varepsilon_h}=u$ and $d_{\varepsilon_h}=b$ on a neighborhood of $\partial Q(x_0,s\varepsilon_h)$ and

$$\mathcal{F}(v_{\varepsilon_h},d_{\varepsilon_h},Q(x_0,s\varepsilon_h)) \le \mathbf{m}(u,b,Q(x_0,s\varepsilon_h))+(\varepsilon_h)^{N+1}.$$

Extend v_{ε_h} as u and d_{ε_h} as b outside $Q(x_0,s\varepsilon_h)$ and consider a layer L around the cube of width $\frac{(1-s)\varepsilon_h}{2}$ and consider a cut off function φ such that $0<\varphi<1$ in L , $\varphi\equiv 1$ on $Q(x_0,s\varepsilon_h)$, $\varphi\equiv 0$ on $Q(x_0,\varepsilon_h)$, $\|D\varphi\|_\infty \le \frac{C}{(1-s)\varepsilon_h}$ and $\left\|D^2\varphi\right\|_\infty \le \frac{C}{(1-2)^2{\varepsilon_h}^2}$. Define

$$w_{\varepsilon_h}(x) := \varphi(x)v_{\varepsilon_h}(x)+(1-\varphi(x))\left(u(x_0)+Du(x_0)(x-x_0)+\frac{1}{2}(x-x_0)^TD^2u(x_0)(x-x_0)\right)$$

and

$$o_{\varepsilon_h}(x) := \varphi(x)d_{\varepsilon_h}(x)+(1-\varphi(x))\left(b(x_0)+Db(x_0)(x-x_0)\right)$$

Since $w_{\varepsilon_h}=u(x_0)+Du(x_0)(\cdot-x_0)+\frac{1}{2}(\cdot-x_0)^TD^2u(x_0)(\cdot-x_0)$ and $d_{\varepsilon_h}=b(x_0)+Db(x_0)(\cdot-x_0)$ on a neighborhood of $\partial Q(x_0,\varepsilon_h)$, arguing as in the proof of Lemma 2.35 we get

$$
\begin{aligned}
&\frac{\mathbf{m}\left(u(x_0)+Du(x_0)(\cdot-x_0)-\frac{1}{2}(\cdot-x_0)D^2u(x_0)(\cdot-x_0), b(x_0)+Db(x_0)(\cdot-x_0); Q(x_0,\varepsilon_h)\right)}{(\varepsilon_h)^N}\\
&\leq \frac{\mathbf{m}(u,b,Q(x_0,s\varepsilon_h))}{\varepsilon_h}+\varepsilon_h+C(1-s^n)+\frac{C}{\varepsilon_h^N}\int_{Q(x_0,\varepsilon_h)\setminus Q(x_0,s\varepsilon_h)}\left(\left|D^2u\right|^p+|Db|^p\right)dx\\
&\quad+\frac{C}{(1-s)^p{\varepsilon_h}^{p+N}}\int_{Q(x_0,\varepsilon_h)}\left|Du(x)-Du(x_0)-D^2u(x_0)(x-x_0)\right|^p dx\\
&+\frac{C}{(1-s)^p{\varepsilon_h}^{p+N}}\int_{Q(x_0,\varepsilon_h)}|b(x)-b(x_0)-Db(x_0)(x-x_0)|^p dx\\
&\quad+\frac{C}{(1-s)^{2p}{\varepsilon_h}^{N+2p}}\int_{Q(x_0,\varepsilon_h)}\Big|u(x)-u(x_0)-Du(x_0)(x-x_0)+\\
&\qquad\quad-\frac{1}{2}(x-x_0)^TD^2u(x_0)(x-x_0)\Big|^p dx.
\end{aligned}
$$

Therefore

$$
f(x_0,u(x_0),Du(x_0),D^2u(x_0),b(x_0),Db(x_0))\leq \liminf_{s\to1^-}\lim_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u,b,Q(x_0,s\varepsilon_h))}{(\varepsilon_h)^N}
$$

and this concludes our proof. □

Clearly Lemmas 2.35 and 2.36 now yield Theorem 2.31.

3 $\mathcal{A}$-quasiconvexity in the variable coefficients

In this chapter we generalize some of the results of [40] to the case of variable coefficients, precisely

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)}(x)\frac{\partial v}{\partial x_i}, \tag{3.1}$$

where $A^{(i)} \in C^{\infty}(\Omega; \mathbb{M}^{l\times d}) \cap W^{1,\infty}$, and

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^{(i)}(x)\omega_i\right) = \text{const} \tag{3.2}$$

for every $x \in \Omega$ and all $\omega \in R^N \setminus \{0\}$.

Given $x_0 \in \Omega$, denote by $\mathcal{A}_{x_0}$ the partial differential operator with constant coefficients that we obtain by freezing x_0, i.e.,

$$\mathcal{A}_{x_0}v := \sum_{i=1}^{N} A^{(i)}(x_0)\frac{\partial v}{\partial x_i}.$$

The following sufficient condition for lower semicontinuity holds.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $1 < q < +\infty$, and let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ be a Caratheódory function, with a growth condition $0 \le f(x, u, v) \le a(x, u)(1 + |v|^q)$, for some locally bounded function $a : \Omega \times \mathbb{R}^m \to [0, +\infty)$ and for all $v \in \mathbb{R}^d$, a.e. $x \in \Omega$. Suppose $f(x, u, .)$ is $\mathcal{A}_x$-quasiconvex for a.e. x in Ω and all $u \in \mathbb{R}^m$. Then*

$$\liminf_{n\to+\infty} \int_{\Omega} f(x, u_n(x), v_n(x))\, dx \ge \int_{\Omega} f(x, u(x), v(x))\, dx$$

whenever $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^q(\Omega; \mathbb{R}^d)$, $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega; \mathbb{R}^l)$.

For the necessary condition we have the following.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $1 < q < +\infty$, and let $f : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be a continuous function satisfying the q-Lipschitz continuity condition*

$$|f(x, v_1) - f(x, v_2)| \le a(x)\left(1 + |v_1|^{q-1} + |v_2|^{q-1}\right)|v_1 - v_2|, \tag{3.3}$$

where $a \in L^{\infty}_{\mathrm{loc}}(\Omega)$. Suppose that we have lower semicontinuity of the integral

$$\liminf \int_{\Omega} f(x, v_n(x))\, dx \ge \int_{\Omega} f(x, v(x))\, dx$$

for sequences $v_n \rightharpoonup v$ in $L^q(\Omega; \mathbb{R}^m)$, constrained by the system of PDEs in the following sense

$$\mathcal{A}v_n := \sum_{i=1}^{N} A^{(i)}(x)\frac{\partial v_n}{\partial x_i} \to 0 \quad in \quad W^{-1,q}(\Omega; \mathbb{R}^l). \tag{3.4}$$

Then $f(x, .)$ is $\mathcal{A}_x$-quasiconvex for all $x \in \Omega$.

We could not prove the necessary condition for exact solutions of the PDE, but only under the more restrictive condition (3.4). In the case of constant coefficients Fonseca and Müller [40] were able to prove the necessary condition for sequences in the kernel of $\mathcal{A}$. Using Fourier series representation they could construct a projection P onto the kernell of $\mathcal{A}$, using algebraic computations on the symbols, and to prove the estimate (continuity of the inverse)

$$||v - Pv||_{L^q} \le C_q||\mathcal{A}v||_{W^{-1,q}}. \tag{3.5}$$

One difficult that arises when we deal with the variable coefficients is that to the composition of operators does not correspond the multiplication of symbols any more, only up to a regularizing operator, thus in our case, using also Fourier analysis, we were just able to prove the estimate

$$||v - P_\eta v||_{L^q} \leq C_q \left(||\mathcal{A}v||_{W^{-1,q}} + ||v||_{W^{-1,q}}\right), \tag{3.6}$$

where P_η is not a projection, $AP_\eta v \neq 0$, in general, but $AP_\eta v_n \to 0$ in $W^{-1,q}$ whenever $v_n \to 0$ in $W^{-1,q}$. We also emphasize that at least in the case $q = 2$ there exits a continuous projection onto the kernel of $\mathcal{A}$ but we were unable to prove the continuity result (3.5), or at least the weaker estimate (3.6), with the projection P_η replaced by P.

We also characterize the Young measures generated by bounded L^q sequences satisfying (3.4). In the case of constant coefficients similar characterization is provided ([40]), for sequences in the kernel of the operator, in this way generalizing the result of the Kinderleher and Pedregal on gradients [48] [49] (in that case $\mathcal{A} = \text{curl}$). For the reasons we explained above we were note able to replace (3.4) by sequences in the kernel of the operator.

Theorem 3.3. *Let $1 < q < +\infty$ and let $\{\nu_x\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. Then there exists a q-equi-integrable sequence $\{v_n\}$ in $L^q(\Omega; \mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega; \mathbb{R}^l)$ if and only if*

i) *there exists $v \in L^q(\Omega; \mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and $v(x) = \langle \nu_x, Id\rangle$ a.e $x \in \Omega$;*

ii) $\int_\Omega \int_{\mathbb{R}^d} |z|^q \, d\nu_x(z)\, dx < +\infty$;

iii) *for a.e. $x \in \Omega$ and all continuous functions g that satisfy $|g(v)| \leq C(1 + |v|^q)$ one has $\langle \nu_x, g\rangle \geq Q_{\mathcal{A}_x} g\left(\langle \nu_x, Id\rangle\right)$.*

3.1 Sufficient condition

We now prove Theorem 3.1.

Proof. By extracting a subsequence (not relabeled) we may assume

$$L := \liminf \int_\Omega f(x, u_n(x), v_n(x))\, dx = \lim \int_\Omega f(x, u_n(x), v_n(x))\, dx.$$

By extracting another subsequence (still not relabeled) we may assume that the pair $\{(u_n, v_n)\}$ generates a Young measure $\{\mu_x = \delta_{u(x)} \otimes \nu_x\}_{x\in\Omega}$, where $\{\nu_x\}_{x\in\Omega}$ is the Young measure associated to v_n. We have

$$L \geq \int_\Omega \int_{\mathbb{R}^m \times \mathbb{R}^d} f(x, \eta, \xi)\, d\mu_x(\eta, \xi) = \int_\Omega \int_{\mathbb{R}^d} f(x, u(x), \xi)\, d\nu_x(\xi).$$

Now we truncate the sequence v_n to get $q-$equi-integrability. As v_n is a bounded sequence in L^q we have

$$\int_\Omega \langle \nu_x, |z|^q\rangle\, dx < +\infty.$$

Consider the following family of truncation functions

$$\tau_k(z) := \begin{cases} z & \text{if} \quad |z| \leq k \\ k\frac{z}{|z|} & \text{if} \quad |z| > k, \end{cases}$$

and we have

$$\lim_k \lim_n \int_\Omega |\tau_k(v_n)|^q\, dx = \lim_k \int_\Omega \langle \nu_x, |\tau_k(.)|^q\rangle\, dx = \int_\Omega \langle \nu_x, |z|^q\rangle\, dx.$$

We can then find a sequence $\hat{v}_k := \tau_k(v_{n_k})$ such that

$$||\hat{v}_k - v_{n_k}||_{L^s} \to 0, \qquad \lim_{k\to+\infty} \int_\Omega |\hat{v}_k|^q\, dx = \int_\Omega \langle \nu_x, |z|^q\rangle\, dx,$$

for $1 < s < p$. The sequence $\hat{v}_k$ also generates the Young measure ν, it is $q-$equi-integrable and

$$\mathcal{A}\hat{v}_k \to 0 \quad \text{in} \quad W^{-1,s}.$$

Now choose a point $x_0 \in \Omega$ such that $f(x_0, u(x_0), .)$ is $\mathcal{A}_{x_0}$-quasiconvex,

$$\begin{aligned} &\lim_{r\to 0} \frac{1}{r^N} \int_{Q(x_0,r)} |\langle \nu_x, |z|^q \rangle - \langle \nu_{x_0}, |z|^q \rangle| \, dx = 0, \\ &\lim_{r\to 0} \frac{1}{r^N} \int_{Q(x_0,r)} |v(x) - v(x_0)|^q \, dx = 0, \end{aligned} \tag{3.7}$$

and

$$\lim_{r\to 0} \int_Q |\langle \nu_{x_0+rz}, \varphi \rangle - \langle \nu_{x_0}, \varphi \rangle| \, dz = 0 \tag{3.8}$$

for a countable number of φ in $C_0(\mathbb{R}^d)$. Define $w_{k,r} \in L^q(Q;\mathbb{R}^d)$ by $w_{k,r}(z) := \hat{v}_k(x_0 + rz)$. Using (3.7) and (3.8), we have

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q |\hat{v}_k(x_0 + rz)|^q \, dz = \langle \nu_{x_0}, |z|^q \rangle,$$

$$\sum_{i=1}^N A^{(i)}(x_0 + rz) \frac{\partial(\hat{v}_k(x_0 + rz))}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s} \quad \text{as } k \to +\infty$$

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q (\hat{v}_k(x_0 + rz) - v(x_0)) \, \Psi(z) \, dz = 0,$$

for every $\Psi \in L^{q'}$,

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q \zeta(z)\varphi\left(\hat{v}_k(x_0 + rz)\right) \, dz = \langle \nu_{x_0}, \varphi \rangle \int_Q \zeta(z) \, dz,$$

for ζ in $C_c(Q)$ and φ in the countable subset of $C_0(\mathbb{R}^d)$ for which (3.8) holds.

Then, using an appropriate diagonalization, we find a sequence $\omega_k \in L^q(Q;\mathbb{R}^d)$ such that

$$\omega_k \rightharpoonup v(x_0) \quad \text{in} \quad L^q, \qquad \sum_{i=1}^N A^{(i)}(x_0 + r_k z) \frac{\partial \omega_k(z)}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s} \tag{3.9}$$

and

$$\lim_{k\to+\infty} \int_Q \eta(z)\varphi(\omega_k) \, dz = \langle \nu_{x_0}, \varphi \rangle \int_Q \eta(z) \, dz,$$

for η and φ in a countable dense subset of $L^1(Q)$ and $C_0(\mathbb{R}^d)$, respectively, and

$$\lim_{k\to+\infty} \int_Q |\omega_k(z)|^q \, dz = \langle \nu_{x_0}, |z|^q \rangle,$$

thus ω_k generates the Young measure ν_{x_0} and it is q-equi-integrable.

Now we prove that

$$\mathcal{A}_{x_0}\omega_k = \sum_{i=1}^N A^{(i)}(x_0) \frac{\partial \omega_k}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s}. \tag{3.10}$$

In fact we have

$$\begin{aligned} \mathcal{A}_{x_0}\omega_k = &\sum_{i=1}^N \frac{\partial}{\partial z_i} \left[\left(A^{(i)}(x_0) - A^{(i)}(x_0 + r_k z) \right) \omega_k(z) \right] \\ &+ r_k \sum_{i=1}^N \frac{\partial A^{(i)}}{\partial x_i}(x_0 + r_k z)\omega_k(z) + \sum_{i=1}^N A^{(i)}(x_0 + r_k z) \frac{\partial \omega_k}{\partial z_i} \end{aligned}$$

and all the terms go to 0 in $W^{-1,s}$. Indeed the first converges to zero due to the s-equi-integrability of ω_k and the continuity of the coefficients which imply

$$\left(A^{(i)}(x_0) - A^{(i)}(x_0 + r_k z)\right) \omega_k(z) \to 0 \quad \text{in} \quad L^s,$$

the second because $r_k \to 0$ and the boundedness of ω_k in L^s, and the third because of (3.9).

Next we modify ω_k in order to get Q-periodicity. We consider an increasing sequence of smooth cut-off functions $\varphi^j \in C_c^\infty(Q)$, $\varphi^j \nearrow 1$ and we do a appropriate diagonalization of $\varphi^j \omega_k$, in order to get a new sequence $\tilde{\omega}_k \in L^q(Q)$, q-equi-integrable, that still generates the homogeneous Young measure ν_{x_0}, and verifies

$$\mathcal{A}_{x_0} \tilde{\omega}_k \to 0 \quad \text{in} \quad W^{-1,s}.$$

Now we just have to project $\{\tilde{\omega}_k\}$ into the kernel of $\mathcal{A}_{x_0}$, i.e., we apply Lemma 2.5 to get

$$\hat{\omega}_k := \mathbb{T}\left[\tilde{\omega}_k - v(x_0) - \int_Q (\tilde{\omega}_k(x) - v(x_0))\, dx\right] + v(x_0),$$

Q-periodic, q-equi-integrable, $\hat{\omega}_k \rightharpoonup v(x_0)$, $\int_Q \hat{\omega}_k(y)\, dy = v(x_0)$, $\hat{\omega}_k$ still generates ν_{x_0} and $\mathcal{A}_{x_0}\hat{\omega}_k = 0$. Thus

$$\int_{\mathbb{R}^d} f(x_0, u(x_0), \xi) d\nu_{x_0}(\xi) = \lim_k \int_Q f(x_0, u(x_0), \hat{\omega}_k(y))\, dy \geq f(x_0, u(x_0), v(x_0)),$$

from which we get

$$L \geq \int_\Omega f(x, u(x), v(x))\, dx.$$

□

Remark 3.4. Using a similar argument one may obtain the same result of Theorem 3.1 for systems in divergence form with L^∞ coefficients, i.e.

$$\mathcal{A}v := \sum_{i=1}^N \frac{\partial \left(A^{(i)}(x)v\right)}{\partial x_i},$$

where $\text{rank}\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right)$ =const, for a.e. $x \in \Omega$ and all $\omega \in \mathbb{R}^N \setminus \{0\}$. Precisely if $f(x, u, .)$ is $\mathcal{A}_x$-quasiconvex for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$ then we have lower semicontinuity for sequences $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^q, $\mathcal{A}v_n \to 0$ in $W^{-1,q}$. In the proof one uses the approximate continuity of the coefficients at a.e. $x \in \Omega$.

However, in this case we were unable to prove that the sufficient condition is also necessary.

3.2 Necessary condition

In this section we prove Theorem 3.2.

Proof. Fix x_0 in Ω, $c \in \mathbb{R}^d$, and let $r > 0$ be such that $Q(x_0, 2r) \subset\subset \Omega$. Let $\omega \in C^\infty(\mathbb{R}^N; \mathbb{R}^m)$, Q-periodic, satisfying

$$\int_Q \omega(y)\, dy = 0 \qquad \mathcal{A}_{x_0}\omega := \sum_{i=1}^N A^{(i)}(x_0) \frac{\partial \omega}{\partial y_i} = 0. \tag{3.11}$$

Using the uniform continuity of f on compact sets we can choose n large enough such that

$$|f(x, v) - f(x', v)| < \varepsilon \quad \text{for } x, x' \in \overline{Q(x_0, r)}, \quad v \in \overline{Q(0, c + ||\omega||_\infty)}, \quad |x - x'| < \frac{1}{n}.$$

Decompose

$$Q(x_0, r) = \cup_{j=1}^{n^N} Q\left(x_j, \frac{r}{n}\right),$$

where the equality above is up to a $\mathcal{L}^N$-negligible set. Given $\varepsilon > 0$ consider $\varphi \in C_c^\infty(Q(x_0,r),[0,1])$ such that $\mathcal{L}^N(Q(x_0,r)\cap\{\varphi \neq 1\}) < \varepsilon r^N$. Define

$$u_m(x) := \begin{cases} \varphi(x)\omega^\star\left(\frac{mn(x-x_j)}{r}\right) & \text{if } x \in Q(x_j,\frac{r}{n}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega^\star(y) := \omega(y+(\frac{1}{2},..,\frac{1}{2}))$. We have

$$\begin{aligned}
\mathcal{A}u_m &= \mathcal{A}u_m - \mathcal{A}_{x_0}u_m + \mathcal{A}_{x_0}u_m \\
&= \sum_{i=1}^{N}\sum_{j=1}^{n^N} \frac{\partial}{\partial x_i}\left(\left(A^{(i)}(x) - A^{(i)}(x_0)\right)\varphi(x)\omega^\star\left(mn\frac{x-x_j}{r}\right)\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad - \sum_{i=1}^{N}\sum_{j=1}^{n^N}\varphi(x)\frac{\partial\left(A^{(i)}(x) - A^{(i)}(x_0)\right)}{\partial x_i}\omega^\star\left(mn\frac{x-x_j}{r}\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{n^N} A^{(i)}(x_0)\frac{\partial\varphi}{\partial x_i}\omega^\star\left(mn\frac{x-x_j}{r}\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad + \varphi(x)\sum_{i=1}^{N}\sum_{j=1}^{n^N} A^{(i)}(x_0)\frac{\partial\left(\omega^\star\left(mn\frac{x-x_j}{r}\right)\right)}{\partial x_i}\chi_{Q(x_j,\frac{r}{n})} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.12}$$

As

$$\omega^\star\left(mn\frac{x-x_j}{r}\right) \rightharpoonup 0 \quad \text{in} \quad L^q(Q(x_j,\frac{r}{n})) \qquad \text{as } m \to +\infty,$$

we have $I_2, I_3 \to 0$ in $W^{-1,q}$ as $m \to +\infty$, and by 3.11 $I_4 = 0$. Moreover

$$\begin{aligned}
||I_1||_{W^{-1,q}} &\leq \sum_{i=1}^{N}\sum_{j=1}^{n^N}\left|\left|\left(A^{(i)}(x) - A^{(i)}(x_0)\right)\omega^\star\left(mn\frac{x-x_j}{r}\right)\varphi(x)\right|\right|_{L^q(Q(x_j,\frac{r}{n}))} \\
&\leq C\sum_{i=1}^{N}\left(\int_{Q(x_0,r)}|A^{(i)}(x) - A^{(i)}(x_0)|^q\,dx\right)^{\frac{1}{q}},
\end{aligned} \tag{3.13}$$

where C is independent of m.

Now consider $\eta \in C_c^\infty(\Omega;[0,1])$, $\eta = 1$ on $Q(x_0,r)$ and define

$$v_m := P_\eta u_m.$$

As P_η is an operator of order 0, by Theorem 2.17, we have

$$||v_m||_{L^q} \leq C||u_m||_{L^q}, \tag{3.14}$$

$$||v_m||_{W^{-1,q}} \leq C||u_m||_{W^{-1,q}},$$

thus, up to a subsequence,

$$v_m \rightharpoonup 0 \quad \text{in} \quad L^q.$$

Moreover, by (2.7),

$$\mathcal{A}v_m \to 0 \quad \text{in} \quad W^{-1,q}.$$

As the pseudodifferential operators are non-local, we need to localize the sequence $\{v_m\}$. For that consider $\eta_r \in C_c^\infty(Q(x_0,2r);[0,1])$, $\eta_r = 1$ in $Q(x_0,r)$, and define

$$\tilde{v}_m := \eta_r v_m.$$

We have

$$\tilde{v}_m \rightharpoonup 0 \quad \text{in} \quad L^q, \qquad \mathcal{A}\tilde{v}_m \to 0 \quad \text{in} W^{-1,q},$$

thus, by the lower semicontinuity we have

$$\liminf \int_\Omega f(x, c + \tilde{v}_m(x))\, dx \geq \int_\Omega f(x,c)\, dx. \tag{3.15}$$

On the other hand, using (3.3), (2.6), (3.13), (3.14), and Hölder's inequality, we get

$$\begin{aligned}
&\left| \int_\Omega f(x, c+\tilde{v}_m(x))\, dx - \int_\Omega f(x, c + u_m(x)) \right| \\
&\leq C \int_\Omega |\tilde{v}_m(x) - u_m(x)| \left(1 + |c + \tilde{v}_m|^{q-1} + |c + u_m|^{q-1}\right) dx \\
&\leq C \int_{Q(x_0,2r)} |\tilde{v}_m(x) - u_m(x)| \left(1 + |\tilde{v}_m|^{q-1} + |u_m|^{q-1}\right) dx \\
&\leq C \left(\int_{Q(x_0,2r)} |\tilde{v}_m(x) - u_m(x)|^q\, dx \right)^{\frac{1}{q}} \left(r^{\frac{N}{q'}} + \left(\int_{Q(x_0,2r)} |\tilde{v}_m|^q\, dx \right)^{\frac{1}{q'}} \right. \\
&\quad \left. + \left(\int_{Q(x_0,2r)} |u_m|^q\, dx \right)^{\frac{1}{q'}} \right) \qquad (3.16) \\
&\leq C \left(\int_\Omega |v_m(x) - u_m(x)|^q\, dx \right)^{\frac{1}{q}} \left(r^{\frac{N}{q'}} + \left(\int_{Q(x_0,r)} |u_m|^q\, dx \right)^{\frac{1}{q'}} \right) \\
&\leq C \left(||\mathcal{A}u_m||_{W^{-1,q}} + ||u_m||_{W^{-1,q}} \right) \left(r^{\frac{N}{q'}} + r^{\frac{N}{q'}} \left(\int_Q |\omega(mz)|^q \right)^{\frac{1}{q'}} dz \right) \\
&\leq C r^{\frac{N}{q'}} \left(\sum_{i=1}^N \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q\, dx \right)^{\frac{1}{q}} + C r^{\frac{N}{q'}} ||u_m||_{W^{-1,q}},
\end{aligned}$$

where C is independent of m. Thus using (3.15) and (3.16) we have

$$\begin{aligned}
\limsup_m \int_\Omega f(x, c + u_m(x))\, dx + C r^{\frac{N}{q'}} \left(\sum_{i=1}^N \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q\, dx \right)^{\frac{1}{q}} \\
\geq \int_\Omega f(x,c)\, dx.
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\limsup_m \int_{Q(x_0,r)} f(x, c + u_m(x))\, dx + C r^{\frac{N}{q'}} \left(\sum_{i=1}^N \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q\, dx \right)^{\frac{1}{q}} \\
\geq \int_{Q(x_0,r)} f(x,c)\, dx.
\end{aligned}$$

We now estimate the first term above, using the continuity of f and Riemann-Lebesgue Lemma,

$$\begin{aligned}
&\limsup_m \int_{Q(x_0,r)} f(x, c + u_m(x))\, dx \\
&\quad \le \limsup_m \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} f\left(x, c + \omega^\star(mn\frac{x-x_j}{r})\right) dx + 2M\varepsilon r^n \\
&\quad \le \limsup_m \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} f\left(x_j, c + \omega^\star(mn\frac{x-x_j}{r})\right) dx + (2M+1)\varepsilon r^N \\
&\quad \le \limsup_m \sum_{j=1}^{n^N} \frac{r^N}{n^N} \int_Q f(x_j, c + \omega(my))\, dy + (2M+1)\varepsilon r^N \qquad (3.17)\\
&\quad \le \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} \left(\int_Q f(x_j, c+\omega(y))\, dy \right) dx + (2M+1)\varepsilon r^N \\
&\quad \le \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} \left(\int_Q f(x, c+\omega(y))\, dy \right) dx + (2M+2)\varepsilon r^N \\
&\quad \le \int_{Q(x_0,r)} \left(\int_Q f(x, c+\omega(y))\, dy \right) dx + O(\varepsilon) r^N,
\end{aligned}$$

where $M := \sup\{f(x,v) : x \in \overline{Q(x_0,r)}, |v| \le c + ||\omega||_\infty\}$. Thus dividing by r^N and using (3.17) we get

$$\begin{aligned}
&\frac{1}{r^N} \int_{Q(x_0,r)} \left(\int_Q f(x, c+\omega(y))\, dy \right) dx + O(\varepsilon) \\
&\quad + C\left(\sum_{i=1}^N \frac{1}{r^N} \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q\, dx \right)^{\frac{1}{q}} \ge \frac{1}{r^N} \int_{Q(x_0,r)} f(x,c)\, dx.
\end{aligned}$$

By letting $r \to 0$ and using the arbitrariness of ε we obtain

$$f(x_0, c) \le \int_Q f(x_0, c + \omega(y))\, dy,$$

i.e., $f(x_0, .)$ is $\mathcal{A}_{x_0}$-quasiconvex. □

3.3 Characterization of Young measures

We now prove Theorem 3.3. The idea is to split the domain into small cubes, approach the variable coefficients operator by one with constant coefficients in each cube, apply in each cube the theorem about characterization of Young measures generated by bounded sequences in L^q that are in the kernel of an operator with constant coefficients (Theorem 2.9), and then use an appropriate diagonalization.

