ON LOWER SEMICONTINUITY AND RELAXATION

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ABSTRACT. Lower semicontinuity and relaxation results in BV are obtained for multiple integrals

$$F(u,\Omega) := \int_\Omega f(x,u(x),\nabla u(x))\,dx, \quad u \in W^{1,1}(\Omega;\mathbb{R}^d),$$

where the energy density f satisfies lower semicontinuity conditions with respect to (x,u) and is not subjected to coercivity hypotheses. These results call for methods recently developed in the Calculus of Variations.

§1. Introduction.

In this paper we address lower semicontinuity and relaxation properties for multiple functionals of the form

$$F(u,\Omega) := \int_\Omega f(x,u(x),\nabla u(x))\,dx,$$

where Ω is an open subset of $\mathbb{R}^N$, and $u(x)$ is a $\mathbb{R}^d$–valued function defined on Ω.

In [30] Serrin considered the scalar–valued case where

$$f \in C(\Omega\times\mathbb{R}\times\mathbb{R}^N;[0,\infty)) \quad \text{and} \quad f(x,u,\cdot) \quad \text{is convex in } \mathbb{R}^N. \tag{1.1}$$

Among his results we select Theorems A and B below.

Theorem A. (cf. [30, Theorem 11]) *Assume that f satisfies* (1.1). *Let $u \in BV_{\text{loc}}(\Omega;\mathbb{R})$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\text{loc}}(\Omega;\mathbb{R})$ converging to u in $L^1_{\text{loc}}(\Omega;\mathbb{R})$. Let λ, ρ be moduli of continuity such that*

(i) *$\rho(s) \le C\,s$ for $C>0$ and all $s>0$ large, and*

$$|f(x,u,\xi) - f(x_0,u_0,\xi)| \le \lambda(|x-x_0|)(1+f(x,u,\xi)) + \rho(|u-u_0|)$$

for all (x,u), $(x_0,u_0) \in \Omega\times\mathbb{R}$, and for all $\xi\in\mathbb{R}^N$;

or

(ii) *$|f(x,u,\xi) - f(x_0,u_0,\xi)| \le \lambda(|x-x_0|+|u-u_0|)(1+f(x,u,\xi))$ for all (x,u), $(x_0,u_0)\in\Omega\times\mathbb{R}$, and for all $\xi\in\mathbb{R}^N$. Suppose, in addition, that $u(x)$ is continuous.*

Then

$$\int_\Omega f(x,u(x),\nabla u(x))\,dx \le \liminf_{n\to\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx.$$

Here ∇u is the Radon–Nikodym derivative of the distributional derivative Du of u, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$. Also, we intend by *modulus of continuity* a nonnegative, increasing, continuous function ρ such that $\rho(0)=0$.

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Theorem B. (cf. [30, Theorem 12]) *Assume that f satisfies* (1.1) *and any one of the following conditions:*

(i) $f(x,u,\xi) \to \infty$ *as* $|\xi| \to \infty$ *for each* $(x,u) \in \Omega \times \mathbb{R}$.
(ii) $f(x,u,\cdot)$ *is strictly convex in* $\mathbb{R}^N$ *for each* $(x,u) \in \Omega \times \mathbb{R}$.
(iii) *The derivatives* f_x, f_ξ *and* $f_{\xi\, x}$ *exist and are continuous.*

Then $F(u,\Omega)$ *is lower semicontinuous in* $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ *with respect to local convergence in* L^1.

The prototype of integrands that we want to study is represented by $f = h(x,u)\,|\xi|$, where $h \geq 0$, for which conditions (i)–(iii) of Theorem B may be violated; hence, in this paper we will focus our attention mainly on Theorem A. Note also that while conditions (i) and (ii) of Theorem A are trivially satisfied when $f = f(\xi)$, so that L^1_{loc} lower semicontinuity holds in this case only under assumption (1.1), Theorem B is more stringent as it imposes extra conditions on the dependence of f on the gradient variable ξ.