Proof. We assume without loss of generality that

$$\langle \nu_x, Id \rangle = 0.$$

.

Consider $\{\xi_h\}_{h=1}^{+\infty}$ a dense countable subset of $L^1(\Omega)$, $\xi_0(x) = 1$, $\{\varphi_l\}_{l=1}^{+\infty}$ a dense countable subset of $C_0(\mathbb{R}^d)$ and $\varphi_0(z) = |z|^q$. Given $\alpha \in \mathbb{N}$ we can find $\gamma > 0$ such that

$$\int_B |\xi_h(x)|\, dx ||\varphi_l||_\infty < \frac{1}{\alpha} \quad \text{for } h, l = 1, ..., \alpha, \qquad (3.18)$$

and
$$\int_B \langle \nu_x, |z|^q \rangle \, dx < \frac{1}{\alpha} \tag{3.19}$$
when $\mathcal{L}^N(B) < \gamma$. For each $\alpha \in \mathbb{N}$ we consider a compact set K_α such that $\mathcal{L}^N(\Omega \setminus K_\alpha) < \min\{\frac{1}{\alpha^2}, \gamma/3\}$ and the functions
$$x \to \langle \nu_x, |z|^q \rangle, \qquad x \to \langle \nu_x, \varphi_l \rangle \quad l = 1, ..., \alpha,$$
are continuous in K_α. We consider disjoint cubes $Q_i \subset\subset \Omega$ of side $\frac{1}{m_\alpha}$, for an appropriate integer m_α, such that $\mathcal{L}^N(\Omega \setminus \cup Q_i) < \min\{\frac{1}{\alpha^2}, \gamma/3\}$,
$$\sup_{x,x' \in Q_i \cap K_\alpha} |A^{(j)}(x) - A^{(j)}(x')|^q < \frac{1/\alpha}{NC_1} \quad j = 1, .., N, \tag{3.20}$$
$$\sup_{x,x' \in Q_i \cap K_\alpha} |\langle \nu_x, \varphi_l \rangle - \langle \nu_{x'}, \varphi_l \rangle| < \frac{1/\alpha}{||\xi_h||} \quad h, l = 1, ..., \alpha, \tag{3.21}$$
and
$$\sup_{x,x' \in Q_i \cap K_\alpha} |\langle \nu_x, |z|^q \rangle - \langle \nu_{x'}, |z|^q \rangle| < \frac{1/\alpha}{|\Omega|}, \tag{3.22}$$
where $C_1 := 2 \int_\Omega \langle \nu_y, |z|^p \rangle \, dy + 1$. By considering less cubes and a smaller compact set $\hat{K}_\alpha$, if necessary, we may assume that for each cube Q_i we have
$$\mathcal{L}^N(Q_i \cap \hat{K}_\alpha) \geq \frac{\mathcal{L}^N(Q_i)}{2}, \tag{3.23}$$
and $\hat{K}_\alpha \subset \overline{\cup Q_i}$. It is easy to check that
$$\mathcal{L}^N(\Omega \setminus \hat{K}_\alpha) < \min\{3/\alpha^2, \gamma\}, \qquad \mathcal{L}^N(\Omega \setminus \cup Q_i) < \min\{3/\alpha^2, \gamma\}.$$

In each cube Q_i we pick up a point $x_i \in \hat{K}_\alpha \cap Q_i$ that fulfills the conditions below
$$\langle \nu_{x_i}, |z|^q \rangle \leq \frac{1}{\mathcal{L}^N(\hat{K}_\alpha \cap Q_i)} \int_{\hat{K}_\alpha \cap Q_i} \langle \nu_y, |z|^q \rangle \, dy, \qquad \langle \nu_{x_i}, \mathrm{Id} \rangle = 0,$$
$$\langle \nu_{x_i}, g \rangle \geq Q_{\mathcal{A}_{x_i}} g(0), \tag{3.24}$$
for every continuous g satisfying $|g(v)| \leq C(1+|v|^q)$. Now we apply Theorem 2.9 and get a $q-$equi-integrable sequence $\hat{v}^i_{\alpha,n} \in L^p(Q_i; \mathbb{R}^d)$ that generates the homogeneous Young measure ν_{x_i} and satisfies $\mathcal{A}_{x_i} v^i_{\alpha,n} = 0$. Using an appropriate sequence of cut-off functions, $\eta^s \in C^\infty_c(Q_i)$, $\eta^s \nearrow 1$, and diagonalizing $\eta^s \hat{v}^i_{\alpha,n}$, one can construct a new sequence, $v^i_{\alpha,n}$, such that $v^i_{\alpha,n} = 0$ on a neighborhood of ∂Q_i, q-equi-integrable, also generating ν_{x_i} and
$$\mathcal{A}_{x_i} v^i_{\alpha,n} \to 0 \quad \text{in} \quad W^{-1,q}(Q_i; \mathbb{R}^l) \quad \text{as} \quad n \to +\infty.$$
Define
$$v_{\alpha,n} := \begin{cases} v^i_{\alpha,n} & \text{if } x \in Q_i, \\ 0 & \text{otherwise.} \end{cases}$$
We have
$$\mathcal{A}_\alpha v_{\alpha,n} := \sum_i \left(\sum_{j=1}^N \frac{\partial \left(A^{(j)}(x_i) v^i_{\alpha,n} \right)}{\partial x_j} \right) \to 0 \quad \text{in} \quad W^{-1,q}$$

We claim that
$$\int_\Omega |v_{\alpha,n}|^q \, dx \leq C_1 \tag{3.25}$$
for all $\alpha \in \mathbb{N}$ and n large enough. As $\{v^i_{\alpha,n}\}_n$ generates ν_{x_i} and it is $q-$equi-integrable, we know that
$$\int_{Q_i} |v^i_{\alpha,n}|^q \, dx \to \langle \nu_{x_i}, |z|^q \rangle \mathcal{L}^N(Q_i).$$

By (3.23) and (3.24),

$$\begin{aligned}\langle \nu_{x_i}, |z|^q\rangle \mathcal{L}^N(Q_i) &\le \frac{\mathcal{L}^N(Q_i)}{\mathcal{L}^N(Q_i\cap \hat{K}_\alpha)}\int_{Q_i\cap\hat{K}_\alpha}\langle \nu_y, |z|^q\rangle\,dy\\ &\le 2\int_{Q_i}\langle \nu_y, |z|^q\rangle\,dy,\end{aligned}$$

and for n large enough

$$\int_\Omega |v_{\alpha,n}|^q\,dx \le 2\int_\Omega \langle \nu_y, |z|^q\rangle\,dy + 1 = C_1.$$

We claim that

$$\sum_i \int_{Q_i\setminus \hat{K}_\alpha} |v^i_{\alpha,n}|^q\,dx \le F(\alpha), \tag{3.26}$$

for some F satisfying the condition $F(\alpha)\to 0$ as $\alpha\to 0$ and n large enough. Using the $q-$equi-integrability of $\left(v^i_{\alpha,n}\right)_n$ and (3.24) we have

$$\begin{aligned}\int_{Q_i\setminus\hat{K}_\alpha} |v^i_{\alpha,n}|^q\,dx \to \quad & \langle \nu_{x_i}, |z|^q\rangle \mathcal{L}^N(Q_i\setminus \hat{K}_\alpha)\\ &\le \frac{\mathcal{L}^N(Q_i\setminus \hat{K}_\alpha)}{\mathcal{L}^N(Q_i\cap \hat{K}_\alpha)}\int_{Q_i\cap\hat{K}_\alpha}\langle \nu_y, |z|^q\rangle\,dy\end{aligned}$$

Setting $J_\alpha := \{i : \alpha\mathcal{L}^N(Q_i\setminus\hat{K}_\alpha) > \mathcal{L}^N(Q_i\cap\hat{K}_\alpha)\}$, we then have

$$\sum_{i\in J_\alpha}\mathcal{L}^N(Q_i\cap\hat{K}_\alpha) \le \sum_{i\in J}\alpha\mathcal{L}^N(Q_i\setminus\hat{K}_\alpha) \le \alpha\mathcal{L}^N(\Omega\setminus\hat{K}_\alpha) < \frac{1}{\alpha}.$$

Thus

$$\begin{aligned}\sum_i \int_{Q_i\setminus\hat{K}_\alpha} |v^i_{\alpha,n}|^q\,dx &\le \sum_i \frac{\mathcal{L}^N(Q_i\setminus \hat{K}_\alpha)}{\mathcal{L}^N(Q_i\cap \hat{K}_\alpha)}\int_{Q_i\cap\hat{K}_\alpha}\langle \nu_y, |z|^q\rangle\,dy + \frac{1}{\alpha}\\ &\le \sum_{i\in J_\alpha}\int_{Q_i\cap\hat{K}_\alpha}\langle \nu_y, |z|^q\rangle\,dy + \frac{1}{\alpha}\int_\Omega\langle \nu_y, |z|^q\rangle\,dy + \frac{1}{\alpha},\end{aligned}$$

for n large enough, from which we get (3.25).

As

$$\begin{aligned}\mathcal{A}_\alpha v_{\alpha,n} - \mathcal{A}v_{\alpha,n} &= \sum_i\left(\sum_{j=1}^N \left(A^{(j)}(x_i) - A^{(j)}(x)\right)\frac{\partial v^i_{\alpha,n}}{\partial x_j}\right)\\ &= \sum_i\left(\sum_{j=1}^N \frac{\partial\left(\left(A^{(j)}(x_i) - A^{(j)}(x)\right)v^i_{\alpha,n}\right)}{\partial x_j} + \sum_{j=1}^N \frac{\partial A^{(j)}(x)}{\partial x_j}v^i_{\alpha,n}\right)\end{aligned}$$

and

$$\sum_i\sum_j\int_{Q_i\cap\hat{K}_\alpha}\left|A^j(x) - A^j(x_i)\right|^q\left|v^i_{\alpha,n}\right|^q\,dx \le \frac{\frac{1}{\alpha}}{C_1}\int_\Omega |v_{\alpha,n}|^q\,dx < \frac{1}{\alpha},$$

$$\sum_i\sum_j\int_{Q_i\setminus\hat{K}_\alpha}\left|A^j(x) - A^j(x_i)\right|^q\left|v^i_{\alpha,n}\right|^q\,dx \le 2^q||A||^q_\infty NF(\alpha),$$

we conclude that for n large enough

$$||\mathcal{A}_\alpha v_{\alpha,n} - \mathcal{A}v_{\alpha,n}||_{-1,q} \le 2||A||^q_\infty NF(\alpha) + \frac{2}{\alpha}. \tag{3.27}$$

We now prove that for n large enough

$$\left|\int_\Omega \xi_h(x)\varphi_l(v_{\alpha,n})\,dx - \int_\Omega \xi_h(x)\langle\nu_x,\varphi_l\rangle\,dx\right| \le \frac{6}{\alpha} \quad \text{for } h,l = 1,..,\alpha. \tag{3.28}$$

Indeed, as $n \to +\infty$,

$$\int_\Omega \xi_h(x)\varphi_l(v_{\alpha,n})\,dx \to \sum_i \langle \nu_{x_i}, \varphi_l\rangle \int_{Q_i} \xi_h(x)\,dx + \varphi_l(0)\int_{\Omega\setminus\cup Q_i} \xi_h(x)\,dx,$$

and

$$\begin{aligned}
&\left|\int_\Omega \xi_h(x)\langle \nu_x, \varphi_l\rangle\,dx - \sum_i \langle \nu_{x_i}, \varphi_l\rangle \int_{Q_i} \xi_h(x)\,dx - \varphi_l(0)\int_{\Omega\setminus\cup Q_i} \xi_h(x)\,dx\right| \\
&\quad \le \int_{\Omega\setminus\cup Q_i} |\xi_h(x)\langle \nu_x, \varphi_l\rangle|\,dx + \sum_i \int_{Q_i\cap \hat{K}_\alpha} |\xi_h(x)\left(\langle \nu_x, \varphi_l\rangle - \langle \nu_{x_i}, \varphi_l\rangle\right)|\,dx \\
&\quad + \sum_i \int_{Q_i\setminus \hat{K}_\alpha} |\xi_h(x)\langle \nu_{x_i}, \varphi_l\rangle|\,dx + \sum_i \int_{Q_i\setminus \hat{K}_\alpha} |\xi_h(x)\langle \nu_x, \varphi_l\rangle|\,dx \\
&\quad + |\varphi_l(0)|\int_{\Omega\setminus\cup Q_i} |\xi_h(x)|\,dx,
\end{aligned}$$

using (3.18) and (3.21) we get (3.28). A similar argument can be carried out in order to obtain

$$\left|\int_\Omega |v_{\alpha,n}(x)|^q\,dx - \int_\Omega \langle \nu_x, |z|^q\rangle\,dx\right| \le F(\alpha) + \frac{4}{\alpha},$$

for n large enough. Then, using appropriate diagonalization, we may find a sequence $w_\alpha := v_{\alpha,n_\alpha} \in L^q$, $w_\alpha \rightharpoonup 0$ in L^q, $\mathcal{A}w_\alpha \to 0$ in $W^{-1,q}$, verifying

$$\lim_\alpha \int_\Omega \xi_h(x)\varphi_l(w_\alpha(x))\,dx = \int_\Omega \xi_h(x)\langle \nu_x, \varphi_l\rangle\,dx,$$

for all $h, l \in \mathbb{N}$, and

$$\lim_\alpha \int_\Omega |w_\alpha(x)|^q\,dx = \int_\Omega \langle \nu_x, |z|^q\rangle\,dx,$$

thus $\{w_k\}$ generates the Young measure ν and it is q-equi-integrable.

For the necessary condition we may use an argument similar to the one of proof of Theorem 3.1, in order to obtain that at a.e. $x_0 \in \Omega$, the homogeneous Young measure ν_{x_0} is generated by a sequence $\omega_k \in L^q(Q)$, Q-periodic, q-equi-integrable, $\omega_k \rightharpoonup v(x_0)$ in L^q, $\int_Q \omega_k(x)\,dx = v(x_0)$, $\mathcal{A}_{x_0}\omega_k = 0$. Then

$$\lim_k \int_Q g(\omega_k(x))\,dx = \langle \nu_x, g\rangle \ge Q_{\mathcal{A}_{x_0}}(v(x_0)),$$

which proves iii); i) and ii) are trivial. $\square$

Remark 3.5. Using a similar proof one may obtain the same result of Theorem 3.3 for systems written in divergence form with L^∞ coefficients

$$\mathcal{A}v := \sum_{i=1}^N \frac{\partial\left(A^{(i)}(x)v\right)}{\partial x_i}$$

and rank $\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right) = \text{const}$, for a.e. $x \in \Omega$ and all $\omega \in \mathbb{R}^N$. In the proof we use Lusin's Theorem to obtain continuity of the coefficients except on a set of small measure.

4 Relaxation result in SBV for micromagnetics

In this chapter we consider the functional

$$E(m) = \int_\Omega f(x, m(x), \nabla m(x))\, dx + \int_{\Omega\cap S(m)} |[m](x)| d\mathcal{H}^{N-1} + E_1(m), \tag{4.1}$$

where $m \in SBV(\Omega;\mathbb{R}^N)$ is subject to the pointwise constraint $|m(x)| = 1$ a.e. in Ω, and E_1 is a continuous functional with respect to the strong topology of L^2.

The motivation to address this type of energies is drawn from *micromagnetics*, a continuum model to describe the behavior of a ferromagnetic body. According to this theory, the equilibrium states of a body subjet to a given external field h_e correspond to absolute (or local) minimizers of the energy functional

$$m \to \alpha \int_\Omega |\nabla m(x)|^2 + \int_\Omega \varphi(m(x))\, dx - \int_\Omega h_e(x).m(x)\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_m(x)|^2\, dx, \tag{4.2}$$

where $\Omega \subset \mathbb{R}^3$ represents a region occupied by the body and the *magnetization* is a function $m : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$|m(x)| = m_s \chi_\Omega(x) \quad \text{for} \quad \mathcal{L}^3 \quad \text{a.e.} \quad x \in \mathbb{R}^3,$$

and $m_s > 0$, the *saturation magnetization*, is a function of the temperature and of the specific material. The *induced magnetic field* is a function $h : \mathbb{R}^3 \to \mathbb{R}^3$ which is related to m through (distributional) Maxwell's equations, i.e.

$$\begin{cases} \operatorname{curl} h_m = 0 \quad \text{in} \quad \mathbb{R}^3, \\ \operatorname{div}(m + h_m) = 0 \quad \text{in} \quad \mathbb{R}^3. \end{cases}$$

We note that the last two integrals in (4.2) are continuous with respect to the strong topology in L^2 (see [35],[45]), and their sum reduces to the term E_1. By considering a surface term in (4.1) we allow the possibilities of m to have discontinuities (magnetic cracks), the body be made of several magnetic materials, or both (see [2] for some arguments concerning the penalization of formation of interfaces).

When the functional (4.1) is not lower semicontinuous, it is usual to look for its relaxation, i.e., given a magnetization m we want to attain it by spending the least possible energy, and this corresponds to characterizing the functional below

$$F(m) := \inf\left\{ \liminf_{k\to+\infty} E[m_k] : m_k \in SBV(\Omega;\mathbb{R}^N), \quad |m_k(x)| = 1 \quad \text{a.e. in } \Omega, \quad m_k \to m \quad \text{in} \quad L^1(\Omega;\mathbb{R}^N) \right\}.$$

We consider here a surface term that will induce interaction, i.e., for sequences $\{m_k\}$ with bounded energy we may not have $\nabla m_k \rightharpoonup \nabla m$ and $D^s m_k \rightharpoonup D^s m$, instead it may happen that part of ∇m is approached in a more economical way using jumps. Adopting a point of view similar to Choksi and Fonseca in [27] we consider a relaxed energy which also take in account the limits of the gradients

$$\begin{aligned} \mathcal{E}(m, M) := \inf\Big\{ \liminf_{k\to+\infty} E[m_k] : m_k \in SBV(\Omega;\mathbb{R}^N), \quad |m_k(x)| = 1 \quad \text{a.e. in } \Omega, \\ m_k \to m \quad \text{in} \quad L^1(\Omega;\mathbb{R}^N), \quad \nabla m_k \rightharpoonup M \quad \text{in} \quad L^2(\Omega;\mathbb{R}^N)\Big\}. \end{aligned}$$

Similar relaxed energies were considered in [27], where they studied relaxed energies associated with structured deformations of continua, a concept introduced by Del Piero and Owen in [36] for taking into account situations where the deformation of a body can be attained via a diffusion of cracks (microscopic disarrangements). In [36] they treat a triples (K, g, G), where K is the macroscopic crack, g is the macroscopic deformation, G is the deformation without disarrangements, and the regions where there is presence of microscopic disarrangements are identified with $\{\nabla g \neq G\}$. In [27] these triplets are reduced to pairs (g, G), and the set K is incorporated in g by identifying crack sites with jump sets of SBV functions.

Here we make the parallel of what was done in [27] to the case of micromagnetics. The minimum energy can be attained by sequences such that there is a diffusion of discontinuities in some region (microscopic disarrangements), which in the limit can be identified by considering pairs (m, M) and through the inequality

$\nabla m \neq M$. Indeed, if $\nabla m \neq M$ in some open set A and if we have a sequence of magnetizations $\{m_k\} \subset L^1(\Omega; S^{N-1})$, with $m_k \to m$ in L^1, $\nabla m_k \rightharpoonup M$ in L^2, we know by the compactness Theorem (2.27) of Ambrosio [9] [14] that necessarily $\mathcal{H}^{N-1}(A \cap S(m_k)) \to +\infty$, i.e., there is a diffusion of discontinuities through A. A pair (m, M) gives a more complete description of the minimizers, because we not only know what is the magnetization, but also we get information about the microscopic disarrangements.

The function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to [0, +\infty)$ is assumed to be Carathеódory and satisfies the growth condition

$$\text{(H1)} \qquad \frac{1}{C}|v|^2 - C \leq f(x, y, v) \leq C\left(1 + |v|^2\right)$$

for some $C > 0$. In addition, the following hold:
for every $(x_0, u_0) \in \Omega \times \mathbb{R}^N$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{(H2)} \qquad f(x_0, u_0, v) - f(x, u, v) \leq \varepsilon(1 + f(x, u, v))$$

for all $(x, u) \in \Omega \times \mathbb{R}^N$ with $|x - x_0| + |u - u_0| < \delta$ and all v, $v' \in \mathbb{R}^{N \times N}$,

for every compact $K \subset \Omega \times \mathbb{R}^N$ exits $L_K > 0$ such that

$$\text{(H3)} \qquad |f(x, u, v) - f(x, u, v')| \leq L_K(1 + |v| + |v'|)|v - v'|$$

for all $(x, u) \in K$ and v, $v' \in \mathbb{R}^{N \times N}$.

Since $m_k \to m$ in $L^2(\Omega; \mathbb{R}^N)$ (what follows immediately from the L^1 convergence and the bounds) implies $E_1(m_k) \to E(m)$, we have that

$$\mathcal{E}(m, M) = \mathcal{F}(m, M) + E_1(m),$$

where $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}(m, M) := \inf \Bigg\{ & \liminf_{k \to +\infty} \int_\Omega f(x, m_k(x), \nabla m_k(x))\, dx + \int_{\Omega \cap S(m_k)} |[m_k](x)| d\mathcal{H}^{N-1} : \\ & m_k \in SBV(\Omega; \mathbb{R}^N), \quad |m_k(x)| = 1 \quad \text{a.e. in } \Omega \\ & m_k \to m \quad \text{in} \quad L^1(\Omega; \mathbb{R}^N), \quad \nabla m_k \rightharpoonup M \quad \text{in} \quad L^2(\Omega; \mathbb{R}^N) \Big\}\end{aligned}$$

Thus, since strong convergence in m entails the convergence of the nonlocal term $\int_\Omega |h_m|^2\, dx$, the relaxation will not involve directly Maxwell's equations. Relaxation results in the context of micromagnetics were also studied by Fonseca and Leoni in [39], where they consider the non-exchange model (without the first term in (4.2), i.e. the exchange energy, see [35]) and they find the relaxation of the functional

$$G(m) := \int_{\mathbb{R}^N} g(x, \chi_\Omega(x) m(x), u(x), \nabla u(x))\, dx,$$

with respect to L^∞-weak* convergence for m, where $(\chi_\Omega m, \nabla u)$ satisfies the Maxwell's equations, i.e. $u \in H^1(\mathbb{R}^N)$ is the unique solution of $\triangle u + \operatorname{div}(\chi_\Omega m) = 0$ in $\mathbb{R}^N$. In [39] they obtained a representation formula involving a quasiconvexification of g which takes into account the underlying partial differential equation.

The main result of this chapter is

Theorem 4.1. *The functional $\mathcal{F}$ has an integral representation of the form*

$$\mathcal{F}(m, M) = \int_\Omega H(x, m(x), \nabla m(x), M(x))\, dx + \int_{\Omega \cap S(m)} |[m](x)| \mathcal{H}^{N-1}.$$

where for $x_0 \in \Omega$, $a \neq 0$, A, $B \in \mathbb{R}^{N \times N}$,

$$\begin{aligned}H(x_0, a, A, B) := \inf \Bigg\{ & \int_Q \bar{f}(x_0, a, \nabla u(x))\, dx + \int_{Q \cap S(u)} |[u](x)|\, d\mathcal{H}^{N-1}(x) : \\ & u \in SBV(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N), \quad u|_{\partial Q} = Ax, \\ & \nabla u \in L^2, \quad \int_Q \nabla u = B \Bigg\},\end{aligned}$$

$\bar{f}(x_0, a, v) := f(x_0, a, P_a(v))$ and P_a is the orthogonal projection of $\mathbb{R}^N$ onto $T_a(S^{N-1})$ (the tangent space to $|a|S^{N-1}$ at the point a).

The problem of relaxing a functional under a manifold constraint was already treated by Dacorogna, Fonseca, Malý and Trivisa in [33], where they obtained the representation result

$$\mathcal{F}(u) := \inf \Big\{ \liminf_{n\to+\infty} \int_\Omega f(\nabla u_n(x))\, dx : u_n \rightharpoonup u \quad \text{in} \quad W^{1,p},$$
$$u(x) \in \mathcal{M} \quad \text{for a.e} \quad x \in \Omega \Big\} = \int_\Omega Q_T f(u(x), \nabla u(x))\, dx,$$

with Q_T the *tangential quasiconvexification* defined by

$$Q_T(a, v) := \inf \left\{ \int_Q f(v + \nabla\varphi(x))\, dx : \varphi \in W_0^{1,\infty}(Q; T_a(\mathcal{M}) \right\},$$

for $a \in \mathcal{M}$, $v \in T_a(\mathcal{M})^d$, and $\mathcal{M}$ is a C^1 manifold. As shown in [33], an alternative formula for Q_T is

$$Q_T(a, v) = Q\bar{f}(a, v),$$

where $\bar{f}(a, v) = f(a, P_a(v))$, P_a denotes the projection into $T_a(\mathcal{M})$ and Q refers to the usual quasiconvexification ([31],[55]), i.e.

$$Q\bar{f}(a, v) := \inf \left\{ \int_Q \bar{f}(a, v + \nabla\varphi(x))\, dx : \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^d) \right\}.$$

Relaxation of functionals under constraints are also treated in [23].

4.1 Integral representation result

As it is usual in relaxation theory, we start by localizing $\mathcal{F}$, precisely, for every open subset $A \subset \mathbb{R}^N$ we define

$$\mathcal{F}[(m, M); A] := \inf \Big\{ \liminf_{k\to+\infty} \int_A f(x, m_k(x), \nabla m_k(x))\, dx + \int_{A\cap S(m_k)} |[m_k](x)| d\mathcal{H}^{N-1} :$$
$$m_k \in SBV(A; \mathbb{R}^N), \quad |m_k(x)| = 1 \quad \text{a.e. in } A,$$
$$m_k \to m \quad \text{in} \quad L^1(A; \mathbb{R}^N), \quad \nabla m_k \rightharpoonup M \quad \text{in} \quad L^2(A; \mathbb{R}^N) \Big\}.$$

We note that there is a compatibility condition linking m to M, precisely, from the condition that $\nabla m_k(x) \in T_{m_k(x)}$ for a.e. $x \in \Omega$, which can be expressed by $m_k(x)^T \nabla m_k(x) = 0$ a.e., passing to the limit we obtain $m(x)^T M(x) = 0$ a.e. in Ω. In view of this remark, in what follows we say that (m, M) is *admissible pair* if $m \in SBV(\Omega; \mathbb{R}^N)$, $M \in L^2(\Omega; \mathbb{R}^{N^2})$, $|m(x)| = 1$ for a.e. $x \in \Omega$, and $m(x)^T M(x) = 0$ a.e. in Ω.

The goal is now to prove that for every admissible pair, $(m, M) \in SBV(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathbb{R}^{N^2})$, $|m(x)| = 1$ a.e. in Ω, $m(x)^T M(x) = 0$ a.e. in Ω, $\mathcal{F}[(m, M); .]$ is the restriction of a Radon measure to $\mathcal{O}(\Omega)$, the set of all open subsets contained in Ω. Once this is established, the integral representation will follow from the Radon-Nikodym Theorem.

The lemma below provides an alternative characterization of the density H.

Lemma 4.2. *Under conditions (H2), (H3) we have*

$$H(x_0, a, A, B) = \inf \Big\{ \liminf_{n\to+\infty} \int_Q \bar{f}(x_0, a, \nabla u_n(y))\, dy + \int_{Q\cap S(u_n)} |[u_n](y)| d\mathcal{H}^{N-1}(y) :$$
$$u_n \in SBV(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N), \quad u_n \to Ax \quad \textit{in} \quad L^1,$$
$$\nabla u_n \rightharpoonup B \quad \textit{in} \quad L^2 \Big\}, \tag{4.3}$$

and

$$H(x_0,a,A,B)=\inf\left\{\int_Q f(x_0,a,\nabla u(x))\,dx+\int_{Q\cap S(u)}|[u](x)|\,d\mathcal{H}^{N-1}:\right.$$
$$u\in SBV(Q;T_a(S^{N-1}))\cap L^\infty,\quad u|_{\partial Q}=Ax, \tag{4.4}$$
$$\left.\nabla u\in L^2,\quad \int_Q\nabla u(x)\,dx=B\right\}$$

Proof. The proof of (4.3) is similar to the proof of Proposition 3.1 in [27].

We start by noticing that one inequality ($\le$) in (4.4) is trivial. For the converse inequality, we fix A, B, $N\times N$ matrices with columns in $T_a(S^{N-1})$, and we consider a test function $u\in SBV(Q;\mathbb{R}^N)\cap L^\infty$ such that $\nabla u\in L^2$, $u|\partial Q=Ax$, $\int_Q u(x)\,dx=B$. Set $\hat u:=P_a u$, and notice that $\hat u\in SBV(Q;T_a(S^{N-1}))\cap L^\infty$, $\hat u|_{\partial Q}=Ax$, $\nabla\hat u\in L^2$ and $\int_Q\nabla\hat u(x)\,dx=B$. Moreover, using the fact that $|[\hat u](x)|\le|[u](x)|$ and the definition of $\bar f$, we get

$$\int_Q f(x_0,a,\nabla\hat u(x))\,dx+\int_{Q\cap S(\hat u)}|[\hat u](x)|\mathcal{H}^{N-1}(x)\le\int_Q\bar f(x_0,a,\nabla u(x))\,dx+\int_{Q\cap S(u)}|[u](x)|\mathcal{H}^{N-1}(x),$$

which proves the other inequality. $\square$

We now prove that any admissible pair (m,M), may be attained by an admissible sequence, and we also obtain an upper bound for the relaxed energy.

Lemma 4.3. *Let $A\subset\Omega$ be open, and let (m,M) be an admissible pair. Then there exists a sequence $\{m_k\}\subset SBV(A;\mathbb{R}^N)$, $|m_k(x)|=1$ a.e. in A, such that*

$$m_k\to m\quad in\quad L^1(A;\mathbb{R}^N),\qquad \nabla m_k\to M\quad in\quad L^2(A;\mathbb{R}^{N^2}).$$

Moreover,

$$\mathcal{F}[(m,M);A]\le C\int_A\left(1+|\nabla m(x)|^2+|M(x)|^2\right)\,dx+\int_{A\cap S(m)}|[m](x)|\mathcal{H}^{N-1}(x). \tag{4.5}$$

Proof. Using Theorem 2.28 we can find a function $h\in SBV(A;\mathbb{R}^N)$ such that

$$\nabla h=M-\nabla m,\qquad |Dh|(A)\le C\int_A|M(x)-\nabla m(x)|\,dx. \tag{4.6}$$

By Lemma 2.29 there exists a sequence $\{h_k\}\subset SBV(A;\mathbb{R}^N)$, $\nabla h_k=0$, such that

$$h_k-h\to 0\quad\text{in}\quad L^\infty(A;\mathbb{R}^N),\qquad |Dh_k|(A)\to|Dh|(A). \tag{4.7}$$

We consider

$$m_k:=\Pi(m+h-h_k)$$

where $\Pi(x)=x/|x|$ is a projection on the $N-1$-dimensional unit sphere. For k large enough the sequence m_k is well defined and belongs to $SBV(A;\mathbb{R}^N)$, moreover it satisfies the constraint $|m_k(x)|=1$ a.e. in A. It is easy to check, taking into account that $\nabla\Pi(m(x)).M(x)=M(x)$ for a.e. $x\in A$, that $m_k\to m$ in $L^1(A;\mathbb{R}^N)$ and $\nabla m_k\to M$ in $L^2(A;\mathbb{R}^N)$. Given $\delta>0$, restrict Π to a neighborhood of S^{N-1} of the form $N_\eta:=\{x\in\mathbb{R}^N:1-\eta<|x|<1+\eta\}$, with η small enough, so that the Lipschitz constant of $\Pi|_{N_\eta}$ will be smaller than $1+\delta$. Then we have, for k large enough,

$$\begin{aligned}\mathcal{F}[(m,M);A]&\le\liminf_{k\to+\infty}\left\{\int_A f(x,m_k(x),\nabla m_k(x))\,dx+\int_{A\cap S(m_k)}|[m_k](x)|d\mathcal{H}^{N-1}\right\}\\&\le\limsup_{k\to+\infty}\left\{\int_A f(x,m_k(x),\nabla m_k(x))\,dx+(1+\delta)\int_{A\cap S(m)}|[m](x)|\mathcal{H}^{N-1}(x)\right.\\&\left.+(1+\delta)\left(\int_{A\cap S(h)}|[h](x)|\mathcal{H}^{N-1}(x)+\int_{A\cap S(h_k)}|[h_k](x)|\mathcal{H}^{N-1}(x)\right)\right\}\end{aligned}$$

and using $(H1)$, (4.6) and (4.7), and the fact that δ is arbitrary we deduce the upper bound (4.5). $\square$

Now we prove a subadditivity property for $\mathcal{F}[(m,M);.]$

Lemma 4.4. *Let A, B, C be open subsets of Ω such that $C \subset\subset B \subset\subset A$. Then*

$$\mathcal{F}[(m,M);A] \leq \mathcal{F}[(m,M);B] + \mathcal{F}[(m,M);A\setminus\overline{C}].$$

Proof. Fix $\varepsilon > 0$. Let $\{m_{1,k}\} \subset SBV(A\setminus\overline{C};\mathbb{R}^N)$, $\{m_{2,k}\} \subset SBV(B;\mathbb{R}^N)$ be sequences such that

$$\mathcal{F}[(m,M);A\setminus\overline{C}] + \varepsilon \geq \lim\left\{\int_{A\setminus\overline{C}} f(x, m_{1,k}(x), \nabla m_{1,k}(x))\,dx \right.$$
$$\left. + \int_{A\setminus\overline{C}\cap S(m_{1,k})} |[m_{1,k}](x)| d\mathcal{H}^{N-1}(x)\right\},$$

$|m_{1,k}(x)| = 1$ a.e. in $A\setminus\overline{C}$, $m_{1,k} \to m$ in $L^1(A\setminus\overline{C};\mathbb{R}^N)$, $\nabla m_{1,k} \rightharpoonup M$ in $L^2(A\setminus\overline{C};\mathbb{R}^{N^2})$, and

$$\mathcal{F}[(m,M);B] + \varepsilon \geq \lim\{\int_B f(x, m_{2,k}(x), \nabla m_{2,k}(x))\,dx$$
$$+ \int_{B\cap S(m_{2,k})} |[m_{2,k}](x)| d\mathcal{H}^{N-1}(x)\},$$

$|m_{2,k}(x)| = 1$ a.e. in B, $m_{2,k} \to m$ in $L^1(B;\mathbb{R}^N)$, $\nabla m_{2,k} \rightharpoonup M$ in $L^2(B;\mathbb{R}^{N^2})$.

Up to a subsequence, we can find bounded Radon measures, ν, μ_1 and μ_2 such that

$$\left(|\nabla m_{1,k}|^2 + |\nabla m_{2,k}|^2\right)\mathcal{L}^N\lfloor(B\setminus\bar{C}) \overset{\star}{\rightharpoonup} \nu$$
$$|[m_{1,k}]|\mathcal{H}^{N-1}\lfloor\big(S(m_{1,k})\cap(B\setminus\bar{C})\big) \overset{\star}{\rightharpoonup} \mu_1$$
$$|[m_{2,k}]|\mathcal{H}^{N-1}\lfloor\big(S(m_{2,k})\cap(B\setminus\bar{C})\big) \overset{\star}{\rightharpoonup} \mu_2 \tag{4.8}$$

Let $S_\delta := \{x \in B : \mathrm{dist}(x,\partial C) < \delta\}$ and choose δ_0 such that $\nu(\partial S_{\delta_0}) = \mu_1(\partial S_{\delta_0}) = \mu_2(\partial S_{\delta_0}) = 0$ and $\int_{\partial S_{\delta_0}\cap S(m)} |[m](x)| d\mathcal{H}^{N-1} = 0$. Let $S := S_{\delta_0}$ and set $S^i := \{x \in B : \mathrm{dist}(x,S) < \frac{1}{i}\}$. We consider a family of cut-off functions $\varphi^i \in C^\infty(A;[0,1])$, $\varphi^i(x) = 1$ on $A\setminus S^i$, $\varphi^i = 0$ in S^{i+1}, $||\nabla\varphi^i||_\infty \leq Ci^2$. Let

$$m_{i,k} := \varphi^i m_{1,k} + (1-\varphi^i)m_{2,k}.$$

Clearly $m_{i,k} \to m$ in $L^1(A;\mathbb{R}^N)$, $\nabla m_{i,k} \rightharpoonup M$ in $L^2(A;\mathbb{R}^{N^2})$ as $k \to +\infty$, and now we need to modify this sequence in order to have its range on S^{N-1}. We will do that in two steps: first we modify the sequence $\{m_{i,k}\}$ into a new one $\{\bar{m}_{i,k}\}$ of the form

$$\bar{m}_{i,k}(x) := \begin{cases} m_{i,k}(x) & \text{if } |m_{i,k}(x)| > \eta_{ik}, \\ m(x) & \text{otherwise,} \end{cases}$$

for suitable η_{ik} in order to have $|\bar{m}_{i,k}(x)| \geq \eta > 0$, and afterwords we project $\{\bar{m}_{i,k}\}$ onto the unit sphere by considering $\tilde{m}_{i,k} := \Pi(\bar{m}_{i,k})$.

We consider the Lipschitz function

$$f(x) := \begin{cases} 1 & \text{if } |x| > 1, \\ |x| & \text{if } \frac{1}{2} < |x| \leq 1, \\ \frac{1}{2} & \text{if } |x| \leq \frac{1}{2}. \end{cases}$$

The composite function $w_{i,k} := f(m_{i,k})$ belongs to SBV (see [14], [12]). Set $E^\eta_{ik} := \{x \in S^i \setminus \overline{S^{i+1}} : |m_{ik}(x)| > \eta\}$. By Theorem 2.20 E^η_{ik} has finite perimeter for a.e. η and all (i,k) and

$$\begin{aligned}\int_{\frac{1}{2}}^{1} P(E^\eta_{ik}; S^i \setminus \overline{S^{i+1}})\, d\eta = |Dw_{i,k}|(S^i \setminus \overline{S^{i+1}}) &\le \int_{S^i\setminus\overline{S^{i+1}}} |\nabla m_{i,k}|\, dx + |D^s w_{i,k}|(S^i \setminus \overline{S^{i+1}})\\ &\le \int_{S^i\setminus\overline{S^{i+1}}} |\nabla m_{i,k}|\, dx + \int_{(\overline{S^i}\setminus S^{i+1})\cap S(m_{1,k})} |[m_{1,k}](x)|\, d\mathcal{H}^{N-1}\\ &\quad + \int_{(\overline{S^i}\setminus S^{i+1})\cap S(m_{2,k})} |[m_{2,k}](x)|\, d\mathcal{H}^{N-1}.\end{aligned} \tag{4.9}$$

For every (i,k) we can find numbers $\eta_{ik} \in \left(\frac{1}{2}, 1\right)$ such that $E^{\eta_{ik}}_{ik}$ has finite perimeter and

$$P(E^{\eta_{ik}}_{ik}; S^i \setminus \overline{S^{i+1}}) \le 2 \int_{\frac{1}{2}}^{1} P(E^\eta_{ik}; S^i \setminus \overline{S^{i+1}})\, d\eta.$$

In view of (4.8) and (4.9), it follows that

$$\lim_{i\to+\infty} \lim_{k\to+\infty} P(E^{\eta_{ik}}_{ik}; S^i \setminus \overline{S^{i+1}}) = 0. \tag{4.10}$$

We have

$$\begin{aligned}\mathcal{F}[(m,M);A] &\le \liminf_{i\to+\infty} \liminf_{k\to+\infty} \left\{ \int_A f(x, \tilde m_{i,k}(x), \nabla \tilde m_{i,k}(x))\, dx + \int_{A\cap S(\tilde m_{i,k})} |[\tilde m_{i,k}](x)| d\mathcal{H}^{N-1}(x) \right\}\\ &\le \lim_{k\to+\infty} \left\{ \int_{A\setminus\overline{C}} f(x, m_{1,k}(x), \nabla m_{1,k}(x))\, dx + \int_{(A\setminus\overline{C})\cap S(m_{1,k})} |[m_{1,k}](x)| d\mathcal{H}^{N-1}(x) \right\}\\ &\quad + \lim_{k\to+\infty} \left\{ \int_B f(x, m_{2,k}(x), \nabla m_{2,k}(x))\, dx + \int_{B\cap S(m_{2,k})} |[m_{2,k}](x)| d\mathcal{H}^{N-1}(x) \right\}\\ &\quad + \limsup_{i\to+\infty} \limsup_{k\to+\infty} \left\{ C \int_{S^i\setminus\overline{S^{i+1}}} (1 + |\nabla \tilde m_{ik}(x)|^2)\, dx + \int_{(S^i\setminus\overline{S^{i+1}})\cap S(\tilde m_{ik})} |[\tilde m_{ik}](x)| d\mathcal{H}^{N-1}(x) \right\}\\ &\le \mathcal{F}[(m,M); A\setminus\overline{C}] + \mathcal{F}[(m,M);B] + 2\varepsilon\\ &\quad + \limsup_{i\to+\infty} \limsup_{k\to+\infty} \left\{ C \int_{S^i\setminus\overline{S^{i+1}}} (1 + |\nabla \bar m_{ik}(x)|^2)\, dx + C\int_{(S^i\setminus\overline{S^{i+1}})\cap S(\bar m_{ik})} |[\tilde m_{ik}](x)| d\mathcal{H}^{N-1}(x) \right\}\\ &\le \mathcal{F}[(m,M); A\setminus\overline{C}] + \mathcal{F}[(m,M);B] + 2\varepsilon\\ &\quad + \limsup_{i\to+\infty} \limsup_{k\to+\infty} \left\{ C \int_{S^i\setminus\overline{S^{i+1}}} (1 + |\nabla m_{1,k}(x)|^2 + |\nabla m_{2,k}(x)|^2)\, dx \right.\\ &\quad + C\int_{S^i\setminus\overline{S^{i+1}}} |\nabla\varphi^i(x)|^2 |m_{1,k}(x) - m_{2,k}(x)|^2\, dx + C\int_{S^i\setminus\overline{S^{i+1}}} |\nabla m(x)|^2\, dx\\ &\quad + C\int_{(S^i\setminus\overline{S^{i+1}})\cap S(m_{1,k})} |[m_{1,k}](x)| d\mathcal{H}^{N-1}(x) + C\int_{(S^i\setminus\overline{S^{i+1}})\cap S(m_{2,k})} |[m_{2,k}](x)| d\mathcal{H}^{N-1}(x)\\ &\quad \left. + C\int_{(S^i\setminus\overline{S^{i+1}})\cap S(m)} |[m](x)| d\mathcal{H}^{N-1}(x) + CP(E^{\eta_{ik}}_{ik}, A) \right\}.\end{aligned}$$

By (4.8) and (4.10) we obtain

$$\mathcal{F}[(m,M);A] \le \mathcal{F}[(m,M); A\setminus\bar C] + \mathcal{F}[(m,M);B] + 2\varepsilon,$$

and since ε is an arbitrary positive number, we deduce the subadditivity. □

Next we prove that $\mathcal{F}[(m,M);.]$ is the trace on the open subsets of Ω of a bounded Radon measure. The argument is exactly similar to that used in Proposition 2.22 in [27] and we include it here for the convenience of the reader.

Proposition 4.5. *There exists a bounded Radon measure μ such that*

$$\mathcal{F}[(m,M);A] = \mu(A)$$

for every open set $A \subset \Omega$. Moreover $\mu << \mathcal{L}^N \lfloor \Omega + |D^s m|$.

Proof. We can find a sequence $\{m_k\}$ such that

$$\mathcal{F}[(m,M);\Omega] = \lim_{k\to+\infty} \left\{ \int_\Omega f(x, m_k(x), \nabla m_k(x))\, dx + \int_{\Omega\cap S(m_k)} |[m_k](x)| d\mathcal{H}^{N-1} \right\}$$

and

$$f(x, m_k(x), \nabla m_k(x))\mathcal{L}^N\lfloor\Omega + |[m_k](x)|\mathcal{H}^{N-1}\lfloor(\Omega\cap S(m_k)) \overset{\star}{\rightharpoonup} \mu,$$

where $\mu \in \mathcal{M}(\mathbb{R}^N)$. We note that we extended the integrands above to all $\mathbb{R}^N$ by zero.

Let $V \subset\subset \Omega$ be an open set, fix $\varepsilon > 0$, and let W be open, with $W \subset\subset V$, and $\mu(V\setminus W) < \varepsilon$. We have

$$\begin{aligned}\mu(V) &\le \mu(W) + \varepsilon = \mu(\mathbb{R}^N) - \mu(\mathbb{R}^N\setminus W) + \varepsilon \\ &\le \mathcal{F}[(m,M);\Omega] - \mathcal{F}[(m,M);\Omega\setminus\bar{W}] + \varepsilon \\ &\le \mathcal{F}[(m,M);V] + \varepsilon.\end{aligned}$$

As ε is arbitrary we conclude that $\mu(V) \le \mathcal{F}[(m,M);V]$. If we only have V open, $V \subset \Omega$, then consider $V' \subset\subset V$, apply the inequality just proved to V', i.e.

$$\mu(V') \le \mathcal{F}[(m,M);V'] \le \mathcal{F}[(m,M);V],$$

and then take the supremum on the left hand-side over all such V's.

Now we prove the reverse inequality. Given an open set V, there is a compact $K \subset\subset V$ such that $\left(C(1+|M|^2)\mathcal{L}^N + C|Dm|\right)(V\setminus K) < \varepsilon$. Let W be an open set verifying $K \subset\subset W \subset\subset V$. By 4.5 and Lemma 4.4,

$$\mathcal{F}[(m,M);V] \le \mathcal{F}[(m,M);W] + \mathcal{F}[(m,M);V\setminus K] \le \mu(\bar{W}) + \varepsilon \le \mu(V) + \varepsilon,$$

and letting $\varepsilon \to 0$ we get $\mathcal{F}[(m,M);V] \le \mu(V)$. □

Next we characterize the densities

$$\frac{d\mathcal{F}[(m,M);.]}{d\mathcal{L}^N}, \qquad \frac{d\mathcal{F}[(m,M);.]}{d\left(|[m]|\mathcal{H}^{N-1}\lfloor S(m)\right)}$$

Proof. **STEP 1: Lower bound**

We consider a sequence $\{m_k\} \subset SBV(\Omega;\mathbb{R}^N)$ such that $m_k \to m$ in L^1, $\nabla m_k \rightharpoonup M$ in L^2 and

$$\sup_k \left\{ \int_\Omega f(x, m_k(x), \nabla m_k(x)) + \int_{\Omega\cap S(m_k)} |[m_k](x)|\mathcal{H}^{N-1}(x) \right\} < +\infty.$$

Up to the extraction of a subsequence (not relabeled), we assume further that

$$\mu_k := f(x, m_k(x), \nabla m_k(x))\mathcal{L}^N\lfloor\Omega + |[m_k](x)|\mathcal{H}^{N-1}\lfloor S(m_k) \overset{\star}{\rightharpoonup} \mu,$$

where μ is a bounded Radon measure. We now decompose the measure μ relatively to $\mathcal{L}^N\lfloor\Omega$ and $|[m](x)|\mathcal{H}^{N-1}\lfloor S(m)$, denoting the respective densities by μ_a and μ_s. We start by establishing a lower bound for μ_a.