It is worth noticing that Theorem A requires no *coercivity hypothesis*, i.e., a condition of the type

$$f(x,u,\xi) \geq C|\xi| - \frac{1}{C}$$

for some constant $C > 0$. One of the main purposes of this paper is to try to understand the deep relation between coercivity (or the lack of it) and lower semicontinuity. A drawback in Theorem A(ii) is that, in practice, one seldom knows whether the target function $u(x)$ is continuous or not. Important examples of integrands which satisfy (i) and (ii) of Theorem A are given by

$$f = f(x,\xi) = h(x)g(\xi), \qquad f = f(x,u,\xi) = h(x,u)g(\xi),$$

where $h(x)$ and $h(x,u)$ are uniformly continuous functions bounded away from zero and g is a nonnegative convex function. Conditions (i) and (ii) of Theorem A appear often in the study of lower semicontinuity and relaxation and were exploited by several authors. Dal Maso [9] obtained an integral representation formula for the relaxation $\mathcal{F}(u,\Omega)$ with respect to the L^1_{loc} topology of the functional F, namely

$$\mathcal{F}(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F(u_n,\Omega) : u_n \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}),\ u_n \to u \quad \text{in} \quad L^1_{\mathrm{loc}}(\Omega;\mathbb{R}) \right\},$$

under a weak form of condition (ii) in Theorem A and assuming coercivity. Let

$$H(u,\Omega) := \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,u,dC(u)) + \int_{S(u)\cap\Omega} \left(\int_{u^-(x)}^{u^+(x)} f^\infty(x,s,\nu_u)\,ds \right) d\mathcal{H}^{N-1}(x), \quad (1.2)$$

where f^∞ is the *recession function* of f, that is $f^\infty(x,u,\xi) := \limsup_{t\to\infty} \dfrac{f(x,u,t\,\xi)}{t}$, $C(u)$ is the Cantor part of Du, and $(u^+ - u^-)$ is the jump of u across the interface $S(u)$.

As a corollary of Dal Maso's general results[1], we have the following theorem

Theorem C. (cf. [9, Theorem 3.2]) *Assume that* Ω *is bounded and that* f *is a Borel function which satisfies* (1.1) *for* $\mathcal{H}^N$ *a.e* $(x,u) \in \Omega \times \mathbb{R}$. *Suppose also that for every* $r > 0$ *there exist* $C > 0$ *and three functions* c, a *and* $A \in C(\Omega;[0,\infty)) \cap L^1(\Omega;[0,\infty))$, *with* $c(x) > 0$ *in* Ω, *such that*

$$c(x)|\xi| - a(x) \leq f(x,u,\xi) \leq C|\xi| + A(x) \quad (1.3)$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ *with* $|u| \leq r$. *Finally, assume that for* $\mathcal{H}^N$ *a.e* $(x_0,u_0) \in \Omega \times \mathbb{R}$ *and for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|f(x,u,\xi) - f(x_0,u_0,\xi)| \leq \varepsilon(1 + |\xi|) \quad (1.4)$$

[1] Dal Maso's results are given in terms of Γ–convergence of a family of functionals $F_n(u,\Omega) := \int_\Omega f_n(x,u(x),\nabla u(x))\,dx$. We take here $f_h \equiv f$ and refer to [9] for the more general statement of Theorems 3.2 and 3.5.

for all $(x,u) \in \Omega \times \mathbb{R}$ *with* $|x-x_0|+|u-u_0| \leq \delta$ *and for all* $\xi \in \mathbb{R}^N$. *Then for all* $u \in BV(\Omega;\mathbb{R}) \cap L^\infty(\Omega;\mathbb{R})$ *we have*

$$\mathcal{F}(u,\Omega) = H(u,\Omega). \tag{1.5}$$

Furthermore, if there exist $b \in L^1(\Omega;[0,\infty))$, $C \geq 0$, $\alpha \geq 1$, *such that* $f(x,u,0) \leq C|u|^\alpha + b(x)$ *for all* $(x,u) \in \Omega \times \mathbb{R}$, *then* (1.5) *holds for all* $u \in BV(\Omega;\mathbb{R}) \cap L^\alpha(\Omega;\mathbb{R})$.

Note that (1.5) implies, in particular, that F is L^1-lower semicontinuous in $W^{1,1}$. Indeed, as $f^\infty \geq 0$ it follows that $\int_\Omega f(x,u(x),\nabla u(x))\,dx \leq \mathcal{F}(u,\Omega)$ for $u \in BV(\Omega;\mathbb{R})$. Therefore, Dal Maso's result extends Theorem A(ii) of Serrin to target functions u of bounded variation *which are not necessarily continuous.* However, the "price" to pay for this extension is the *coercivity and growth assumption* (1.3). Dal Maso established also an integral representation result for functionals which are not necessarily coercive.