Fix $x_0 \in \Omega$ such that

$$\lim_{r\to 0} \frac{\mu(Q(x_0,r))}{r^N}$$

exists and is finite,

$$\lim_{r\to 0}\frac{1}{r^{N+1}}\int_{Q(x_0,r)}|m(x)-m(x_0)-\nabla m(x_0)(x-x_0)|\,dx=0,$$

and

$$\lim_{r\to 0}\frac{1}{r^{N}}\int_{Q(x_0,r)}|M(x)-M(x_0)|^2\,dx=0.$$

The set of points that do not satisfy all these conditions has Lebesgue measure zero.

We choose a sequence $r_n\to 0$ such that $\mu(\partial Q(x_0,r_n))=0$ for all n, and we have

$$\begin{aligned}
\mu_a(x_0)=\frac{d\mu}{d\mathcal{L}^N}(x_0)&=\lim_{n\to+\infty}\frac{\mu(Q(x_0,r_n))}{{r_n}^N}=\lim_{n\to+\infty}\frac{1}{{r_n}^N}\lim_{k\to+\infty}\mu_k(Q(x_0,r_n))\\
&=\lim_{n\to+\infty}\lim_{k\to+\infty}\frac{1}{r_n^N}\left\{\int_{Q(x_0,r_n)}f(x,m_k(x),\nabla m_k(x))\,dx+\int_{Q(x_0,r_n)\cap S(m_k)}|[m_k](x)|d\mathcal{H}^{N-1}\right\}\\
&=\lim_{n\to+\infty}\lim_{k\to+\infty}\frac{1}{r_n^N}\left\{\int_{Q(x_0,r_n)}\bar{f}(x,m_k(x),\nabla m_k(x))\,dx+\int_{Q(x_0,r_n)\cap S(m_k)}|[m_k](x)|d\mathcal{H}^{N-1}\right\}\\
&=\lim_{n\to+\infty}\lim_{k\to+\infty}\left\{\int_Q\bar{f}\left(x_0+r_ny,m_k(x_0+r_ny),\nabla m_k(x_0+r_ny)\right)\,dy\right.\\
&\qquad\left.+\frac{1}{r_n}\int_{Q\cap\frac{S(m_k)-x_0}{r_n}}|[m_k](x_0+r_ny)|d\mathcal{H}^{N-1}\right\}<+\infty.
\end{aligned}$$

Set

$$\omega_{k,n}(y):=\frac{m_k(x_0+r_ny)-m(x_0)}{r_n}.$$

It can be easily checked that $\limsup_{n\to+\infty}\limsup_{k\to+\infty}||\nabla\omega_{k,n}||_{L^2}\le C$, $\lim_{n\to+\infty}\lim_{k\to+\infty}\int_Q|\omega_{k,n}(y)-\nabla m(x_0)y|\,dy=0$ and $\lim_{n\to+\infty}\lim_{k\to+\infty}\int_Q(\omega_{k,n}-M(x_0))\varphi(y)\,dy=0$ for every $\varphi\in L^2$. Using a diagonalizing procedure and the separability of L^2, we can find a sequence $\omega_k\in SBV(Q;\mathbb{R}^N)\cap L^\infty(Q;\mathbb{R}^N)$, verifying the conditions

$$\omega_k\to\nabla m(x_0)y\quad\text{in}\quad L^1,\qquad\nabla\omega_k\rightharpoonup M(x_0)\quad\text{in}\quad L^2,$$

and

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0)\ge\lim_{k\to+\infty}&\left\{\int_Q\bar{f}(x_0+r_ky,m(x_0)+r_k\omega_k(y),\nabla\omega_k(y))\,dy\right.\\
&\left.+\int_{Q\cap S(\omega_k)}|[\omega_k](y)|d\mathcal{H}^{N-1}\right\}.
\end{aligned}$$

Now we prove the existence of another sequence $\bar{\omega}_k\in SBV(Q;\mathbb{R}^N)\cap L^\infty(Q;\mathbb{R}^N)$, $\nabla\bar{\omega}_k\in L^2(Q;\mathbb{R}^{N^2})$, verifying the conditions

$$\bar{\omega}_k\to\nabla m(x_0)y\quad\text{in}\quad L^1,\qquad\nabla\bar{\omega}_k\rightharpoonup M(x_0)\quad\text{in}\quad L^2$$

and

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0)&\ge\liminf_{k\to+\infty}\left\{\int_Q\bar{f}(x_0+r_ky,m(x_0)+r_k\omega_k(y),\nabla\omega_k(y))\,dy+\int_{Q\cap S(\omega_k)}|[\omega_k](y)|d\mathcal{H}^{N-1}\right\}\\
&\ge\liminf_{k\to+\infty}\left\{\int_Q\bar{f}(x_0,m(x_0),\nabla\bar{\omega}_k(y))\,dy+\int_{Q\cap S(\bar{\omega}_k)}|[\bar{\omega}_k](y)|d\mathcal{H}^{N-1}\right\}.
\end{aligned}\tag{4.11}$$

To this end, we consider the family of Lipschitz continuous functions $\varphi_i:\mathbb{R}^N\to\mathbb{R}^N$,

$$\varphi_i(x):=\begin{cases}x & \text{if } |x|\le\frac{1}{2^{i+1}}\\ -x+\frac{1}{2^i}\frac{x}{|x|} & \text{if } \frac{1}{2^{i+1}}<|x|\le\frac{1}{2^i}\\ 0 & \text{otherwise}\end{cases}$$

We note φ_i has Lipschitz constant 1 and we define

$$\hat{\omega}_{i,k} := \nabla m(x_0)y + \varphi_i(\omega_k(y) - \nabla m(x_0)y).$$

Using the chain rule (see [14], [12]) we find

$$\nabla\hat{\omega}_{i,k} = \nabla m(x_0) + \nabla^\tau \varphi_i(\omega_k(y) - \nabla m(x_0)y).(\nabla\omega_k(y) - \nabla m(x_0))$$

and

$$|D^s\hat{\omega}_{i,k}|(Q) \le |D^s\omega_k|(Q), \tag{4.12}$$

where $\nabla^\tau\varphi_i$ denotes $\nabla(\varphi_i|A(y))$, with $A(y) := \tilde{\omega}_k(y) - \nabla m(x_0)y + \left\{(\nabla\omega_k(y) - \nabla m(x_0))v : v \in \mathbb{R}^N\right\}$ (see [14] pag. 193, Thm. 3.101).

We have

$$\begin{aligned}
&\int_Q \bar{f}(x_0 + r_k y, m(x_0) + r_k\hat{\omega}_{i,k}(y), \nabla\hat{\omega}_{i,k}(y))\,dy \\
&\quad\le \int_{Q\cap\{y\in Q:|\omega_k(y)-\nabla m(x_0)y|\le\frac{1}{2^{i+1}}\}} \bar{f}(x_0 + r_k y, m(x_0) + r_k\omega_k(y), \nabla\omega_k(y))\,dy \\
&\quad\quad + C\int_{Q\cap\left\{y\in Q:\frac{1}{2^{i+1}}<|\omega_k(y)-\nabla m(x_0)y|\le\frac{1}{2^i}\right\}} \left(1 + |\nabla\hat{\omega}_{i,k}|^2\right)\,dy \\
&\quad\quad + C\int_{Q\cap\{y\in Q:|\omega_k(y)-\nabla m(x_0)y|>\frac{1}{2^i}\}} \left(1 + |\nabla m(x_0)|^2\right)\,dy.
\end{aligned} \tag{4.13}$$

We show that the last two terms tend to 0 if we choose (i,k) going to infinity in a suitable way. Let j_k be the greatest even number verifying the inequality

$$2^j \le \frac{1}{\sqrt{||\omega_k - \nabla m(x_0)y||_{L^1(Q)}}}.$$

For every $i \le j_k$

$$\begin{aligned}
|\{y \in Q : |\omega_k(y) - \nabla m(x_0)y| > \frac{1}{2^i}\}| &\le 2^i \int_Q |\omega_k - \nabla m(x_0)y|\,dy \\
&\le \sqrt{||\omega_k - \nabla m(x_0)y||_{L^1(Q)}} \to 0
\end{aligned}$$

as $k \to +\infty$. From the bound

$$\begin{aligned}
&\sum_{i=\frac{j_k}{2}}^{j_k} \int_{Q\cap\{y\in Q:\frac{1}{2^{i+1}}\le|\omega_k(y)-\nabla m(x_0)y|<\frac{1}{2^i}\}} C\left(1 + |\nabla\hat{\omega}_{i,k}|^2\right)\,dy \\
&\qquad\le C\int_Q \left(1 + |\nabla\omega_k(y) - \nabla m(x_0)|^2\right)\,dy \le C,
\end{aligned}$$

we get the existence of an index $i_k \in \left[\frac{j_k}{2}, j_k\right]$ such that

$$\int_{Q\cap\{y\in Q:\frac{1}{2^{i_k+1}}\le|\omega_k(y)-\nabla m(x_0)y|<\frac{1}{2^{i_k}}\}} C\left(1 + |\nabla\hat{\omega}_{i_k,k}|^2\right)\,dy \le \frac{2C}{j_k + 2}. \tag{4.14}$$

We define $\bar{\omega}_k := \hat{\omega}_{i_k,k}$, and it is easy to check that

$$\bar{\omega}_k \to \nabla m(x_0)y \quad \text{in} \quad L^\infty, \qquad \nabla\bar{\omega}_k \rightharpoonup M(x_0) \quad \text{in} \quad L^2,$$

and, by (4.13) and (4.14),

$$\begin{aligned}
&\liminf_{k\to+\infty} \int_Q \bar{f}(x_0 + r_k y, m(x_0) + r_k\bar{\omega}_k(y), \nabla\bar{\omega}_k(y))\,dy \\
&\le \liminf_{k\to+\infty} \int_Q \bar{f}(x_0 + r_k y, m(x_0) + r_k\omega_k(y), \nabla\omega_k(y))\,dy.
\end{aligned}$$

Now using hypothesis $(H2), (H3)$, and by (4.12), we deduce the inequality (4.11), which, in turn, yields

$$\mu_a(x_0) \geq H(x_0, m(x_0), \nabla m(x_0), M(x_0)) \quad \text{for} \quad \mathcal{L}^N \text{a.e. } x \text{ in } \Omega. \tag{4.15}$$

Next we obtain a lower bound for the density μ_s. Fix $x_0 \in S(m)$ such that

$$\lim_{r \to 0} \frac{\mu(Q(x_0, r))}{\int_{Q(x_0,r) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}}$$

exists and is finite. The complementer to this set of points in $S(m)$ has zero $|[m]|\mathcal{H}^{N-1} \lfloor S(m)$-measure. We have

$$\begin{aligned}
\mu_s(x_0) &= \frac{d\mu}{d(|[m](x)|\mathcal{H}^{N-1} \lfloor S(m))} = \lim_{r_n \to 0} \frac{\mu(Q(x_0, r_n))}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} \\
&= \lim_{r_n \to 0} \lim_{k \to +\infty} \frac{\mu_k(Q(x_0, r_n))}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} \\
&= \lim_{r_n \to 0} \lim_{k \to +\infty} \left\{ \frac{\int_{Q(x_0,r_n)} f(x, m_k(x), \nabla m_k(x))\, dx}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} + \frac{\int_{Q(x_0,r_n) \cap S(m_k)} |[m_k](x)| d\mathcal{H}^{N-1}}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} \right\} \\
&\geq \liminf_{r_n \to 0} \liminf_{k \to +\infty} \frac{\int_{Q(x_0,r_n) \cap S(m_k)} |[m_k](x)| d\mathcal{H}^{N-1}}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} \\
&\geq \liminf_{r_n \to 0} \frac{\int_{Q(x_0,r_n)} |\nabla m - M|\, dx + \int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}}{\int_{Q(x_0,r_n) \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}} \geq 1,
\end{aligned}$$

where we have used the fact that $[m_k]\mathcal{H}^{N-1} \lfloor S(m_k) \overset{\star}{\rightharpoonup} (\nabla m - M)\mathcal{L}^N \lfloor \Omega + [m]\mathcal{H}^{N-1} \lfloor S(m_k)$ and the lower semicontinuity of the total variation with respect to weak* convergence. This, together with (4.15), entails

$$\begin{aligned}
&\liminf_{k \to +\infty} \int_\Omega f(x, m_k(x), \nabla m_k(x)) + \int_{\Omega \cap S(m_k)} |[m_k](x)| \mathcal{H}^{N-1}(x) \\
&\qquad \geq \mu(\Omega) \geq \int_\Omega H(x, m(x), \nabla m(x), M(x))\, dx + \int_{\Omega \cap S(m)} |[m](x)| d\mathcal{H}^{N-1}(x).
\end{aligned}$$

STEP 2: Upper bound

Fix (m, M). As the function $(x, v) \to f(x, m(x), v)$ is Carathéodory, using Scorza-Dragoni Theorem, we can find sets K_j such that $|\Omega \setminus K_j| < \frac{1}{j}$ and $(x, v) \to f(x, m(x), v)$ is continuous on $K_j \times \mathbb{R}^{N^2}$. We denote by K_j^* the set of Lebesgue points of χ_{K_j} and we define $\omega := \cup (K_j \cap K_j^*)$. Fix $x_0 \in \omega$ a Lebesgue point for m, ∇m and M.

Let $a := m(x_0)$, $A := \nabla m(x_0)$, $B := M(x_0)$. Given $\delta > 0$ we can find $u \in SBV(Q; \mathbb{R}^N) \cap L^\infty$, $\nabla u \in L^2$, $u|_{\partial Q} = Ax$, $\int_Q \nabla u(x)\, dx = B$, such that

$$\int_Q f(x_0, a, \nabla u(x))\, dx + \int_{Q \cap S(u)} |[u](x)| \mathcal{H}^{N-1} \leq H(x_0, a, A, B) + \delta. \tag{4.16}$$

We write

$$u(x) = Ax + \Phi(x),$$

where $\Phi|_{\partial Q} = 0$, $\int_Q \nabla \Phi(x)\, dx = B - A$ and Φ is extended to all $\mathbb{R}^N$ by periodicity. We define

$$m_{\varepsilon,n} := \Pi \left(m(x) + \frac{\varepsilon}{n} \Phi_{\varepsilon,n}(x) + h_\varepsilon - h_{\varepsilon,n} \right), \qquad x \in Q(x_0, \varepsilon)$$

where $\Phi_{\varepsilon,n}(x) := \Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)$, $h_\varepsilon \in SBV(Q(x_0, \varepsilon); \mathbb{R}^N)$ is such that (see Theorem 2.28)

$$\nabla h_\epsilon = M(x) - \nabla m(x) + \nabla m(x_0) - M(x_0),$$

and

$$|Dh_\varepsilon|(Q(x_0,\varepsilon)) \le C \int_{Q(x_0,\varepsilon)} |M(x) - M(x_0) - \nabla m(x) + \nabla m(x_0)|\, dx, \tag{4.17}$$

and $h_{\varepsilon,n} \in SBV(Q(x_0,\varepsilon))$ are such that (see Theorem 2.29)

$$\nabla h_{\varepsilon,n} = 0, \quad h_{\varepsilon,n} - h_\varepsilon \to 0 \quad \text{in} \quad L^\infty, \quad |Dh_{\varepsilon,n}|(Q(x_0,\varepsilon)) \to |Dh_\varepsilon|(Q(x_0,\varepsilon)). \tag{4.18}$$

We note that for fixed ε and n large enough $m_{\varepsilon,n}$ is well defined, $m_{\varepsilon,n} \in SBV(Q(x_0,\varepsilon);\mathbb{R}^N)$, $|m_{\varepsilon,n}(x)| = 1$ a.e., $m_{\varepsilon,n} \to m$ in L^∞, $\nabla m_{\varepsilon,n} \rightharpoonup M$ in L^2 and $\{\nabla m_{\varepsilon,n}\}$ is 2-equi-integrable. We then have

$$\begin{aligned}\mathcal{F}[(m,M);Q(x_0,\varepsilon)] \le \liminf_{n\to+\infty} \Bigg\{ &\int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx \\ &+ \int_{Q(x_0,\varepsilon)\cap S(m_{\varepsilon,n})} |[m_{\varepsilon,n}](x)| \mathcal{H}^{N-1}(x) \Bigg\}.\end{aligned}$$

We start by treating the volume part, and we prove that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx \\ &\le \limsup_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{\epsilon^N} \int_{Q(x_0,\epsilon)} f(x, m(x), \nabla m_{\epsilon,n}(x))\, dx.\end{aligned} \tag{4.19}$$

For fixed ε and using the 2-equi-integrability of $\{\nabla m_{\varepsilon,n}\}$, for each $\delta' > 0$ we can find $\gamma > 0$ such that

$$\int_{Q(x_0,\varepsilon)\cap A} C(1 + |\nabla m_{\varepsilon,n}(x)|^2)\, dx < \delta' \varepsilon^N$$

whenever $|A| < \gamma$, where C comes from $(H1)$. Since f is Carathéodory, find a compact set K_ϵ, with $|Q(x_0,\varepsilon) \setminus K_\varepsilon| < \gamma$, such that $f|K_\varepsilon \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ is continuous, and define

$$E_{\epsilon,n} := \{x \in Q(x_0,\epsilon) : |\nabla m_{\epsilon,n}(x)| < L\},$$

where L is large enough to guarantee that $|Q(x_0,\varepsilon) \setminus E^c_{\varepsilon,n}| < \gamma$. For $x \in Q(x_0,\varepsilon) \cap K_\varepsilon \cap E_{\varepsilon,n}$ we have

$$|f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x)) - f(x, m(x), \nabla m_{\varepsilon,n}(x))| < \delta',$$

for n large enough because of the uniform continuity of f on compact subsets of $K_\varepsilon \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ and the L^∞ convergence of $m_{\varepsilon,n}$ to m.

We then have

$$\begin{aligned}&\int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx \\ &\quad= \int_{Q(x_0,\varepsilon)\cap K_\varepsilon} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx + \int_{Q(x_0,\varepsilon)\setminus K_\varepsilon} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx \\ &\quad\le \int_{Q(x_0,\varepsilon)\cap K_\varepsilon} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx + \delta' \varepsilon^N \\ &\quad= \int_{Q(x_0,\varepsilon)\cap K_\varepsilon \cap E_{\varepsilon,n}} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx + \int_{Q(x_0,\varepsilon)\cap K_\varepsilon \cap E^c_{\varepsilon,n}} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\, dx + \delta' \varepsilon^N \\ &\quad\le \int_{Q(x_0,\varepsilon)\cap K_\varepsilon \cap E_{\varepsilon,n}} f(x, m(x), \nabla m_{\varepsilon,n}(x))\, dx + 3\delta' \varepsilon^N \\ &\quad\le \int_{Q(x_0,\varepsilon)} f(x, m(x), \nabla m_{\varepsilon,n}(x))\, dx + 3\delta' \varepsilon^N.\end{aligned}$$

Dividing through by ε^N, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n}(x))\,dx\\ &\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f(x, m(x), \nabla m_{\varepsilon,n}(x))\,dx + 3\delta',\end{aligned}$$

and letting $\delta' \to 0$, we deduce (4.19). Now we prove that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\epsilon)} f(x, m(x), \nabla m_{\epsilon,n}(x))\,dx\\ &\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi\left(m(x)+\frac{\epsilon}{n}\Phi_{\epsilon,n}(x)+h_\epsilon(x)-h_{\epsilon,n}(x)\right)\right.\\ &\qquad\left.\left(\nabla\Phi\left(\frac{n(x-x_0)}{\epsilon}\right)+\nabla m(x_0)\right)\right)dx.\end{aligned}$$

This follows from $(H3)$, which allow us to assert that the error in passing from the left integral to the right integral, for ε fixed and n large enough, is given by

$$\begin{aligned}&\frac{C}{\epsilon^N}\int_{Q(x_0,\epsilon)}\left(1+\left|M(x)+\nabla\Phi\left(\frac{n(x-x_0)}{\epsilon}\right)+\nabla m(x_0)-M(x_0)\right|\right.\\ &\qquad\left.+\left|\nabla\Phi\left(\frac{n(x-x_0)}{\epsilon}\right)+\nabla m(x_0)\right|\right)|M(x)-M(x_0)|\,dx.\end{aligned}\tag{4.20}$$

Using the fact that x_0 is a 2-Lebesgue point for M, Hölder's Inequality and the Riemman Lebesgue Lemma, we obtain that $\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}(4.20)=0$.

Using again $(H3)$ and the L^∞ convergence of $m(x)+\frac{\varepsilon}{n}\Phi_{\varepsilon,n}(x)+h_\varepsilon-h_{\varepsilon,n}$ to $m(x)$, we deduce that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi\left(m(x)+\frac{\varepsilon}{n}\Phi_{\varepsilon,n}(x)+h_\varepsilon(x)-h_{\varepsilon,n}(x)\right)\right.\\ &\qquad\left.\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx\\ &\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi(m(x))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx.\end{aligned}$$

Now using $(H3)$ and the fact that x_0 is a Lebesgue point for ∇m, Hölder's Inequality and Riemman Lebesgue Lemma, we deduce that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi(m(x))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx\\ &\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx.\end{aligned}$$

We next prove that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx\\ &\le \limsup_{\epsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\epsilon^N}\int_{Q(x_0,\epsilon)} f\left(x_0, m(x_0), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right)dx.\end{aligned}\tag{4.21}$$

Let j_0 be such that $x_0 \in K_{j_0}\cap K^*_{j_0}$. We define the set

$$E_{\varepsilon,n} := \left\{x\in Q(x_0,\varepsilon) : \left|\nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right)+\nabla m(x_0)\right)\right| < L\right\},$$

where $L > 0$ will be defined latter. We have

$$
\begin{aligned}
|Q(x_0,\varepsilon) \setminus E_{\varepsilon,n}| &\leq \int_{Q(x_0,\varepsilon)} \frac{\left|\nabla\Pi(m(x_0))(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0))\right|}{L}\, dx \\
&\leq \frac{C'}{L}\varepsilon^N \int_Q |\nabla\Phi(ny) + \nabla m(x_0)|\, dy \leq \frac{C'\varepsilon^N}{L}.
\end{aligned}
$$

As

$$
\begin{aligned}
&\frac{1}{\epsilon^N}\int_{Q(x_0,\varepsilon)\setminus E_{\varepsilon,n}} C\left(1 + \left|\nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right|^2\right) dx \\
&\qquad = \int_{Q\setminus \frac{E_{\varepsilon,n}-x_0}{\varepsilon}} C(1 + |\nabla\Pi(m(x_0)(\nabla\Phi(ny) + \nabla m(x_0))|^2)\, dy,
\end{aligned}
$$

and since of $|Q \setminus \frac{E_{\varepsilon,n}-x_0}{\varepsilon}| < \frac{C'}{L}$, we can take L large enough and use the 2-equi-integrability to get the estimate (independent of ε)

$$
\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\setminus E_{\varepsilon,n}} C\left(1 + \left|\nabla\Pi(m(x_0))\left(\nabla\Phi(\frac{n(x-x_0)}{\varepsilon}) + \nabla m(x_0)\right)\right|^2\right) dx < \delta'.
$$

Also, using the fact that x_0 is a Lebesgue point for $\chi_{K_{j_0}}$ we have $|\frac{Q(x_0,\varepsilon)\setminus K_{j_0}}{\varepsilon^N}| \to 0$, and an argument similar to the one above yields

$$
\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\setminus K_{j_0}} C\left(1 + \left|\nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right|^2\right) dx < \delta'.
$$

We can use the uniform continuity of $f|K_{j_0} \times \overline{B(0,L)}$ to ensure that for ε small enough we have

$$
|f(x, m(x), v) - f(x_0, m(x_0), v)| < \delta' \quad \text{for} \quad |v| \leq L.
$$

We deduce that

$$
\begin{aligned}
&\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) \\
&\quad \leq \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_{j_0}} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) \\
&\qquad + \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\setminus K_{j_0}} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) \\
&\quad \leq \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_{j_0}\cap E_{\varepsilon,n}} f\left(x, m(x), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) \\
&\qquad + 2\delta' \\
&\quad \leq \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_{j_0}\cap E_{\varepsilon,n}} f\left(x_0, m(x_0), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) \\
&\qquad + 3\delta' \\
&\quad \leq \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x_0, m(x_0), \nabla(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\varepsilon}\right) + \nabla m(x_0)\right)\right) + 3\delta',
\end{aligned}
$$

Taking limits in both sides, and in view of the fact that δ' is arbitrary positive number, we reach the inequality (4.21). Changing variables and using the Riemann-Lebesgue Lemma

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\epsilon^N}\int_{Q(x_0,\epsilon)} f\left(x_0, m(x_0), \nabla\Pi(m(x_0))\left(\nabla\Phi\left(\frac{n(x-x_0)}{\epsilon}\right)+\nabla m(x_0)\right)\right)\\
&=\limsup_{n\to+\infty}\int_Q f\left(x_0, m(x_0), \nabla\Pi(m(x_0))(\nabla\Phi(ny)+\nabla m(x_0)\right)\, dy\\
&=\int_Q f(x_0, m(x_0), \nabla\Pi(m(x_0))(\nabla\Phi(y)+\nabla m(x_0)))\, dy\\
&=\int_Q f(x_0, m(x_0), \nabla\Pi(m(x_0))\nabla u(y))\, dy = \int_Q \bar{f}(x_0, m(x_0), \nabla u(y))\, dy.
\end{aligned}$$

We now estimate the surface integral. Given $\delta' > 0$, choose $\eta > 0$ small enough so that the Lipschitz constant of $\Pi|N_\eta$, where $N_\eta := \{x \in \mathbb{R}^N : 1-\eta < |x| < 1+\eta\}$, is smaller than $1+\delta'$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap S(m_{\varepsilon,n})}|[m_{\varepsilon,n}](x)|d\mathcal{H}^{N-1}\\
&\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1+\delta'}{\varepsilon^N}\left[\int_{Q(x_0,\varepsilon)\cap S(m)}|[m](x)|d\mathcal{H}^{N-1}\right.\\
&\quad+\frac{\varepsilon}{n}\int_{Q(x_0,\varepsilon)\cap S(\Phi(\frac{n(.-x_0)}{\varepsilon}))}\left|[\Phi]\left(\frac{n(x-x_0)}{\varepsilon}\right)\right| d\mathcal{H}^{N-1}\\
&\qquad\left.+|Dh_\varepsilon|(Q(x_0,\varepsilon))+|Dh_{\varepsilon,n}|(Q(x_0,\varepsilon))\right].
\end{aligned}\tag{4.22}$$

Using the fact that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap S(m)}|[m](x)|d\mathcal{H}^{N-1} = 0$ a.e., a change of variables and the periodicity of Φ for the second term, (4.17), (4.18), and the arbitrariness of δ', yield the estimate

$$\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap S(m_{\varepsilon,n})}|[m_{\varepsilon,n}](x)|\mathcal{H}^{N-1} \le \int_{Q\cap S(u)}|[u](x)|\mathcal{H}^{N-1}(x). \tag{4.23}$$

Putting together the estimates for the volume and the surface integral, we conclude that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\mathcal{F}[(m,M);Q(x_0,\varepsilon)]}{\varepsilon^N} &\le \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\liminf_{n\to+\infty}\left\{\int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n})\, dx\right.\\
&\quad\left.+\int_{Q(x_0,\varepsilon)\cap S(m_{\varepsilon,n})}|[m_{\varepsilon,n}](x)|d\mathcal{H}^{N-1}(x)\right\}\\
&\le \limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f(x, m_{\varepsilon,n}(x), \nabla m_{\varepsilon,n})\, dx\\
&\quad+\limsup_{\varepsilon^N}\limsup_{n\to+\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap S(m_{\varepsilon,n})}|[m_{\varepsilon,n}](x)|d\mathcal{H}^{N(see Theorem 2.29)-1}(x)\\
&\le \int_Q \bar{f}(x_0, m(x_0), \nabla u(y))\, dy + \int_{Q\cap S(u)}|[u](y)|d\mathcal{H}^{N-1}(y)\\
&\le H(x_0, m(x_0), \nabla m(x_0), M(x_0))+\delta,
\end{aligned}$$

and letting $\delta \to 0$ we obtain the upper bound for $\frac{d\mathcal{F}[(m,M);.]}{d\mathcal{L}^N}$.

The upper bound for the singular part of the measure $\mathcal{F}[(m,M);.]$ follows immediately from the bound (4.5). ☐

5 Second Order Analysis for Thin Structures

A commonly used approach to the thin film theory consists of dimensional reduction through asymptotic analysis from $3D$ models to $2D$ ones. Often in the applications it is essential to determine exactly the relation between the thin film theory and the $3D$ theory. To this end, this subject has recently been the target of intensive study, both in the linear and in the non linear settings, and one can trace back some of the arguments to [1], where fully nonlinear beam models were obtained. Results deriving plate models from thin structures through $3D - 2D$ dimensional reduction are due to Fox, Raoult and Simo in [41], a detailed analysis in a Γ- convergence setting can be found in [50] and [51], and for what concerns optimal design in [38]. The overall picture was completed in [20], in what concerns energy densities depending on the gradient of the displacement, and taking in account both material heterogeneity and rapidly varying profiles. Actually those techniques apply as well to $nD - (n-d)D$ reduction, for any $n \geq d > 0$. Techniques for recognizing of bending effect in thin films have been recently developed in [43], [44] and [42].

The situation is different in the case where the energy contains also interfacial terms, i.e. terms which depend on the second order derivatives. This case has been considered in [17], where to a classical elastic energy depending on the gradient is added a quadratic term of the second derivative. Also in [60], Shu considered dimensional reduction depending on several length scales, and taking into account a quadratic term for the interfacial energy.

In this chapter we study $3D$-$2D$ dimensional reduction for models with non convex energies involving second order derivatives. Thus, we generalize the model of Bhattacharya and James ([17]) refered above, and as for the lower order terms there is strong convergence we just focus on a energy depending only on the second derivative. We follow very closely the paper of Braides, Fonseca and Francfort [20], where with a volume energy depending only on the gradient material heterogeneity and varying profiles are treated. Using Γ- convergence techniques, we first give a general result for $2D$ models for energy densities which allow material heterogeneity and varying profiles. Next, we specialize the study considering transversally inhomogeneous thin domains and deriving the homogeneous model from those. Finally, we also consider a model in which microstructure and profile oscillate on a scale which is comparable to the thickness of the domain. We are able to provide an integral representation result for the limiting energy, under quite general assumptions on the initial density.

The limiting energy is determined by two vector fields u and b defined on a plane sheet, where u is associated to the deformation of the middle surface and b is the Cosserat vector associated with transverse shear and normal compression, and which keeps memory of the rotation of the original normal vector to the section ω in the $3D$. Since the limit model is not convex and takes into account both on u and on b, it requires a more general notion of convexity, $\mathcal{A}$- quasiconvexity, for a suitable operator $\mathcal{A}$.

The setting is as follows. Let ε be a positive real number, the letter p, as subscript, will run from 1 to 2, thus the coordinates of a point in $\mathbb{R}^3$ will be denoted by (x_p, x_3), and D_p and D_p^2 stand, respectively, for the gradient and the Hessian tensor with respect to the planar variables $x_p := (x_1, x_2)$. We denote a general element of $E_2^s(\mathbb{R}^m)$ by $H = (H^i)_{jk}$, where for $i \in \{1, \ldots, s\}$ $(H^i)_{jk}$ is a symmetric $M^{m\times m}$ matrix, i.e. $H^i_{jk} = H^i_{kj}$, for every i, j, k. Given $H \in E_2^3(\mathbb{R}^3)$ we consider a triple $(h, \xi, c) \in E_2^3(\mathbb{R}^2) \times M^{3\times 2} \times \mathbb{R}^3$ defined by

$$\begin{cases} h^i_{jk} := H^i_{jk} & i = 1,2,3, \quad 1 \leq j,k \leq 2, \\ \xi_{ik} := H^i_{k3} & 1 \leq i \leq 3, \quad 1 \leq k \leq 2, \\ c_i := H^i_{33} & i = 1,2,3. \end{cases} \tag{5.1}$$

Let us consider a thin domain $\Omega(\varepsilon)$ in $\mathbb{R}^3$,

$$\Omega(\varepsilon) := \{(x_1, x_2, x_3) : (x_1, x_2) \in \omega \text{ and } |x_3| < \varepsilon f_\varepsilon(x_1, x_2)\}$$

where ω is a bounded Lipschitz domain in $\mathbb{R}^2$ and $f_\varepsilon(x_1, x_2)$ determines the ε-dependent profile $x_3 = \pm f_\varepsilon(x_1, x_2)$. We assume that the domain is filled with a material energy density $W(\varepsilon)(x_1, x_2, x_3, \cdot)$. Equilibrium correspond to the transformation fields $u(\varepsilon)$ which minimize

$$w \mapsto \int_{\Omega(\varepsilon)} W(\varepsilon)(x_1, x_2, x_3, D^2 w) dx$$

among all the kinematically admissible fields w.

As it is usual, one tries to reformulate the problem on a fixed domain through the re-scaling $\frac{1}{\varepsilon}$ in the transverse direction x_3. Set

$$\begin{aligned}&\Omega := \omega\times(-1,1)\\&\Omega_\varepsilon := \{\ (x_1,x_2,x_3) : (x_1,x_2,\varepsilon x_3) \in \Omega(\varepsilon)\}\,,\\&u_\varepsilon(x_1,x_2,x_3) := u(\varepsilon)(x_1,x_2,\varepsilon x_3),\\&W_\varepsilon(x_1,x_2,x_3,\cdot) := W(\varepsilon)(x_1,x_2,\varepsilon x_3,\cdot).\end{aligned}$$

Clearly u_ε minimizes

$$v \mapsto \int_{\Omega_\varepsilon} W_\varepsilon\left(x_1,x_2,x_3,{D_p}^2 v\,\middle|\,\frac{1}{\varepsilon}D_p(v_{,3})\middle|\,\frac{1}{\varepsilon^2}v_{,3,3}\right)dx$$

among all the kinematically admissible fields v on Ω_ε, where $(h|\xi|c)$ stands for a triple in $E_2^3(\mathbb{R}^2)\times M^{3\times 2}\times\mathbb{R}^3$ according to the notations in (5.1).

Under suitable coercivity conditions on W_ε (or $W(\varepsilon)$), it can be verified (cf. Proposition 5.5) that for a sequence $\{u_\varepsilon\}$ with bounded energy and prescribed boundary values, there exist a subsequence $\{\varepsilon_k\}$, $u \in W^{2,p}(\omega;\mathbb{R}^3)$ and $b \in W^{1,p}(\omega;\mathbb{R}^3)$ such that $(u_{\varepsilon_k}-u)\chi_{\Omega_{\varepsilon_k}} \to 0$, $(Du_{\varepsilon_k}-Du)\chi_{\Omega_{\varepsilon_k}} \to 0$, and $(\frac{1}{\varepsilon_k}u_{\varepsilon_k,3} - b)\chi_{\Omega_{\varepsilon_k}} \to 0$ in $L^p(\mathbb{R}^3)$, where $\chi_{\Omega_{\varepsilon_k}}$ denotes the characteristic function of Ω_{ε_k}. Therefore we are led to investigate, using the Γ- convergence approach, the asymptotic behavior of the family of functionals

$$J_\varepsilon(v,\omega) := \int_{\Omega_\varepsilon} W_\varepsilon\left(x_1,x_2,x_3,{D_p}^2 v\,\middle|\,\frac{1}{\varepsilon}D_p(v_{,3})\middle|\,\frac{1}{\varepsilon^2}v_{,3,3}\right)dx$$

under the above type of convergence.

Define

$$A_\varepsilon := \{(x_p,x_3) : x_p \in A, |x_3| \le f_\varepsilon(x_p)\},$$

and

$$\partial_t A_\epsilon := \{(x_p,x_3) : |x_3| < f_\epsilon(x_p), x_p \in \partial A\}.$$

Set, for any $v \in W^{1,p}(\Omega;\mathbb{R}^3)$,

$$J_\varepsilon(v,A) := \begin{cases} \int_{A_\varepsilon} W_\varepsilon\left(x_p,x_3;D_p^2 v,\frac{1}{\varepsilon}D_p v_{,3},\frac{1}{\varepsilon^2}v_{,3,3}\right)dx_p dx_3 & \text{if } v \in W^{2,p}(A_\varepsilon,\mathbb{R}^3),\\ +\infty & \text{otherwise,}\end{cases}$$

and for any $u \in W^{1,p}(\Omega,\mathbb{R}^3)$, $b \in L^p(\Omega,\mathbb{R}^3)$, and any decreasing sequence $\{\varepsilon\}$ converging to 0,

$$\begin{aligned}J_{\{\varepsilon\}}(u,b,A) := \inf\Big\{ \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon;A) : v_\varepsilon \in W^{2,p}(A_\varepsilon,\mathbb{R}^3), (v_\varepsilon - u)\chi_{A_\varepsilon} \to 0 \text{ in } L^p(\Omega,\mathbb{R}^3),\\ (Dv_\varepsilon - Du)\chi_{A_\varepsilon} \to 0 \text{ in } L^p(\Omega,\mathbb{R}^3),\\ \text{and } \left(\frac{1}{\varepsilon}v_{\varepsilon,3} - b\right)\chi_{A_\varepsilon} \to 0 \text{ in } L^p(\Omega,\mathbb{R}^3)\Big\}.\end{aligned} \tag{5.2}$$

In the sequel we state the main results of this chapter.

Theorem 5.1. *For any decreasing sequence $\{\varepsilon\}$ converging to 0, there exists a subsequence $\{\varepsilon^{\mathcal{R}}\}$ such that for every $u \in W^{2,p}(\omega,\mathbb{R}^3)$ and $b \in W^{1,p}(\omega,\mathbb{R}^3)$, and for every open subset A of ω, one can determine $\{w_{\varepsilon^{\mathcal{R}}}\}$ in $W^{2,p}(A_{\varepsilon^{\mathcal{R}}})$ such that*

$$\begin{cases} (w_{\varepsilon^{\mathcal{R}}} - u)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3),\\ (Dw_{\varepsilon^{\mathcal{R}}} - Du)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3),\\ \left(\frac{1}{\varepsilon^{\mathcal{R}}}w_{\varepsilon^{\mathcal{R}},3} - b\right)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega;\mathbb{R}^3),\\ J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) = \lim_{\varepsilon^{\mathcal{R}}\to 0} J_{\varepsilon^{\mathcal{R}}}(w_{\varepsilon^{\mathcal{R}}},A). \end{cases} \tag{5.3}$$

Furthermore, there exists a Carathéodory function $W_{\{\varepsilon^{\mathcal{R}}\}} : \mathbb{R}^2\times E_2^3(\mathbb{R}^2)\times M^{3\times 2} \to \mathbb{R}$ such that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) = 2\int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_p,D^2u,Db)dx_p. \tag{5.4}$$

In particular, if the energy densities $W_\varepsilon(x,H)$ do not depend on ε and if they coincide with a Carathéodory function of the type $W(x_3,H)$, then Theorem 5.2 below describes the asymptotic behavior of the whole sequence $J_\varepsilon(v,\omega)$. To this end, define for every $h\in E_2^3(\mathbb{R}^2)$ and $d\in M^{3\times 2}$,

$$\begin{aligned}\underline{W}(h,d) := \inf_{\lambda>0}\inf_{\phi}\Big\{\frac{1}{2}\int_{Q'\times(-1,1)} & W(x_3,h+D_p^2\phi,d+\lambda D_p\phi_{,3},\lambda^2\phi_{,3,3})dx_pdx_3 : \\ & \phi\in W^{2,p}(Q'\times(-1,1);\mathbb{R}^3),\phi=0 \text{ on } \partial Q'\times(-1,1)\}.\end{aligned} \tag{5.5}$$

Theorem 5.2. *For almost any $x_p\in\omega$, for all $h\in E_2^3(\mathbb{R}^2)$ and $d\in M^{3\times 2}$, $W_{\{\varepsilon^{\mathcal{R}}\}}(x_p,h,d)=\underline{W}(h,d)$. Consequently, for every $u\in W^{2,p}(\omega,\mathbb{R}^3)$, $b\in W^{1,p}(\omega,\mathbb{R}^3)$ and any A open subset of ω,*

$$J_{\{\varepsilon\}}(u,b,A)=2\int_A \underline{W}(D_p^2u,D_pb)dx_p,$$

where the functional $J_{\{\varepsilon\}}(u,b,A)$ is defined in (5.2).

Finally, in the periodic case, i.e. when $W_\varepsilon(x_p,x_3)=W\left(\frac{x_p}{\varepsilon},x_3,H\right)$ and $f_\varepsilon(x_p)=f\left(\frac{x_p}{\varepsilon}\right)$, the limiting energy density is obtained in Theorem 5.3.

Theorem 5.3. *If $u\in W^{2,p}(\omega,\mathbb{R}^3), b\in W^{1,p}(\omega,\mathbb{R}^3)$, and if A is an open subset of ω, then*

$$J_{\{\varepsilon\}}(u,b,A)=\int_A W_{\mathrm{hom}}(D^2u,Db)dx_p. \tag{5.6}$$

where, for any $(h,d)\in E_2^3(\mathbb{R}^2)\times M^{3\times 2}$,

$$W_{\mathrm{hom}}(h,d):=\liminf_{t\nearrow\infty}V(t) \tag{5.7}$$

and, for every $t>0$,

$$\begin{aligned}V(t):=\frac{1}{t^2}\inf_{\phi}\Big\{\int_{(tQ')^f} & W(x_p,x_3,h+D_p^2\phi,d+D_p\phi_{,3},\phi_{,3,3})dx_pdx_3 : \\ \phi\in W^{2,p}((tQ')^f,\mathbb{R}^3), & \phi(x_p,x_3)=0 \textit{ if } x_p\in\partial(tQ'),|x_3|<f(x_p)\Big\},\end{aligned} \tag{5.8}$$

and where, for $A\subset\mathbb{R}^2$, $A^f:=\{(x_p,x_3):x_p\in A,|x_3|<f(x_p)\}$.

5.1 Integral representation result

Let $\{\varepsilon\}$ be any decreasing sequence of real numbers converging to 0. Assume that $\{W_\varepsilon(x,H)\}_\varepsilon$ is a sequence of nonnegative Carathéodory functions on $\Omega\times E_2^3(\mathbb{R}^3)$ such that

$$\beta'|H|^p\le W_\varepsilon(x;H)\le\beta(1+|H|^p) \tag{5.9}$$

for some $0<\beta'\le\beta<\infty$, $1<p<\infty$. In what follows we write $W_\varepsilon(h|\xi|c)$ to designate $W_\varepsilon(H)$.

For each ε let $f_\varepsilon(x_p)$ be a continuous function on ω such that, for some $\gamma>0$ independent of ε,

$$0<\gamma\le f_\varepsilon(x_p)\le 1, \text{ for all } x_p\in\omega. \tag{5.10}$$

It can be easily verified that for any $u\in W^{2,p}(\omega;\mathbb{R}^3), b\in W^{1,p}(\omega;\mathbb{R}^3)$, the energy $J_{\{\varepsilon\}}(u,b;A)$, defined in (5.2) is finite. Indeed, by using in the definition as test sequence $v_\varepsilon:=u+\varepsilon x_3b_\varepsilon$, with $b_\varepsilon\in C^\infty(\omega,\mathbb{R}^3)\cap W^{1,p}(\omega,\mathbb{R}^3)$ in order to guarantee $u_\varepsilon\in W^{2,p}(\Omega;\mathbb{R}^3)$, and with $b_\varepsilon\to b\in W^{1,p}(\omega,\mathbb{R}^3)$ as $\varepsilon\to 0$, by the growth condition (5.9) we get

$$J_{\{\varepsilon\}}(u,b,A)\le C\beta\int_A(1+|D^2u|^p+|Db|^p)dx_p. \tag{5.11}$$

Remark 5.4. $J_{\{\varepsilon\}}(u,b,\cdot)$ is an increasing set function on the open subsets of ω.

The proof of result below follows the steps of the analogous result in [20].

Proposition 5.5. *Let $p>1$, $u_0\in W^{2,p}(\omega,\mathbb{R}^3)$ and $b_0\in W^{1,p}(\omega,\mathbb{R}^3)$. Assume that $\{v_\varepsilon\}$ is a sequence in $W^{2,p}(\Omega_\varepsilon,\mathbb{R}^3)$ such that*

$$v_\varepsilon \equiv u_0(x_p)+\varepsilon x_3 b_0(x_p) \text{ on } \partial_t\omega_\varepsilon \tag{5.12}$$

and

$$\sup_\varepsilon \int_{\Omega_\varepsilon} W_\varepsilon\left(x_p,x_3,D_p^2v,\frac{1}{\varepsilon}D_pv_{,3},\frac{1}{\varepsilon^2}v_{,3,3}\right)dx_pdx_3<\infty.$$

Then there exist $u\in W^{2,p}(\omega,\mathbb{R}^3)$ with trace u_0 on $\partial\omega$, $b\in W^{1,p}(\omega,\mathbb{R}^3)$ with trace b_0 on $\partial\omega$, and a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that

$$(v_{\varepsilon_k}-u)\chi_{\Omega_{\varepsilon_k}}\to 0,(Dv_{\varepsilon_k}-Du)\chi_{\Omega_{\varepsilon_k}}\to 0 \text{ and } \left(\frac{1}{\varepsilon_k}v_{\varepsilon_k,3}-b\right)\chi_{\Omega_{\varepsilon_k}}\to 0 \text{ in } L^p(\Omega,\mathbb{R}^3). \tag{5.13}$$

Proof. In view of (5.9) and (5.10), one has

$$\begin{aligned}&\int_{\omega\times(-\gamma,\gamma)}\left(|D_p^2v_\varepsilon|^p+\frac{1}{\varepsilon^p}|D_pv_{\varepsilon,3}|^p+\frac{1}{\varepsilon^{2p}}|v_{\varepsilon,3,3}|^p\right)dx_pdx_3\\ &\leq \int_{\omega_\varepsilon}\left(|D_p^2v_\varepsilon|^p+\frac{1}{\varepsilon^p}|D_pv_{\varepsilon,3}|^p+\frac{1}{\varepsilon^{2p}}|v_{\varepsilon,3,3}|^p\right)dx_pdx_3<\infty,\end{aligned} \tag{5.14}$$

so that Poincaré Inequality and Rellich Theorem guarantee that there exists a function u in $W^{2,p}(\omega\times(-\gamma,\gamma);\mathbb{R}^3)$, and a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that

$$\begin{cases} v_{\varepsilon_k}\rightharpoonup u \text{ in } W^{2,p}(\omega\times(-\gamma,\gamma);\mathbb{R}^3),\\ v_{\varepsilon_k}(x_p,\pm\gamma)\to u(x_p,\pm\gamma) \text{ in } W^{1,p}(\omega,\mathbb{R}^3).\end{cases} \tag{5.15}$$

Note that, in order to have (5.15), f_ε must be such that the trace of v_ε is meaningful on $\partial_t\omega_\varepsilon$. Moreover, (5.14) implies that $u_{,3,3}=0$. Hence $u_{,3}=A(x_p)$ and so $u=A(x_p)x_3+B(x_p)$, and $D_pu=D_pA(x_p)x_3+D_pB(x_p)$. In addition $(D_pu)_{,3}=0$, $A(x_p)=C$. Thus

$$u=Cx_3+B(x_p) \text{ and } u=u_0 \text{ on } \partial\omega\times(-\gamma;\gamma)\text{, that is } C=0,$$

and we deduce that $u-u_0\in W_0^{2,p}(\omega,\mathbb{R}^3)$.

Moreover, observe that

$$\begin{aligned}&\int_{\omega_{\varepsilon k}}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3=\int_{\omega\times(-\gamma,\gamma)}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3\\ &\quad+\int_\omega\int_\gamma^{f_{\varepsilon_k}(x_p)}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3+\int_\omega\int_{-f_{\varepsilon_k}(x_p)}^{-\gamma}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3\\ &=\int_{\omega\times(-\gamma,\gamma)}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3+\\ &\quad+\int_\omega\int_\gamma^{f_{\varepsilon_k}(x_p)}\left|\int_\gamma^{x_3}Dv_{\varepsilon_k,3}(x_p,s)ds+Dv_{\varepsilon_k}(x_p,\gamma)-D_pu(x_p)\right|^pdx_pdx_3\\ &\quad+\int_\omega\int_{-f_{\varepsilon_k}(x_p)}^{-\gamma}\left|\int_{-\gamma}^{x_3}Dv_{\varepsilon_k,3}(x_p,s)ds+Dv_{\varepsilon_k}(x_p,-\gamma)-D_pu(x_p)\right|^pdx_pdx_3\\ &\leq\int_{\omega\times(-\gamma,\gamma)}|Dv_{\varepsilon_k}-D_pu|^pdx_pdx_3+\\ &\quad+C\left\{\int_\omega|Dv_{\varepsilon_k}(x_p,\pm\gamma)-Du(x_p)|^p\,dx_p+\int_{\omega_{\varepsilon k}}|Dv_{\varepsilon_k,3}|^pdx_pdx_3\right\},\end{aligned} \tag{5.16}$$

so that (5.14) and (5.15) imply that $(Dv_{\varepsilon_k}-Du)\chi_{\Omega_{\varepsilon_k}}\to 0$ in $L^p(\Omega;\mathbb{R}^3)$. Arguing as in (5.16), it can be shown that $\|v_{\varepsilon_k,3}\|_{L^p(\omega_\varepsilon)}\to 0$, and using this argument, and analogously to (5.16), one can easily see that $\int_{\omega_{\varepsilon k}}|v_{\varepsilon_k}-u|^pdx_pdx_3\to 0$, thus the first and the second convergences in (5.13) follow. In what concerns the last convergence in (5.13), (5.14) and (5.12) entail that the sequences $\left\{D_p \frac{1}{\varepsilon} v_{\varepsilon,3}\right\}$ and $\left\{\frac{1}{\varepsilon} v_{\varepsilon,3,3}\right\}$ are bounded in $L^p(\omega_\varepsilon, \mathbb{R}^3)$ and $\frac{1}{\varepsilon} v_{\varepsilon,3}|_{\partial_t \omega_\varepsilon} = b_0(x_p)$. Thus, applying again Poincaré Inequality and Rellich's Theorem, we get the existence of a function $b \in W^{1,p}(\omega \times (-\gamma, \gamma), \mathbb{R}^3)$ and of a subsequence of $\{\varepsilon_k\}$, still denoted by $\{\varepsilon_k\}$, such that

$$\begin{cases} \dfrac{1}{\varepsilon_k} v_{\varepsilon_k,3} \rightharpoonup b \text{ in } W^{1,p}(\omega \times (-\gamma, \gamma), \mathbb{R}^3), \\ \dfrac{1}{\varepsilon_k} v_{\varepsilon_k,3}(x_p, \pm\gamma) \to b(x_p, \pm\gamma) \text{ in } L^p(\omega, \mathbb{R}^3), \end{cases}$$

and, in addition, as $b_{,3} = 0$, $b - b_0 \in W^{1,p}(\omega, \mathbb{R}^3)$. Thus, as in (5.16), we obtain

$$\begin{aligned} \int_{\omega_{\varepsilon k}} \left| \frac{1}{\varepsilon_k} v_{\varepsilon_k,3} - b \right|^p dx_p dx_3 &\leq \int_{\omega \times (-\gamma,\gamma)} \left| \frac{1}{\varepsilon_k} v_{\varepsilon_k,3} - b \right|^p dx_p dx_3 \\ + C \Bigg\{ \int_\omega \left| \frac{1}{\varepsilon_k} v_{\varepsilon_k,3}(x_p, \pm\gamma) - b(x_p) \right|^p dx_p &+ \int_{\omega_{\varepsilon k}} \left| \frac{1}{\varepsilon_k} v_{\varepsilon_k,3} \right|^p dx_p dx_3 \Bigg\} \end{aligned}$$

which, together with (5.14) and (5.15), yields the last relation in (5.13) and this concludes the proof. □

Next introduce a countable collection $\mathcal{C}$ of open subsets of ω such that for all $\delta > 0$ and $A \in \mathcal{A}(\omega)$ there exists a finite union C_A of disjoint elements of $\mathcal{C}$ such that

$$\begin{cases} \overline{C_A} \subset A, \\ \mathcal{L}^2(A) \leq \mathcal{L}^2(C_A) + \delta. \end{cases}$$

Denote by $\mathcal{R}$ the countable collection of all finite unions of elements of $\mathcal{C}$, i.e. $\mathcal{R} := \left\{\cup_{i=1}^k C_i : k \in \mathbb{N}, C_i \in \mathcal{C}\right\}$.

Since $W^{1,p}$ and L^p are separable metric spaces, by using a diagonal argument, and in the spirit of Γ-convergence (see [18] Proposition 7.9), we can assert that there exists a subsequence $\{\varepsilon^\mathcal{R}\} \subset \{\varepsilon\}$, $\varepsilon^\mathcal{R} \to 0$, such that, upon setting

$$\begin{aligned} J_{\{\varepsilon^\mathcal{R}\}}(u, b, A) := \inf_{v_{\varepsilon^\mathcal{R}}} \Bigg\{ &\liminf_{\varepsilon^\mathcal{R} \to 0} J_{\varepsilon^\mathcal{R}}(v_{\varepsilon^\mathcal{R}}, A) : v_{\varepsilon^\mathcal{R}} \text{ in } W^{2,p}(A_{\varepsilon^\mathcal{R}}, \mathbb{R}^3), (v_{\varepsilon^\mathcal{R}} - u)\chi_{A_{\varepsilon^\mathcal{R}}} \to 0 \\ &(Dv_{\varepsilon^\mathcal{R}} - Du)\chi_{A_{\varepsilon^\mathcal{R}}} \to 0 \text{ and } \left(\frac{1}{\varepsilon^\mathcal{R}} v_{\varepsilon^\mathcal{R},3} - b\right) \chi_{A_{\varepsilon^\mathcal{R}}} \to 0 \text{ in } L^p(\Omega, \mathbb{R}^3) \Bigg\}, \end{aligned} \tag{5.17}$$

for every $C \in \mathcal{R}$ and for every $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ and $b \in L^p(\Omega, \mathbb{R}^3)$ there exists a sequence $\{v^C_{\varepsilon^\mathcal{R}}\}$ in $W^{2,p}(C_{\varepsilon^\mathcal{R}})$ such that

$$\begin{cases} (v_{\varepsilon^\mathcal{R}} - u)\chi_{C_{\varepsilon^\mathcal{R}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ (Dv_{\varepsilon^\mathcal{R}} - Du)\chi_{C_{\varepsilon^\mathcal{R}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ \left(\frac{1}{\varepsilon^\mathcal{R}} v_{\varepsilon^\mathcal{R},3} - b\right) \chi_{C_{\varepsilon^\mathcal{R}}} \to 0 & \text{in } L^p(\Omega; \mathbb{R}^3), \\ J_{\{\varepsilon^\mathcal{R}\}}(u, b, C) = \lim_{\varepsilon^\mathcal{R} \to 0} J_{\varepsilon^\mathcal{R}}(v_{\varepsilon^\mathcal{R}}, C). \end{cases} \tag{5.18}$$

Next we seek to extend (5.18) to every A open subset of ω.

Proof of Theorem 5.1. The proof relies on the techniques of the Γ- convergence, and the structure follows step by step the proof of Theorem 2.5 in [20]. First it is observed that the test sequences can be chosen with prescribed boundary values. Then (5.3) is asserted, in the sense that $J_{\{\varepsilon^\mathcal{R}\}}(u, b, A)$ can be achieved. Finally, we prove that $J_{\{\varepsilon^\mathcal{R}\}}$ admits the integral representation (5.4), as it satisfies all the assumptions of Theorem 2.31.

Step 1. The next result shows that the approximating sequences $\{v_\varepsilon\}$ may take the boundary value $u + \varepsilon x_3 b$ on the lateral boundary A_ε. Actually, with no loss of generality, it can assumed the function b regular, since b can be approximated by smooth functions in $C_c^\infty(\omega)$.

Lemma 5.6. *Let A be an open subset of ω. Let $u \in W^{2,p}(\Omega, \mathbb{R}^3)$ and let $b \in W^{1,p}(\Omega, \mathbb{R}^3)$. If $\{\overline{\varepsilon}\} \subset \{\varepsilon^\mathcal{R}\}$ and $\{v_{\overline{\varepsilon}}\} \in W^{2,p}(A_{\overline{\varepsilon}}, \mathbb{R}^3)$ are such that*

$$\begin{cases} (v_{\overline{\varepsilon}} - u)\chi_{A_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ (Dv_{\overline{\varepsilon}} - Du)\chi_{A_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ \left(\frac{1}{\overline{\varepsilon}} v_{\overline{\varepsilon},3} - b\right) \chi_{A_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega; \mathbb{R}^3), \\ J_{\{\varepsilon^\mathcal{R}\}}(u, b, A) = \lim_{\overline{\varepsilon} \to 0} J_{\overline{\varepsilon}}(v_{\overline{\varepsilon}}, A), \end{cases} \tag{5.19}$$

then there exists a sequence $\{w_{\overline{\varepsilon}}\} \subset W^{1,p}(A_{\overline{\varepsilon}}, \mathbb{R}^3)$ *which satisfies (5.19) and is such that*

$$w_{\overline{\varepsilon}} = u + \overline{\varepsilon} x_3 b \text{ in } \{(x_p, x_3) : x_p \in A \backslash K^{\overline{\varepsilon}} \text{ and } |x_3| < f_{\overline{\varepsilon}}(x_p)\}$$

for some compact set $K_{\overline{\varepsilon}} \subset A$.

Proof. Set

$$C := \sup_{\overline{\varepsilon}} \int_{A_{\overline{\varepsilon}}} \left(1 + |D_p^2 v_{\overline{\varepsilon}}|^p + \frac{1}{\varepsilon^p}|D_p v_{\overline{\varepsilon},3}|^p + \frac{1}{\varepsilon^{2p}}|v_{\overline{\varepsilon},3,3}|^p\right) dx_p dx_3.$$

From (5.9) it results $C < \infty$. Define

$$K(\overline{\varepsilon}) := \left|\left[\frac{1}{\|v_\varepsilon - u - \varepsilon x_3 b\|^{\frac{1}{4}}_{L^p(A_{\overline{\varepsilon}})} + \|D_p v_\varepsilon - D(u + \varepsilon x_3 b)\|^{\frac{1}{4}}_{L^p(A_{\overline{\varepsilon}})} \|\frac{1}{\overline{\varepsilon}} v_{\overline{\varepsilon},3} - b\|^{\frac{1}{4}}_{L^p(A_{\overline{\varepsilon}})}}\right]\right| \tag{5.20}$$

where $|[a]|$ denotes the integer part of a, and $M(\overline{\varepsilon}) := |[\sqrt{K(\overline{\varepsilon})}]|$. In order to apply the De Giorgi's slicing argument, define

$$A(\overline{\varepsilon}) := \left\{x_p \in A : \operatorname{dist}(x_p, \partial A) < \frac{M(\overline{\varepsilon})}{K(\overline{\varepsilon})}\right\}$$

Note that, in view of (5.20), $K(\overline{\varepsilon}) \nearrow \infty$ while $\mathcal{L}^2(A(\overline{\varepsilon})) \searrow 0$ as $\overline{\varepsilon} \nearrow 0$. Subdivide $A(\overline{\varepsilon})$ into $M(\overline{\varepsilon})$ disjoint open subsets

$$A_i^{\overline{\varepsilon}} := \left\{x_p \in A : \operatorname{dist}(x_p, \partial A) \in \left[\frac{i}{K(\overline{\varepsilon})}, \frac{i+1}{K(\overline{\varepsilon})}\right)\right\}, i = 0, \dots, M(\overline{\varepsilon}) - 1.$$

Then, there exists $i(\overline{\varepsilon}) \in \{0, \dots, M(\overline{\varepsilon}) - 1\}$ such that

$$\int_{(A_i^{\overline{\varepsilon}})_{\overline{\varepsilon}}} \left(1 + |D_p^2 v_{\overline{\varepsilon}}|^p + \frac{1}{\varepsilon^p}|D_p v_{\overline{\varepsilon},3}|^p + \frac{1}{\varepsilon^{2p}}|v_{\overline{\varepsilon},3,3}|^p\right) dx_p dx_3 \le \frac{C}{M(\overline{\varepsilon}),} \tag{5.21}$$

where $(A_i^{\overline{\varepsilon}})_{\overline{\varepsilon}} := \left\{(x_p, x_3) : x_p \in A_i^{\overline{\varepsilon}}, |x_3| < f_{\overline{\varepsilon}}(x_p)\right\}$. Let $\phi(\overline{\varepsilon}) \in C_0^\infty(A)$ be such that

$$\begin{cases} 0 \le \phi(\overline{\varepsilon}) \le 1, \|D_p \phi(\overline{\varepsilon})\|_{L^\infty} \le 2K(\overline{\varepsilon}), \|D_p^2 \phi(\overline{\varepsilon})\|_{L^\infty} \le 2K^2(\overline{\varepsilon}), \\ \phi(\overline{\varepsilon}) = \begin{cases} 1, \text{ if } \operatorname{dist}(x_p, \partial A) > \frac{i(\overline{\varepsilon})+1}{K(\overline{\varepsilon})}, \\ 0, \text{ if } \operatorname{dist}(x_p, \partial A) \le \frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon}),} \end{cases} \end{cases} \tag{5.22}$$

and set

$$w_{\overline{\varepsilon}} := \phi(\overline{\varepsilon}) v_{\overline{\varepsilon}} + (1 - \phi(\overline{\varepsilon}))(u + \overline{\varepsilon} x_3 b). \tag{5.23}$$

It results $w_{\overline{\varepsilon}} \in W^{2,p}(A_{\overline{\varepsilon}}, \mathbb{R}^3)$ and $w_{\overline{\varepsilon}} = u + \overline{\varepsilon} x_3 b$ in $\{(x_p, x_3) : x_p \in A \backslash K^{\overline{\varepsilon}}$ and $|x_3| < f_{\overline{\varepsilon}}(x_p)\}$, where $K^{\overline{\varepsilon}}$ is defined as $\left\{x_p \in A : \operatorname{dist}(x_p, \partial A) \ge \frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon})}\right\}$. Moreover, from (5.19)

$$(w_{\overline{\varepsilon}} - u)\chi_{A_{\overline{\varepsilon}}} \to 0, (Dw_{\overline{\varepsilon}} - Du)\chi_{A_{\overline{\varepsilon}}} \to 0 \text{ and } \left(\frac{1}{\overline{\varepsilon}} w_{\overline{\varepsilon},3} - b\right) \chi_{A_{\overline{\varepsilon}}} \to 0 \text{ in } L^p(\Omega, \mathbb{R}^3). \tag{5.24}$$

From the bound from above (5.9) and (5.22), (5.23) we have

$$
\begin{aligned}
&J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)\\
&\geq \limsup_{\overline{\varepsilon}\to 0}\int_{A_{\overline{\varepsilon}}\cap[\{x_p:\mathrm{dist}(x_p,\partial A)>\frac{i(\overline{\varepsilon})+1}{K(\overline{\varepsilon})}\}\times(-1,1)]} W_{\overline{\varepsilon}}\left(x_p,x_3,D_p^2 v_{\overline{\varepsilon}},\frac{1}{\overline{\varepsilon}}D_p v_{\overline{\varepsilon},3},\frac{1}{\overline{\varepsilon}^2}v_{\overline{\varepsilon},3,3}\right)dx_p dx_3\\
&\geq \limsup_{\overline{\varepsilon}\to 0}\left\{\int_{A_{\overline{\varepsilon}}} W_{\overline{\varepsilon}}\left(x_p,x_3,D_p^2 w_{\overline{\varepsilon}},\frac{1}{\overline{\varepsilon}}D_p w_{\overline{\varepsilon},3},\frac{1}{\overline{\varepsilon}^2}w_{\overline{\varepsilon},3,3}\right)dx_p dx_3\right.\\
&\quad -C\int_{A_{\overline{\varepsilon}}\cap[\{x_p:\mathrm{dist}(x_p,\partial A)<\frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon})}\}\times(-1,1)]}\left(1+|D_p^2 u+\varepsilon x_3 D_p^2 b|^p+|D_p b|^p\right)dx_p dx_3\\
&\quad -\beta\int_{(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}}}\left(1+|D_p^2 v_{\overline{\varepsilon}}|^p+\frac{1}{\varepsilon^p}|D_p v_{\overline{\varepsilon},3}|^p+\frac{1}{\varepsilon^{2p}}|v_{\overline{\varepsilon},3,3}|^p\right)dx_p dx_3\\
&\quad -C|K(\overline{\varepsilon})|^{2p}\int_{(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}}}|v_{\overline{\varepsilon}}-u-\overline{\varepsilon}x_3 b|^p dx_p dx_3-\\
&\quad -C|K(\overline{\varepsilon})|^{p}\int_{(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}}}\left(|D_p v_{\overline{\varepsilon}}-D_p(u+\overline{\varepsilon}x_3 b)|^p+\left|\frac{1}{\overline{\varepsilon}}v_{\overline{\varepsilon},3}-b\right|^p\right)dx_p dx_3\\
&\geq \limsup_{\overline{\varepsilon}\to 0} J_{\overline{\varepsilon}}(w_{\overline{\varepsilon}},A)-C\liminf_{\overline{\varepsilon}\to 0}\mathcal{L}^2(A(\overline{\varepsilon}))-C\beta\liminf_{\overline{\varepsilon}\to 0}\frac{1}{M(\overline{\varepsilon})}\\
&\quad -C'\liminf_{\overline{\varepsilon}\to 0}\|v_{\overline{\varepsilon}}-u-\overline{\varepsilon}x_3 b\|^{\frac{p}{2}}_{L^p(A(\overline{\varepsilon}))}-C'\liminf_{\overline{\varepsilon}\to 0}\|D_p v_{\overline{\varepsilon}}-D_p(u+\overline{\varepsilon}x_3 b)\|^{\frac{3p}{4}}_{L^p(A(\overline{\varepsilon}))}\\
&\quad -C'\liminf_{\overline{\varepsilon}\to 0}\|\frac{1}{\overline{\varepsilon}}v_{\overline{\varepsilon},3}-b\|^{\frac{3p}{4}}_{L^p(A(\overline{\varepsilon}))}=\limsup_{\overline{\varepsilon}\to 0} J_{\overline{\varepsilon}}(w_{\overline{\varepsilon}},A),
\end{aligned}
\tag{5.25}
$$

where (5.20) and (5.21) have been used to obtain the last inequality. Further, from (5.3) and (5.24) we have

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)\leq \liminf_{\overline{\varepsilon}\to 0} J_{\overline{\varepsilon}}(w_{\overline{\varepsilon}},A)$$

which, together with (5.25), completes the proof. □

Step 2. Let A be an open subset of ω, $u\in W^{2,p}(\omega,\mathbb{R}^3)$ and $b\in W^{1,p}(\omega,\mathbb{R}^3)$. We prove that there exists a sequence $\{v_{\varepsilon^{\mathcal{R}}}^{C^\delta(\varepsilon^{\mathcal{R}})}\}$ converging in the sense of (5.18) for which $J_{\{\overline{\varepsilon}^{\mathcal{R}}\}}(u,b,A)$ is attained.

Fix $\delta>0$ and choose a subset C^δ of A in $\mathcal{R}$ such that

$$\begin{cases}\overline{C^\delta}\subset A,\\ \int_{A\setminus C^\delta}\left(1+\left|D_p^2 u\right|^p+|D_p b|^p\right)dx\leq \frac{\delta}{2\beta}.\end{cases}$$

Consider a sequence $\{v_{\varepsilon^{\mathcal{R}}}^{C^\delta}\}$ in $W^{2,p}(C^\delta)$ satisfying

$$\lim_{\varepsilon^{\mathcal{R}}\to 0^+} J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}^{C^\delta},C^\delta)=J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,C^\delta).$$

Without loss of generality we may assume that b is a smooth function, thus by Lemma 5.6 we can suppose that $v_{\varepsilon^{\mathcal{R}}}^{C^\delta}=u+\varepsilon^{\mathcal{R}}x_3 b$ nearby ∂C^δ, and extend $v_{\varepsilon^{\mathcal{R}}}^{C^\delta}$ as $u+\varepsilon^{\mathcal{R}}x_3 b$ outside C^δ, so that the extension (not relabelled) $v_{\varepsilon^{\mathcal{R}}}^{C^\delta}\in W^{2,p}(A_{\varepsilon^{\mathcal{R}}})$ and it is still admissible. Since $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,C^\delta)\leq J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)$, for every $\delta>0$ we have

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+} J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}^{C^\delta},A)\\
&\leq \limsup_{\delta\to 0^+}\lim_{\varepsilon^{\mathcal{R}}\to 0^+}\left\{J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}^{C^\delta},C^\delta)+C\int_{A_{\varepsilon^{\mathcal{R}}}-C^\delta}\left(1+\left|D_p^2 u\right|^p+|D_p b|^p\right)dx\right\}\\
&=\limsup_{\delta\to 0^+} J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,C^\delta)\leq J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)\leq \liminf_{\delta\to 0^+}\liminf_{\varepsilon^{\mathcal{R}}\to 0^+} J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}^{C^\delta},A).
\end{aligned}
$$

a diagonalization result (cf. Lemma 7.1 [20]) concludes the proof, since there exists a decreasing sequence

$\{\delta(\varepsilon^{\mathcal{R}})\} \searrow 0$ such that

$$\begin{cases} (v_{\varepsilon^{\mathcal{R}}}^{C^\delta(\varepsilon^{\mathcal{R}})} - u)\chi_{A_\varepsilon^{\mathcal{R}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ (Dv_{\varepsilon^{\mathcal{R}}}^{C^\delta(\varepsilon^{\mathcal{R}})} - Du)\chi_{A_\varepsilon^{\mathcal{R}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ \left(\frac{1}{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}},3}^{C^\delta(\varepsilon^{\mathcal{R}})} - b\right)\chi_{A_\varepsilon^{\mathcal{R}}} \to 0 & \text{in } L^p(\Omega; \mathbb{R}^3), \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A) = \lim_{\varepsilon^{\mathcal{R}} \to 0} J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}^{C^\delta(\varepsilon^{\mathcal{R}})}, A). \end{cases} \tag{5.26}$$

Step 3. We prove that $J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, \cdot)$ is a measure. To this end, first we can observe that from the proof of Step 2, for $u \in W^{2,p}(\omega, \mathbb{R}^3)$ and $b \in W^{1,p}(\omega, \mathbb{R}^3)$, $J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, \cdot)$ is inner regular. Thus, let $\delta > 0$, and find $C^\delta \in \mathcal{R}$ such that

$$\begin{cases} \overline{C}^\delta \subset A, \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A) \le J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, C^\delta) + \delta. \end{cases} \tag{5.27}$$

Since from (5.2) and (5.26), it results

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A) \ge J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A \backslash \overline{C}^\delta) + J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, C^\delta),$$

from (5.27) we obtain

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A \backslash \overline{C}^\delta) \le \delta. \tag{5.28}$$

Next we prove that $J_{\{\varepsilon^{\mathcal{R}}\}}$ is subadditive. Indeed let A, B, C be open subsets in ω such that $C \subset\subset B \subset A$. We have to show that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A) \le J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, B) + J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A \backslash \overline{C}). \tag{5.29}$$

To this end, consider an open set $D \subset\subset A \backslash \overline{C}$. We can find two sequences $\{v_{\varepsilon^{\mathcal{R}}}^B\}$, $\{v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}}\}$ such that (5.26) is verified on B and $A \backslash \overline{C}$ respectively, and, in view of Lemma 5.6,

$$v_{\varepsilon^{\mathcal{R}}}^B = u + \varepsilon^{\mathcal{R}} x_3 b \text{ on } \partial_t B_{\varepsilon^{\mathcal{R}}} \text{ and } v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}} = u + \varepsilon^{\mathcal{R}} x_3 b \text{ on } \partial_t (A \backslash \overline{C})_{\varepsilon^{\mathcal{R}}}.$$

Next define the sequence of Radon measures

$$\begin{aligned} \lambda_{\varepsilon^{\mathcal{R}}} := \Big\{ 1 + |D_p^2 v_{\varepsilon^{\mathcal{R}}}^D|^p + |D_p^2 v_{\varepsilon^{\mathcal{R}}}^B|^p + \left(\frac{1}{\varepsilon^{\mathcal{R}}}\right)^p |D_p v_{\varepsilon^{\mathcal{R}},3}^D|^p \\ + \left(\frac{1}{\varepsilon^{\mathcal{R}}}\right)^p |D_p v_{\varepsilon^{\mathcal{R}},3}^B|^p + \left(\frac{1}{\varepsilon^{\mathcal{R}}}\right)^{2p} \left(|v_{\varepsilon^{\mathcal{R}},3,3}^B|^p + |v_{\varepsilon^{\mathcal{R}},3,3}^D|^p \right) \Big\} \chi_{((A - \overline{C})\backslash D)_{\varepsilon^{\mathcal{R}}}} \mathcal{L}^3 \end{aligned}$$

where $((A \backslash \overline{C}) \backslash D)_{\varepsilon^{\mathcal{R}}} = \{(x_p, x_3), x_p \in (A \backslash \overline{C}) \backslash D, |x_3| < f_{\varepsilon^{\mathcal{R}}}(x_p)\}$. From (5.9) $\{\lambda_{\varepsilon^{\mathcal{R}}}\}$ is a bounded sequence of finite Radon measures on $\mathbb{R}^3$, hence there exists a subsequence $\{\overline{\varepsilon}\}$ of $\{\varepsilon^{\mathcal{R}}\}$ and a finite nonnegative Radon measure λ such that

$$\lambda_{\overline{\varepsilon}} \rightharpoonup^* \lambda \text{ weakly*- in the sense of measures.}$$

Define $\widehat{\lambda}(X) := \lambda(X \times [-1, 1])$ for any Borel subset X of ω, and, for $0 < \eta < 1$, set

$$S_\eta := \{x \in (A - \overline{C}) \backslash D : \mathrm{dist}(x_p, \partial(A \backslash \overline{C})) = \eta\},$$

The sets S_η are pairwise disjoint for every η and there is $\eta \in (0, 1)$ such that

$$\widehat{\lambda}(S_{\eta_0}) = 0.$$

Let L_ζ be a layer around S_{η_0}, i.e.,

$$L_\zeta := \{x_p \in (A \backslash \overline{C}) \backslash D : \mathrm{dist}(x_p, S_{\eta_0}) \le \zeta\}.$$

Consider a smooth cut off function $\phi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} \|\phi\|_{L^\infty} \le 1, \|D_p \phi\|_{L^\infty} \le \dfrac{C}{\mathrm{dist}(D, A \backslash \overline{C})}, \|D_p^2 \phi\|_{L^\infty} \le \dfrac{C}{\mathrm{dist}^2(D, A \backslash \overline{C})} \\ \phi = \begin{cases} 1 \text{ if } x_p \in D, \\ 0 \text{ if } x_p \notin A \backslash \overline{C}. \end{cases} \end{cases} \tag{5.30}$$

Setting

$$v_{\varepsilon^{\mathcal{R}}} := \phi v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}} + (1-\phi)v_{\varepsilon^{\mathcal{R}}}^{B},$$

we have $v_{\varepsilon^{\mathcal{R}}} \in W^{2,p}(A_{\varepsilon^{\mathcal{R}}}, \mathbb{R}^3)$ and

$$(v_{\varepsilon^{\mathcal{R}}} - u)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0, (Dv_{\varepsilon^{\mathcal{R}}} - Du)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 \text{ and } \left(\frac{1}{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}},3} - b\right)\chi_{A_{\varepsilon^{\mathcal{R}}}} \text{ in } L^p(\Omega, \mathbb{R}^3).$$

Hence, from (5.3) and (5.30) it results

$$\begin{aligned}
&J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) \leq \liminf_{\varepsilon^{\mathcal{R}}\to 0} J_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}}, A) \leq J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,B) + J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A\backslash\overline{C}) \\
&+\beta \limsup_{\varepsilon^{\mathcal{R}}\to 0} \lambda_{\varepsilon^{\mathcal{R}}}(L_\zeta \times(-1,1)) \\
&+\beta \frac{C}{\operatorname{dist}^p(D, \partial(A\backslash\overline{C}))} \limsup_{\varepsilon^{\mathcal{R}}\to 0} \int_{((A\backslash\overline{C})\backslash D)_{\varepsilon^{\mathcal{R}}}} \left(|D_p v_{\varepsilon^{\mathcal{R}}}^{B} - D_p v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}}|^p + \frac{1}{\varepsilon^{\mathcal{R}p}} |v_{\varepsilon^{\mathcal{R}},3}^{B} - v_{\varepsilon^{\mathcal{R}},3}^{A\backslash\overline{C}}|^p\right) dx_p dx_3 \\
&+\beta \frac{C}{\operatorname{dist}^{2p}(D, \partial(A\backslash\overline{C}))} \limsup_{\varepsilon^{\mathcal{R}}\to 0} \int_{((A\backslash\overline{C})\backslash D)_{\varepsilon^{\mathcal{R}}}} |v_{\varepsilon^{\mathcal{R}}}^{B^\delta} - v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}}|^p dx_p dx_3.
\end{aligned}$$

The last two terms reduce to zero since

$$|v_{\varepsilon^{\mathcal{R}}}^{B} - v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}}|\chi_{((A\backslash\overline{C})\backslash D)_{\varepsilon}^{\mathcal{R}}} \leq |v_{\varepsilon^{\mathcal{R}}}^{B^\delta} - u|\chi_{B_{\varepsilon^{\mathcal{R}}}} + |v_{\varepsilon^{\mathcal{R}}}^{D^\delta} - u|\chi_{A\backslash\overline{C}_{\varepsilon^{\mathcal{R}}}},$$

$$|D_p v_{\varepsilon^{\mathcal{R}}}^{B} - D_p v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}}|\chi_{((A\backslash\overline{C})\backslash D)_{\varepsilon^{\mathcal{R}}}} \leq |D_p v_{\varepsilon^{\mathcal{R}}}^{B} - D_p u|\chi_{B^\delta_{\varepsilon^{\mathcal{R}}}} + |D_p v_{\varepsilon^{\mathcal{R}}}^{A\backslash\overline{C}} - D_p u|\chi_{(A\backslash\overline{C})_{\varepsilon^{\mathcal{R}}}}$$

and

$$\frac{1}{\varepsilon^{\mathcal{R}}} |v_{\varepsilon^{\mathcal{R}},3}^{B} - v_{\varepsilon^{\mathcal{R}},3}^{A\backslash\overline{C}}|\chi_{((A\backslash\overline{C})\backslash D)_{\varepsilon^{\mathcal{R}}}} \leq \left|\frac{1}{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}},3}^{B} - b\right|\chi_{B_{\varepsilon^{\mathcal{R}}}} + \left|\frac{1}{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}},3}^{A\backslash\overline{C}} - b\right|\chi_{(A\backslash\overline{C})_{\varepsilon^{\mathcal{R}}}}.$$

As

$$\limsup_{\zeta\to 0} \limsup_{\varepsilon^{\mathcal{R}}\to 0} \lambda_{\varepsilon^{\mathcal{R}}}(L_\zeta \times(-1,1)) \leq \limsup_{\zeta\to 0} \hat{\lambda}(\bar{L_\zeta}) \leq \hat{\lambda}(S_{\eta_0}) = 0$$

we can let first $\varepsilon^{\mathcal{R}} \to 0$ and then $\zeta \to 0$ to get

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) \leq J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,B) + J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A\backslash\overline{C}),$$

which proves (5.29).

The definition (5.3) of $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,\omega)$ implies the existence of a subsequence $\{\overline{\varepsilon}\}$ of $\{\varepsilon^{\mathcal{R}}\}$ and of an associated subsequence $\{v_{\overline{\varepsilon}}\}$ in $W^{2,p}(\omega_{\overline{\varepsilon}}, \mathbb{R}^3)$ such that

$$\begin{cases} (v_{\overline{\varepsilon}} - u)\chi_{\omega_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ (Dv_{\overline{\varepsilon}} - Du)\chi_{\omega_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega, \mathbb{R}^3), \\ \left(\frac{1}{\overline{\varepsilon}} v_{\overline{\varepsilon},3} - b\right)\chi_{\omega_{\overline{\varepsilon}}} \to 0 & \text{in } L^p(\Omega; \mathbb{R}^3), \\ J_{\{\overline{\varepsilon}\}}(u,b,\omega) = \lim_{\overline{\varepsilon}\to 0} J_{\varepsilon^{\mathcal{R}}}(v_{\overline{\varepsilon}}, \omega). \end{cases} \tag{5.31}$$

Up to the choice of a subsequence, still denoted by $\{\overline{\varepsilon}\}$, there exists a Radon measure μ such that

$$W_{\overline{\varepsilon}}\left(x_p, x_3, D_p^2 v_{\overline{\varepsilon}}, \frac{1}{\overline{\varepsilon}} D_p v_{\overline{\varepsilon},3}, \frac{1}{\overline{\varepsilon}^2} v_{\overline{\varepsilon},3,3}\right)\chi_{\omega_{\overline{\varepsilon}}}\mathcal{L}^3 \overset{*}{\rightharpoonup} \mu \tag{5.32}$$

Let $X \subset \mathbb{R}^2$ and define $\widehat{\mu}(X) := \mu(X\times[-1,1])$. Then (5.31) and (5.32) imply that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,\omega) \geq \widehat{\mu}(\mathbb{R}^2), \tag{5.33}$$

while, for each open subset $A \subset \omega$ we have

$$\begin{aligned}
J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) &\leq \liminf_{\overline{\varepsilon}\to 0} J_{\overline{\varepsilon}}(v_{\overline{\varepsilon}}, A) = \liminf_{\overline{\varepsilon}\to 0} \int_{A_{\overline{\varepsilon}}} W_{\overline{\varepsilon}}\left(x_p, x_3, D_p^2 v_{\overline{\varepsilon}}, \frac{1}{\overline{\varepsilon}} D_p v_{\overline{\varepsilon},3}, \frac{1}{\overline{\varepsilon}^2} v_{\overline{\varepsilon},3,3}\right) dx_p dx_3 \\
&\leq \mu(A\times[-1,1]) = \widehat{\mu}(\overline{A}).
\end{aligned}$$

Thus, in view of (5.28), (5.29), (5.33), (5.32), Lemma 7.3 in [20] permits to conclude that $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,\cdot)$ is the trace on the open subsets of ω of a finite nonnegative Radon measure. Moreover, the bound from above in (5.9) guarantees that it is absolutely continuous with respect to the $\mathcal{L}^2|_\omega$.

Step 4. In order to apply Theorem 2.31 it can be easily verified that the functional $J_{\{\varepsilon^{\mathcal{R}}\}}$ maps any triple $(u,b,A), u \in W^{2,p}(\omega,\mathbb{R}^3), b \in W^{1,p}(\omega,\mathbb{R}^3)$, and A any open subset of ω, into $\overline{\mathbb{R}}$, and

(i) $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) = J_{\{\varepsilon^{\mathcal{R}}\}}(v,d,A)$ whenever $u=v$ and $b=d$ a.e. on A,

(ii) $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,\cdot)$ is a finite nonnegative Radon measure,

(iii) $C'\beta' \int_A (|D_p^2 u|^p + |D_p b|^p) dx_p \le J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) \le C\beta \int_A (1 + |D_p^2 u|^p + |D_p b|^p) dx_p$,

(iv) $J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)$ is $W^{2,p}(A) \times W^{1,p}(A)$ weakly lower semicontinuous.

Property (iii) follows from (5.11) and the lower weak semicontinuity of the norm in the Sobolev spaces, while (iv) is a consequence of (5.9) and it can be easily verified through a diagonal argument. Indeed, let $\{(u_n,b_n)\} \in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ converging weakly to $(u,b) \in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$. For any $n \in \mathbb{N}$ consider $\{w^n_{\varepsilon^{\mathcal{R}}}\} \subset W^{2,p}(A_{\varepsilon^{\mathcal{R}}})$ such that (5.3) holds for $J_{\varepsilon^{\mathcal{R}}(n)}(u_n,b_n,A)$. It can be constructed a diagonal sequence $\{w^n_{\varepsilon^{\mathcal{R}}(n)}\}$ such that

$$\begin{array}{ll} (w^n_{\varepsilon^{\mathcal{R}}(n)} - u)\chi_{A_{\varepsilon^{\mathcal{R}}(n)}} \to 0 & \text{in } L^p(\Omega;\mathbb{R}^3), \\ (Dw^n_{\varepsilon^{\mathcal{R}}(n)} - Du)\chi_{A_{\varepsilon^{\mathcal{R}}(n)}} \to 0 & \text{in } L^p(\Omega;\mathbb{R}^3), \\ \left(\frac{1}{\varepsilon^{\mathcal{R}}(n)} w^n_{\varepsilon^{\mathcal{R}}(n),3} - b\right)\chi_{A_{\varepsilon^{\mathcal{R}}(n)}} \to 0 & \text{in } L^p(\Omega;\mathbb{R}^3), \end{array}$$

and

$$\liminf_n J_{\varepsilon^{\mathcal{R}}(n)}(u_n,b_n,A) = \lim_n \int_{A_{\varepsilon^{\mathcal{R}}(n)}} W(D_p^2 w^n_{\varepsilon^{\mathcal{R}}(n)}, \frac{1}{\varepsilon^{\mathcal{R}}(n)} D_p w^n_{\varepsilon^{\mathcal{R}}(n),3}, \frac{1}{\varepsilon^{\mathcal{R}}(n)^2} w^n_{\varepsilon^{\mathcal{R}}(n),3,3}) dx.$$

$$\lim_n \int_{A_{\varepsilon^{\mathcal{R}}(n)}} W(D_p^2 w^n_{\varepsilon^{\mathcal{R}}(n)}, \frac{1}{\varepsilon^{\mathcal{R}}(n)} D_p w^n_{\varepsilon^{\mathcal{R}}(n),3}, \frac{1}{\varepsilon^{\mathcal{R}}(n)^2} w^n_{\varepsilon^{\mathcal{R}}(n),3,3}) dx \ge J_{\{\varepsilon^{\mathcal{R}}(n)\}}(u,b,A)$$

and since

$$J_{\{\varepsilon^{\mathcal{R}}(n)\}}(u,b,A) \ge J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)$$

the lower semicontinuity is asserted.

A direct application of Theorem 2.31 entails the existence of a function $W_{\{\varepsilon^{\mathcal{R}}\}}$ in Theorem 5.1 which still satisfies the growth condition (5.9). Moreover, since $J_{\{\varepsilon^{\mathcal{R}}\}}(u + c + Bx, b + d, A) = J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A)$ for every $c, d \in \mathbb{R}^3$, $B \in M^{3\times 2}$, the function $W_{\{\varepsilon^{\mathcal{R}}\}}$ depends just on x, on the second derivatives of u and on the first gradient of b. In order to see that the W_{ε^R} is Carathéodory is enough to use an argument similar to the one in Theorem 20.1 and Lemma 20.2 in [34].

□

Remark 5.7. For the sake of completeness, we also observe that, under the additional assumption (5.12), there is convergence (in the sense of (5.13)) of the minimizers (or almost minimizers) $\{(u_{\varepsilon^R}, \frac{1}{\varepsilon^R}\frac{\partial}{\partial x_3}u_{\varepsilon^R})\}$ to a pair (u,b) at which the functional $J_{\{\varepsilon^R\}}((\cdot,\cdot),\Omega)$ achieves the minimum. Indeed, we can observe that the sequence $\{(u_{\varepsilon^R}, \frac{1}{\varepsilon^R}\frac{\partial}{\partial x_3}u_{\varepsilon^R})\}$ is bounded, so up to a subsequence (not relabelled) it converges to (u,b) in the sense 5.13. Thus

$$J_{\{\varepsilon^R\}}((u,b),\Omega) \le \liminf_{\varepsilon^R \to 0} J_{\varepsilon^R}(u_{\varepsilon^R}, \Omega_{\varepsilon^R}) \le \lim_{\varepsilon^R \to 0} J_{\varepsilon^R}(v_{\varepsilon^R}, \Omega_{\varepsilon^R})$$

where $\{(v_{\varepsilon^R}, \frac{1}{\varepsilon^R} v_{\varepsilon^R,3})\}$ is any subsequence converging to a couple $(v,r) \in W^{2,p}(\omega,\mathbb{R}^3)\times W^{1,p}(\omega,\mathbb{R}^3)$ and for which the previous limit coincides with $J_{\{\varepsilon^R\}}((v,r),\Omega)$.

In the sequel we will establish some convexity properties of the energy density $W_{\{\varepsilon^{\mathcal{R}}\}}(x_p,\cdot,\cdot)$. Let $Q' := (0,1)^2$ be the unit cube in $\mathbb{R}^2$, and let T_2 be the 2- dimensional torus.

Given a function $v = (h,\xi) : T_2 \to E_2^3(\mathbb{R}^2) \times M^{3\times 2}$, consider the operator $\mathcal{A}$ given by

$$\mathcal{A}v := (\mathcal{A}_1 h, \mathcal{A}_2 \xi) \tag{5.34}$$

where

$$\mathcal{A}_1 h = \left(\frac{\partial h^i_{j1}}{\partial x_2} - \frac{\partial h^i_{j2}}{\partial x_1}\right), \quad i = 1,2,3, \quad j = 1,2,$$

and

$$\mathcal{A}_2 \xi = \left(\frac{\partial \xi^i_1}{\partial x_2} - \frac{\partial \xi^i_2}{\partial x_1}\right), \quad i = 1,2,3.$$

It is easy to verify (see [40] for details) that

$$\left\{h \in C^\infty(T_2; E^3_2(\mathbb{R}^2)) : \mathcal{A}_1 h = 0, \int_{T_2} h dx = 0\right\} = \left\{D^2_p u : u \in C^\infty(T_2, \mathbb{R}^3)\right\}.$$

In fact, for every $i = 1,2,3$, if $\mathcal{A}_1 h^i = 0$ then $h^i_{jk} = \frac{\partial w^i_j}{\partial x_k}$ for some functions $w^i_j \in C^\infty(Q_p, \mathbb{R}^6)$ with average zero. Note that w^i_j is periodic since h^i is periodic and $\int_{T_2} h^i dx = 0$. Then, by the symmetry of h^i_{jk} with respect to i and j, it results $\mathrm{curl} w^i = 0$ and we conclude that $h^i_{jk} = \frac{\partial^2 u^i}{\partial x_k \partial x_j}$ for some $u^i \in C^\infty(T_2; \mathbb{R})$. The operator is a constant rank operator. Indeed, for every $w \in S^1$ we have

$$\ker \mathbb{A}_1(w) = \left\{X \in E^3_2 : w_i X^l_{jk} - w_j X^l_{ik} = 0, i,j = 1,2, k = 1,2, l = 1,2,3\right\} = \left\{b \otimes w \otimes w, b \in \mathbb{R}^3\right\},$$

so $\dim \mathrm{Ker} \mathbb{A}_1(w) = 3$. Also

$$\left\{\xi \in C^\infty(T_2, M^{3\times 2}) : \mathbb{A}_2 \xi = 0, \int_{T_2} \xi dx = 0\right\} = \left\{D_p \varphi : \varphi \in C^\infty(T_2, \mathbb{R}^3)\right\},$$

and it is easy to see that $\mathbb{A}_2$ is a constant rank operator. In fact for every $w \in S^1$ it results

$$\begin{aligned}\mathrm{Ker}\mathbb{A}_2(w) &= \left\{V \in M^{3\times 2} : \mathbb{A}_2(w) V^l = 0, l = 1,2,3\right\} = \left\{w_i V^l_j - w_j V^l_i = 0, \ \ l = 1,2,3, \ \ i,j = 1,2\right\} \\ &= \left\{a \otimes w, \ \ a \in \mathbb{R}^3\right\}\end{aligned}$$

and $\dim \mathrm{Ker}\mathbb{A}_2(w) = 3$. It follows immediately that $\mathcal{A}$ is a constant rank operator, and for every $w \in S^1$,

$$\mathrm{Ker}\mathbb{A}(w) = \left\{(X,V) \in E^3_2 \times M^{3\times 2} : (X,V) = (b \otimes w^{\otimes 2}, a \otimes w), \, b \in \mathbb{R}^3, a \in \mathbb{R}^3\right\}, \tag{5.35}$$

where $w^{\otimes 2}$ stands for $w \otimes w$. For every $v \in E^3_2(\mathbb{R}^2) \times M^{3\times 2}$, with $v = (h, \xi)$, we have

$$Q_{\mathcal{A}} f(v) = \inf\left\{\int_{Q'} f(v + w(x)) dx \ : w \in C^\infty_{per}(\mathbb{R}^2; E^3_2 \times M^{3\times 2}) \cap \mathrm{Ker}\mathcal{A}, \int_{Q'} w dx = 0,\right\},$$

or, equivalently,

$$Q_{\mathcal{A}} f((h,\xi)) = \inf\left\{\int_{Q'} f((h + D^2_p u, \xi + D_p \varphi)) dx : \varphi \in C^\infty_0(Q'; \mathbb{R}^3), u \in C^\infty_0(Q', \mathbb{R}^3)\right\}. \tag{5.36}$$

Remark 5.8. Since $W_{\{\varepsilon^{\mathcal{R}}\}}(x_p, \cdot, \cdot)$ is the integrand of a weakly lower semicontinuous functional on $W^{1,p}(\omega, \mathbb{R}^3) \times L^p(\omega, \mathbb{R}^3)$, namely $J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A)$ in (5.18),(see *(iv)* in the proof of Theorem 5.1 above), and since the assumptions of Theorem 3.6 in [40] are fulfilled, the energy density $W_{\{\varepsilon^{\mathcal{R}}\}}(x_p, \cdot, \cdot)$ is $\mathcal{A}$-quasiconvex, where $\mathcal{A}$ is the operator introduced by (5.34).

We end this section by observing that the notion of $\mathcal{A}$- quasiconvexity determined by the operator in (5.34) entails fine continuity properties. In particular, the following result holds.

Proposition 5.9. *Let $W : E^3_2(\mathbb{R}^2) \times M^{3\times 2} \to \mathbb{R}$ be an $\mathcal{A}$-quasiconvex function, where $\mathcal{A}$ is the differential operator introduced in (5.34), then W is locally Lipschitz continuous.*

Proof. The proof is entirely similar to that of Theorem 2.8, and it relies on the observation that, in view of (5.35), in the cone $\Lambda = \cup_{w \in S^1} \ker \mathbb{A}(w)$ there are enough directions to generate $E^3_2(\mathbb{R}^2) \times M^{3\times 2}$. □

5.2 Homogeneous and inhomogeneous thin films

We start with the inhomogeneous setting. The homogeneous case will be obtained as a corollary. We denote by Q' the unit cube $\left(-\frac{1}{2}, \frac{1}{2}\right)^2$ in $\mathbb{R}^2$. Assume that the energy density $W_\varepsilon(x_1, x_2, x_3, H)$ does not depend neither on ε nor on the planar variables, i.e. $W_\varepsilon(x_1, x_2, x_3, H) \equiv W(x_3, H)$, where $W(x_3, H)$ is a Carathéodory function defined on $(-1,1)\times E^3_2(\mathbb{R}^3)$ such that

$$\beta'|H|^p \le W(x_3, H) \le \beta(1 + |H|^p),\ 0 < \beta' \le \beta < \infty, \text{ for a.e. } x_3 \in (-1, 1). \tag{5.37}$$

We consider a fixed profile, i.e. $f_\varepsilon(x_p) \equiv 1$, for every $x_p \in \omega$.

Theorem 5.1 states that for any sequence $\{\varepsilon\} \searrow 0$ there exists a subsequence $\{\varepsilon^{\mathcal{R}}\} \searrow 0$ such that $J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A)$ defined in (5.17) is given by

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u, b, A) := 2 \int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_p, D^2_p u, D_p b) dx_p. \tag{5.38}$$

Thus it remains to identify the energy density $W_{\{\varepsilon^{\mathcal{R}}\}}$.

Proof of Theorem 5.2. Consider any sequence $\{\varepsilon\} \searrow 0$ and let $\{\varepsilon^{\mathcal{R}}\}$ be as (5.17) and (5.18). Fix $h \in E^3_2(\mathbb{R}^2)$ and $d \in M^{3\times 2}$ and let x_0 be a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot, h, d)$. Then

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) = \lim_{q\to\infty} q^2 \int_{Q'\left(x_0, \frac{1}{q}\right)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_p, h, d) dx_p, \tag{5.39}$$

where $Q'\left(x_0, \frac{1}{q}\right)$ is a cube in $\mathbb{R}^2$ centered at x_0 with side length $\frac{1}{q}$, with q large enough to ensure $Q'\left(x_0, \frac{1}{q}\right) \subset \omega$. By virtue of (5.38), (5.39) reduces to

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) = \lim_{q\to\infty} \frac{q^2}{2} J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x^T h x, dx, Q'\left(x_0, \frac{1}{q}\right)\right). \tag{5.40}$$

For q large enough, let $\{v^q_{\varepsilon^{\mathcal{R}}}\} \subset W^{2,p}\left(Q'\left(x_0, \frac{1}{q}\right)\times(-1,1), \mathbb{R}^3\right)$ be such that

$$\begin{cases} v^q_{\varepsilon^{\mathcal{R}}} \to 0 \text{ in } W^{1,p}\left(Q'\left(x_0, \dfrac{1}{q}\right)\times(-1,1), \mathbb{R}^3\right), \\ \dfrac{1}{\varepsilon^{\mathcal{R}}} v^q_{\varepsilon^{\mathcal{R}},3} \to 0 \text{ in } L^p\left(Q'\left(x_0, \dfrac{1}{q}\right)\times(-1,1), \mathbb{R}^3\right), \\ J_{\{\varepsilon^{\mathcal{R}}\}}\left(\dfrac{1}{2}x^T h x, dx, Q'\left(x_0, \dfrac{1}{q}\right)\right) = \\ \displaystyle\lim_{\varepsilon^{\mathcal{R}}\to 0} \int_{Q'\left(x_0, \frac{1}{q}\right)\times(-1,1)} W\left(x_3, h + D^2_p v^q_{\varepsilon^{\mathcal{R}}}, d + \frac{1}{\varepsilon^{\mathcal{R}}} D_p v^q_{\varepsilon^{\mathcal{R}},3}, \frac{1}{(\varepsilon^{\mathcal{R}})^2} v^q_{\varepsilon^{\mathcal{R}},3,3}\right) dx_p dx_3. \end{cases} \tag{5.41}$$

The existence of this subsequence $\{v^q_{\varepsilon^{\mathcal{R}}}\}$ is ensured by Theorem 5.1. Further, Lemma 5.6 guarantees that $v^q_{\varepsilon^{\mathcal{R}}}$ can be chosen equal to 0 in a neighborhood of $\partial Q'\left(x_0, \frac{1}{q}\right)\times(-1,1)$. Define

$$v_{q,\varepsilon^{\mathcal{R}}}(x_p, x_3) := q^2 v^q_{\varepsilon^{\mathcal{R}}}\left(x_0 + \frac{x_p}{q}, x_3\right),\ x_p \in Q'.$$

In view of (5.41), (5.40) can be written as

$$\begin{aligned} &W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) \\ &= \frac{1}{2} \lim_{q\to\infty} \lim_{\varepsilon^{\mathcal{R}}\to 0} \int_{Q'\times(-1,1)} W\left(x_3, h + D^2_p v_{q,\varepsilon^{\mathcal{R}}}, d + \frac{1}{q\varepsilon^{\mathcal{R}}} D_p v_{q,\varepsilon^{\mathcal{R}},3}, \frac{1}{(q\varepsilon^{\mathcal{R}})^2} v_{q,\varepsilon^{\mathcal{R}},3,3}\right) dx_p dx_3. \end{aligned} \tag{5.42}$$

Since $v_{q,\varepsilon^{\mathcal{R}}} = 0$ on $\partial Q'\times(-1,1)$, (5.42) becomes

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) \ge \underline{W}(h, d). \tag{5.43}$$

It remains to prove the opposite inequality to (5.43). Since W verifies (5.37), a simple density argument guarantees that for every $\eta > 0$ we can find $\lambda > 0$ and $\phi \in W^{2,\infty}(Q' \times (-1,1), \mathbb{R}^3)$, with $\phi = 0$ on $\partial Q' \times (-1,1)$, such that

$$\underline{W}(h,d) + \eta \geq \frac{1}{2} \int_{Q' \times (-1,1)} W(x_3, h + D_p^2 \phi, d + \lambda D_p \phi_{,3}, \lambda^2 \phi_{,3,3}) dx_p dx_3. \tag{5.44}$$

Set

$$v_{\varepsilon^{\mathcal{R}}}(x_p, x_3) := \frac{1}{2} x_p^t h x_p + \varepsilon^{\mathcal{R}} x_3 d x_p + (\lambda \varepsilon^{\mathcal{R}})^2 \phi\left(\frac{x_p}{\lambda \varepsilon^{\mathcal{R}}}, x_3\right),$$

where it has been assumed that ϕ is laterally extended by Q'-periodicity. Then

$$v_{\varepsilon^{\mathcal{R}}} \to \frac{1}{2} x_p^t h x_p \text{ in } W^{1,p}(\Omega, \mathbb{R}^3), \; \frac{1}{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}},3} \to d x_p \text{ in } L^p(\Omega, \mathbb{R}^3).$$

For each open subset A of ω it results that

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2} x_p^t h x_p, d x_p, A\right) &\leq \liminf_{\varepsilon^{\mathcal{R}} \to 0} \int_{A \times (-1,1)} W\left(x_3, D_p^2 v_{\varepsilon^{\mathcal{R}}}, \frac{1}{\varepsilon^{\mathcal{R}}} D_p v_{\varepsilon^{\mathcal{R}},3}, \frac{1}{\varepsilon^{\mathcal{R}2}} v_{\varepsilon^{\mathcal{R}},3,3}\right) dx_p dx_3 \\ = \liminf_{\varepsilon^{\mathcal{R}} \to 0} \int_{A \times (-1,1)} & W\left(x_3, h + D_p^2 \phi\left(\frac{x_p}{\lambda \varepsilon^{\mathcal{R}}}, x_3\right), d + \lambda D_p \phi_{,3}\left(\frac{x_p}{\lambda \varepsilon^{\mathcal{R}}}, x_3\right) \lambda^2 \phi_{,3,3}\left(\frac{x_p}{\lambda \varepsilon^{\mathcal{R}}}, x_3\right)\right) dx_p dx_3. \end{aligned} \tag{5.45}$$

Observe that the function $\int_{-1}^{1} W\left(x_3, h + D_p^2 \phi(\cdot, x_3), d + \lambda D_p \phi_{,3}(\cdot, x_3), \lambda^2 \phi_{,3,3}(\cdot, x_3)\right) dx_3$ belongs to $L^\infty(\mathbb{R}^2)$ and it is periodic, thus it converges weakly * to its average, and by virtue of (5.44), (5.45) becomes,

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2} x_p^t h x_p, d x_p, A\right) &\leq \mathcal{L}^2(A) \int_{-1}^{1} \int_{Q'} W\left(x_3, h + D_p^2 \phi, d + \lambda D_p \phi_{,3}, \lambda^2 \phi_{,3,3}\right) dx_p dx_3 \\ &\leq 2\mathcal{L}^2(A) \underline{W}(h,d) + 2\eta \mathcal{L}^2(A). \end{aligned}$$

Thus letting η tend to 0,

$$J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2} x_p^t h x_p, d x_p, A\right) \leq 2\mathcal{L}^2(A) \underline{W}(h,d),$$

which, in the light of Theorem 5.1, also reads as

$$\int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) dx_p \leq 2\mathcal{L}^2(A) \underline{W}(h,d).$$

Choosing $x_0 \in \omega$ a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot, h, d)$ and A to be a small ball centered at x_0 with vanishing radius, it follows that

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d) \leq \underline{W}(h,d).$$

We recall that in all the argument above the function $W_{\{\varepsilon^{\mathcal{R}}\}}(x_0, h, d)$ does not depend on the choice of the subsequence $\{\varepsilon^{\mathcal{R}}\}$, so one does not need to extract a subsequence from $\{\varepsilon\}$, and that concludes the proof. □

Remark 5.10. Remark 5.8 and Proposition 5.9 entail that the function defined by (5.5) is $\mathcal{A}$ quasiconvex and locally Lipschitz continuous.

Finally, we obtain a representation result in the homogeneous case, i.e. when W does not depend on x_3. We set $\overline{W}(h,d) := \inf_{c \in \mathbb{R}^3} W(h,d,c)$, with $h \in E_2^3(\mathbb{R}^2)$ and $d \in M^{3\times 2}$, and we recall that $Q_{\mathcal{A}}\overline{W}(h,d)$ is the $\mathcal{A}$ -quasiconvexification of $\overline{W}(\cdot,\cdot)$ introduced in (5.36).

Remark 5.11. If W does not depend on x_3, then

$$\underline{W}(h,d) = Q_{\mathcal{A}}\overline{W}(h,d) \text{ for every } h \in E_2^3(\mathbb{R}^2), d \in M^{3\times 2}.$$

As a consequence, it is possible to extend the results proved in the gradient case in [50] and [51] and recover for second order derivatives, in the quadratic case, the result proved in [17].

Indeed, by (5.5) and (5.36)

$$\underline{W}(h,d) \geq \inf_{\phi} \inf_{\lambda>0} \left\{ \frac{1}{2} \int_{-1}^{1} \int_{Q'} \overline{W}(h + D_p^2\phi, d + \lambda D_p \phi_{,3}) dx_p dx_3 : \phi \in W^{2,p}(Q' \times (-1,1), \mathbb{R}^3), \right.$$

$$\left. \phi = 0 \text{ on } \partial Q' \times (-1,1) \right\} = \frac{1}{2} \int_{-1}^{1} Q_{\mathcal{A}} \overline{W}(h,d) dx_3 = Q_{\mathcal{A}} \overline{W}(h,d),$$

thus $\underline{W}(h,d) \geq Q_{\mathcal{A}}\overline{W}(h,d)$. It remains to prove the opposite inequality. From (5.37) and the measurability selection criterion (cf. Theorem 1.2 p. 236 [37]), one can find functions $\phi^\eta, \psi^\eta, \xi^\eta$ in $W_0^{2,\infty}(Q', \mathbb{R}^3)$ such that

$$Q_{\mathcal{A}}\overline{W}(h,d) + \eta \geq \int_{Q'} W(h + D_p^2 \phi^\eta, d + D_p \psi^\eta, \xi^\eta) dx_p.$$

Extend $\phi^\eta, \psi^\eta, \xi^\eta$ Q'-periodically to $\mathbb{R}^2$ and set

$$\phi_n^\eta(x_p, x_3) := \frac{1}{n^2} \phi^\eta(nx_p) + \frac{1}{n^3} \psi^\eta(nx_p) x_3 + \frac{1}{n^4} x_3^t \xi^\eta(nx_p) x_3.$$

Thus $\phi_n^\eta \in W^{2,p}(Q' \times (-1,1), \mathbb{R}^3)$ with $\phi_n^\eta = 0$ on $\partial Q' \times (-1,1)$, and so

$$\begin{aligned}
\underline{W}(h,d) &\leq \frac{1}{2} \liminf_{n \to +\infty} \int_{Q' \times (-1,1)} W(h + D_p^2 \phi_n^\eta, d + n^2 D_p \phi_{n,3}^\eta, n^4 \phi_{n,3,3}^\eta) dx_p dx_3 \\
&= \frac{1}{2} \liminf_{n \to +\infty} \int_{Q' \times (-1,1)} W \left(h + D_p^2 \phi^\eta(nx_p) + \frac{1}{n} D_p^2 \psi^\eta(nx_p) \cdot x_3 + \frac{1}{2n^2} x_3^t \xi^\eta(nx_p) x_3, \right. \\
&\qquad \left. d + D_p \psi^\eta(nx_p) + \frac{1}{n^2} \xi^\eta(nx_p) x_3, \xi^\eta(nx_p) \right) dx_p dx_3 \\
&\leq \frac{1}{2} \liminf_{n \to +\infty} \int_{Q' \times (-1,1)} W \left(h + D_p^2 \phi^\eta(nx_p), d + D_p \psi^\eta(nx_p), \xi^\eta(nx_p) \right) dx_p dx_3,
\end{aligned}$$

where the uniform continuity of W has been used to derive the last inequality. Next observe that $W(h + D_p^2\phi^\eta(\cdot), d + D_p\psi^\eta(\cdot), \xi^\eta(\cdot))$ is a periodic function in $L^\infty(\mathbb{R}^2)$, thus it weakly-* converges to its average and we obtain

$$\underline{W}(h,d) \leq \int_{Q'} W(h + D_p^2 \phi^\eta, d + D_p \psi^\eta, \xi^\eta) dx_p dx_3 \leq Q_{\mathcal{A}} \overline{W}(h,d) + \eta,$$

and, to conclude, it suffices to let η converge to 0.

5.3 Periodic case

In this section we suppose that $W(x_p, x_3, H)$ is a Carathéodory function from $Q' \times (-1,1) \times E_2^3(\mathbb{R}^3)$ into $\mathbb{R}$ satisfying

$$\beta' |H|^p \leq W(x_p, x_3, H) \leq \beta(1 + |H|^p) \tag{5.46}$$

with $1 < p < \infty$, $\beta', \beta > 0$. W is extended by Q' periodicity to $\mathbb{R}^2 \times (-1,1) \times E_2^3(\mathbb{R}^3)$, thus we set

$$W_\varepsilon(x_p, x_3, H) := W\left(\frac{x_p}{\varepsilon}, x_3, H\right).$$

Let f be a continuous function from Q' into $[0,1]$, extended Q'- periodically to $\mathbb{R}^2$, with $0 < \gamma \leq \min f$ and we set

$$f_\varepsilon(x_p) := f\left(\frac{x_p}{\varepsilon}\right).$$

Remark 5.12. One can easily verify that the function W_{hom} defined by (5.7), satisfies the following relation

$$W_{\text{hom}}(h,d) = \inf_{t>0} V(t),$$

where $V(\cdot)$ introduced in (5.8) Further, in the definition (5.8) Dirichlet boundary conditions on the test functions can be replaced by periodic boundary conditions.

Proof of Theorem 5.3. Let $\{\varepsilon\}$ be a sequence $\searrow 0$. Theorem 5.1 guarantees the existence of a subsequence $\{\varepsilon^{\mathcal{R}}\}$ of $\{\varepsilon\}$ and of a Carathéodory function $W_{\{\varepsilon^{\mathcal{R}}\}}$ such that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u,b,A) = \int_A W_{\{\varepsilon^{\mathcal{R}}\}}(D^2u, Db)dx_p.$$

In order to show the independence of $W_{\{\varepsilon^{\mathcal{R}}\}}$ from x_p, we fix $(h,d) \in E_2^3(\mathbb{R}^2)\times M^{3\times 2}$ and consider x_0, y_0 Lebesgue points in ω for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot,h,d)$, so that

$$\begin{aligned} W_{\{\varepsilon^{\mathcal{R}}\}}(x_0,h,d) &= \lim_{\delta\to 0}\frac{1}{\delta^2}\int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_p,h,d)dx_p = \lim_{\delta\to 0}\frac{1}{\delta^2}J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, dx_p, Q'(x_0,\delta)\right), \\ W_{\{\varepsilon^{\mathcal{R}}\}}(y_0,h,d) &= \lim_{\delta\to 0}\frac{1}{\delta^2}\int_{Q'(y_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_p,h,d)dx_p = \lim_{\delta\to 0}\frac{1}{\delta^2}J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, dx_p, Q'(y_0,\delta)\right). \end{aligned} \tag{5.47}$$

According to Lemma 5.6, there exists a sequence $\{\gamma^\delta_{\varepsilon^{\mathcal{R}}}\}$ such that $\gamma^\delta_{\varepsilon^{\mathcal{R}}} = 0$ on $\{(x_p,x_3) : |x_3| < f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right), x_p \in \partial Q'(x_0,\delta)\}$,

$$\begin{cases} \gamma^\delta_{\varepsilon^{\mathcal{R}}}\chi_{Q'(x_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3), \\ D\gamma^\delta_{\varepsilon^{\mathcal{R}}}\chi_{Q'(x_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3), \\ \frac{1}{\varepsilon^{\mathcal{R}}}\gamma^\delta_{\varepsilon^{\mathcal{R}},3}\chi_{Q'(x_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3), \end{cases}$$

and

$$J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, dx_p, Q'(x_0,\delta)\right) = \lim_{\varepsilon^{\mathcal{R}}\to 0} J_{\varepsilon^{\mathcal{R}}}\left(\frac{1}{2}x_p^t h x_p + \varepsilon^{\mathcal{R}}x_3 d\cdot x_p + \gamma^\delta_{\varepsilon^{\mathcal{R}}}, Q'(x_0,\delta)\right). \tag{5.48}$$

where, as usual $Q'(x_0,\delta)_{\varepsilon^{\mathcal{R}}} := \left\{(x_p,x_3) : x_p \in Q'(x_0,\delta), |x_3| < f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right)\right\}$. Next we can define the vector $\tau_{\varepsilon^{\mathcal{R}}} \in \varepsilon^{\mathcal{R}}\mathbb{Z}^n$ as

$$(\tau_{\varepsilon^{\mathcal{R}}})_i := \varepsilon^{\mathcal{R}}\left|\left[\frac{(y_0-x_0)_i}{\varepsilon^{\mathcal{R}}}\right]\right|, \text{ for } i = 1,\dots,N.$$

Clearly $\tau_{\varepsilon^{\mathcal{R}}} \to y_0 - x_0$ as $\varepsilon^{\mathcal{R}} \to 0$. Consider

$$\phi^\delta_{\varepsilon^{\mathcal{R}}}(x_p,x_3) := \gamma^\delta_{\varepsilon^{\mathcal{R}}}(x_p - \tau_{\varepsilon^{\mathcal{R}}}, x_3)$$

where $\gamma^\delta_{\varepsilon^{\mathcal{R}}}$ has been extended by 0 to $[\mathbb{R}^2 - Q'(x_0,\delta)]_{\varepsilon^{\mathcal{R}}}$, and where

$$[\mathbb{R}^2 - Q'(x_0,\delta)]_{\varepsilon^{\mathcal{R}}} = \left\{(x_p,x_3) : x_p \in [\mathbb{R}^2 - Q'(x_0,\delta)], |x_3| \le f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right)\right\}.$$

Let $r > 1$ and take $\varepsilon^{\mathcal{R}}$ small enough to guarantee

$$Q'(y_0 - \tau_{e^{\mathcal{R}}}, \delta) \subset Q'(x_0, r\delta). \tag{5.49}$$

Since

$$\begin{cases} \phi^\delta_{\varepsilon^{\mathcal{R}}}\chi_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3), \\ D\phi^\delta_{\varepsilon^{\mathcal{R}}}\chi_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3) \\ \frac{1}{\varepsilon^{\mathcal{R}}}\phi^\delta_{\varepsilon^{\mathcal{R}},3}\chi_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3), \end{cases}$$

we have

$$\begin{aligned} &J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, dx_p, Q'(y_0,\delta)\right) \le \\ &\quad \liminf_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}, x_3, h + D_p^2\phi^\delta_{\varepsilon^{\mathcal{R}}}, d + \frac{1}{\varepsilon^{\mathcal{R}}}D_p\phi^\delta_{\varepsilon^{\mathcal{R}},3}, \frac{1}{\varepsilon^{\mathcal{R}2}}\phi^\delta_{\varepsilon^{\mathcal{R}},3,3}\right)dx_p dx_3 \\ &\le \liminf_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'(y_0-\tau_{\varepsilon^{\mathcal{R}}},\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_p+\tau_{\varepsilon^{\mathcal{R}}}}{\varepsilon^{\mathcal{R}}}, x_3, h + D_p^2\phi^\delta_{\varepsilon^{\mathcal{R}}}(x_p+\tau_{\varepsilon^{\mathcal{R}}},x_3), d + \frac{1}{\varepsilon^{\mathcal{R}}}D_p\phi^\delta_{\varepsilon^{\mathcal{R}},3}(x_p+\tau_{\varepsilon^{\mathcal{R}}},x_3),\right. \\ &\quad \left.\frac{1}{\varepsilon^{\mathcal{R}2}}\phi^\delta_{\varepsilon^{\mathcal{R}},3,3}(x_p+\tau_{\varepsilon^{\mathcal{R}}},x_3)\right)dx_p dx_3 \\ &\le \liminf_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'(x_0,r\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}, h + D_p^2\gamma^\delta_{\varepsilon^{\mathcal{R}}}(x_p,x_3), d + \frac{1}{\varepsilon^{\mathcal{R}}}D_p\gamma^\delta_{\varepsilon^{\mathcal{R}},3}(x_p,x_3), \frac{1}{\varepsilon^{\mathcal{R}2}}\gamma^\delta_{\varepsilon^{\mathcal{R}},3,3}\right)dx_p dx_3 \\ &\le J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, d\cdot x_p, Q'(x_0,\delta)\right) + 2\beta(1+|h|^p+|d|^p)\mathcal{L}^2(Q'(x_0,r\delta) - Q'(x_0,\delta)), \end{aligned}$$

where it has been used (5.49) and (5.48) and the periodicity of $W(\cdot, x_3)$. Letting $r \to 1$ we obtain

$$J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, d\cdot x_p, Q'(y_0,\delta)\right) \le J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, d\cdot x_p, Q'(x_0,\delta)\right),$$

and the reverse inequality can be shown in a similar way. Hence, in view of (5.47), we have

$$W_{\{\varepsilon^{\mathcal{R}}\}}(y_0,h,d) = W_{\{\varepsilon^{\mathcal{R}}\}}(x_0,h,d) =: W_{\{\varepsilon^{\mathcal{R}}\}}(h,d).$$

It remains to identify $W_{\{\varepsilon^{\mathcal{R}}\}}(h,d)$. Without loss of generality, we assume that $0 \in \omega$ and $Q' \subset \omega$, by virtue of Lemma 5.6, there exists a sequence $\{\psi_{\varepsilon^{\mathcal{R}}}\}$, such that $\psi_{\varepsilon^{\mathcal{R}}} = 0$ on $\{(x_p,x_3) : |x_3| < f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right), x_p \in \partial Q'\}$,

$$\begin{cases} \psi_{\varepsilon^{\mathcal{R}}}\chi_{Q'_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3),\\ D\psi_{\varepsilon^{\mathcal{R}}}\chi_{Q'_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3),\\ \frac{1}{\varepsilon^{\mathcal{R}}}\psi_{\varepsilon^{\mathcal{R}},3}\chi_{Q'_{\varepsilon^{\mathcal{R}}}} \to 0 & \text{in } L^p(\Omega,\mathbb{R}^3),\end{cases}$$

and

$$W_{\{\varepsilon^{\mathcal{R}}\}}(h,d) = J_{\varepsilon^{\mathcal{R}}}\left(\frac{1}{2}x_p^t h x_p, d\cdot x_p, Q'\right) = \lim_{\varepsilon^{\mathcal{R}}\to 0} J_{\varepsilon^{\mathcal{R}}}\left(\frac{1}{2}x_p^t h x_p + \varepsilon^{\mathcal{R}} x_3 d\cdot x_p + \psi_{\varepsilon^{\mathcal{R}}}, Q'\right).$$

Define

$$\phi_{\varepsilon^{\mathcal{R}}}(x_p,x_3) := \frac{1}{{\varepsilon^{\mathcal{R}}}^2}\psi_{\varepsilon^{\mathcal{R}}}(\varepsilon^{\mathcal{R}}x_p,x_3).$$

Then $\phi_{\varepsilon^{\mathcal{R}}} \in W^{2,p}([(0,\frac{1}{\varepsilon^{\mathcal{R}}})^2]^f,\mathbb{R}^3)$ and it agrees with 0 as soon as $x_p \in \partial\left(0,\frac{1}{\varepsilon^{\mathcal{R}}}\right)^2$, thus it is admissible as test function for $V\left(\frac{1}{\varepsilon^{\mathcal{R}}}\right)$, and we have

$$\begin{aligned} &W_{\hom}(h,d) \le\\ &\quad \limsup_{\varepsilon^{\mathcal{R}}\to 0} V\left(\frac{1}{\varepsilon^{\mathcal{R}}}\right) \le \limsup_{\varepsilon^{\mathcal{R}}\to 0} {\varepsilon^{\mathcal{R}}}^2 \int_{\left[\left(0,\frac{1}{\varepsilon^{\mathcal{R}}}\right)^2\right]^f} W\left(x_p,x_3,h+D_p^2\phi_{\varepsilon^{\mathcal{R}}}, d+D_p\phi_{\varepsilon^{\mathcal{R}},3},\phi_{\varepsilon^{\mathcal{R}},3,3}\right)dx_pdx_3\\ &= \limsup_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3,h+D_p^2\psi_{\varepsilon^{\mathcal{R}}}, d+\frac{1}{\varepsilon^{\mathcal{R}}}D\psi_{\varepsilon^{\mathcal{R}}}, \frac{1}{{\varepsilon^{\mathcal{R}}}^2}\psi_{\varepsilon^{\mathcal{R}},3,3}\right)dx_pdx_3 = W_{\{\varepsilon^{\mathcal{R}}\}}(h,d). \end{aligned} \tag{5.50}$$

Conversely, one can consider $\lambda_n \nearrow \infty$ such that $V(\lambda_n) \to \liminf_{t\nearrow\infty} V(t)$. For each n, take $\phi_n \in W^{2,p}(\{(0,\lambda_n)^2\times(-1,1) : |x_3| < f(x_p)\},\mathbb{R}^3)$ with $\phi_n = 0$ if $x_p \in \partial(0,\lambda_n)^2$ and such that

$$V(\lambda_n) + \frac{1}{\lambda_n^3} \ge \frac{1}{{\lambda_n}^2}\int_{[(0,\lambda_n)^2]^f} W(x_p,x_3,h+D_p^2\phi_n, d+D_p\phi_{n,3},\phi_{n,3,3})dx_pdx_3. \tag{5.51}$$

Set $\psi^n_{\varepsilon^{\mathcal{R}}} := {\varepsilon^{\mathcal{R}}}^2\phi_n\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3\right)$, where ϕ_n has been laterally extended by zero to $((|[\lambda_n+1]|)^2)^f$, and then to the whole of $\mathbb{R}^2$ by $(|[\lambda_n+1]|)^2$ -periodicity. Then, since $\psi^n_{\varepsilon^{\mathcal{R}}} \to 0$ as $\varepsilon^{\mathcal{R}} \to 0$,

$$\begin{aligned} &J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, dx_p, Q'\right) \le \liminf_{\varepsilon^{\mathcal{R}}\to 0} J_{\varepsilon^{\mathcal{R}}}\left(\frac{1}{2}x_p^t h x_p + \varepsilon^{\mathcal{R}}x_3 d\cdot x_p + \psi^n_{\varepsilon^{\mathcal{R}}}, Q'\right)\\ &= \liminf_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3,h+D_p^2\psi^n_{\varepsilon^{\mathcal{R}}}, d+\frac{1}{\varepsilon^{\mathcal{R}}}D_p\psi^n_{\varepsilon^{\mathcal{R}},3},\frac{1}{{\varepsilon^{\mathcal{R}}}^2}\psi^n_{\varepsilon^{\mathcal{R}},3,3}\right)dx_pdx_3\\ &= \liminf_{\varepsilon^{\mathcal{R}}\to 0}\int_{Q'}\int_{-f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right)}^{f\left(\frac{x_p}{\varepsilon^{\mathcal{R}}}\right)} W\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3,h+D_p^2\phi_n\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3\right), d+D_p\phi_{n,3}\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3\right),\ \phi_{n,3,3}\left(\frac{x_p}{\varepsilon^{\mathcal{R}}},x_3\right)\right)dx_pdx_3\\ &= \frac{1}{(|[\lambda_n]|+1)^2}\int_{(0,\lambda_n)^2}\left[\int_{-f(x_p)}^{f(x_p)} W(x_p,x_3,h+D_p^2\phi_n(x_p,x_3), d+D_p\phi_{n,3}(x_p,x_3),\phi_{n,3,3}(x_p,x_3))dx_3\right]dx_p\\ &\quad + \frac{1}{(|[\lambda_n]|+1)^2}\int_{[(0,|[\lambda_n]|+1)\setminus(0,\lambda_n)^2]^f} W(x_p,x_3,h,d,0)dx_pdx_3 \le \frac{\lambda_n^2}{(|[\lambda_n]|+1)^2}\left(V(\lambda_n)+\frac{1}{\lambda_n^3}\right) + o\left(\frac{1}{\lambda_n}\right), \end{aligned}$$

where it has been used (5.51) as well as the $(|[\lambda_n]|+1)^2$ periodicity of

$$\int_{f(\cdot)}^{f(\cdot)} W(\cdot,x_3,h+D_p^2\phi_n(\cdot,x_3), d+D_p\phi_{n,3}(\cdot,x_3),\phi_{n,3,3}(\cdot,x_3))dx_3.$$

So, letting n tend to ∞,

$$J_{\{\varepsilon^{\mathcal{R}}\}}\left(\frac{1}{2}x_p^t h x_p, d \cdot x_p, Q'\right) \leq \liminf_{t\to\infty} V(t),$$

i.e.

$$W_{\{\varepsilon^{\mathcal{R}}\}}(h,d) \leq \liminf_{t\to\infty} V(t). \tag{5.52}$$

From (5.50) and (5.52), we get

$$\liminf_{t\nearrow\infty} V(t) \leq \limsup_{\varepsilon^{\mathcal{R}}\to 0} V\left(\frac{1}{\varepsilon^{\mathcal{R}}}\right) \leq W_{\{\varepsilon^{\mathcal{R}}\}}(h,d) \leq \liminf_{t\nearrow\infty} V(t),$$

which proves the desired result.

The independence of the adopted arguments on the subsequence $\{\varepsilon^{\mathcal{R}}\}$ concludes the proof. $\square$

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