Theorem D. (cf. [9, Theorem 3.5]) *Assume that* Ω *is bounded, that* f *is a Borel function, and that* $f(x,u,\cdot)$ *is positively homogeneous of degree one and convex. Suppose also that there exist* $\lambda : \Omega \times \mathbb{R} \to [0,\infty)$, *with* $\lambda(\cdot,u)$ *continuous and* $\lambda(0,u) = 0$ *for* $\mathcal{L}^1$ *a.e.* $u \in \mathbb{R}$, *and a function* $P \in C(\Omega;[0,\infty))$ *such that*

$$|f(x,u,\xi) - f(x_0,u,\xi)| \leq \lambda(|x-x_0|,u)(1+f(x,u,\xi)), \qquad 0 \leq f(x,u,\xi) \leq P(u)|\xi|, \tag{1.6}$$

for all x, $x_0 \in \Omega$, $u \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^N$. *Then* (1.5) *holds for all* $u \in BV(\Omega;\mathbb{R})$.

The main difference between hypotheses (1.4) and $(1.6)_1$ is that (1.4) is a *local* hypothesis in (x,u), while $(1.6)_1$ may be interpreted as a global restriction in u. The conditions of Theorem D are satisfied by

$$f(x,u,\xi) = h(x)B(u)|\xi|$$

where h is a positive, bounded, uniformly continuous function and B is a nonnegative continuous function. Note also that condition $(1.6)_1$ is trivially satisfied when f does not depend on x, that is $f = f(u,\xi)$. Lower semicontinuity for these integrands of the form $f = f(u,\xi)$ was later studied by De Giorgi, Buttazzo and Dal Maso [13], who proved the following result:

Theorem E. (cf. [13, Theorem 1]) *Assume that* $f = f(u,\xi)$ *is nonnegative, measurable in the variable* u, *and convex in* ξ. *Suppose also that* $f(u,0)$ *is lower semicontinuous and that*

$$\limsup_{\xi \to 0} \frac{(f(u,0) - f(u,\xi))^+}{|\xi|} \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}).$$

Then for every $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ *the function* $f(u(x),\nabla u(x))$ *is measurable and the functional* $F(u,\Omega)$ *is lower semicontinuous in* $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ *with respect to local convergence in* L^1.

Theorem E was extended by De Cicco in [12] to functions of bounded variation. More precisely, De Cicco showed that when $f = f(u,\xi)$ satisfies the hypotheses of Theorem E then the functional $H(u)$ defined in (1.2) is lower semicontinuous in $BV_{\mathrm{loc}}(\Omega;\mathbb{R})$ with respect to local convergence in L^1. The hypotheses of Theorem E were significantly weakened by Ambrosio in [3] (see also [11]), where the sequence $\{u_n\}$ is assumed to be bounded in $W^{1,1}(\Omega;\mathbb{R})$. This condition is somewhat related to coercivity, and we will not dwell more on it here.

Unlike the case where $f = f(u,\xi)$, *without some kind of coercivity* one cannot expect in general lower semicontinuity in the L^1 topology for functionals of the form

$$\int_\Omega f(x,\nabla u(x))\,dx.$$

Indeed, in [9] Dal Maso, following a counterexample of Aronszajn, constructed a continuous function $\omega : \Omega \to \mathbb{R}$, where $\Omega = (0,1)\times(0,1)$ and $x = (x_1, x_2)$, and a sequence of functions $\{u_n\}$ converging to $u(x) = x_2$ in $L^\infty(\Omega;\mathbb{R})$, such that

$$\int_\Omega |(\sin\omega(x), \cos\omega(x)) \cdot \nabla u(x))|\,dx > \liminf_{n\to\infty} \int_\Omega |(\sin\omega(x), \cos\omega(x)) \cdot \nabla u_n(x))|\,dx.$$

Since the target function $u(x_1, x_2) = x_2$ is continuous, $f(x,\xi) = |(\sin\omega(x), \cos\omega(x))\cdot\xi|$ cannot satisfy either condition (i) or (ii) of Theorem A, for in this case we would obtain a contradiction. This example suggests that, when there is no coercivity, lower semicontinuity in the L^1 topology may fail unless we strengthen (1.1) with an uniform continuity condition.

We are now ready to present the main result of the paper.

Theorem 1.1. *Assume that $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ is a Borel integrand, $f(x,u,\cdot)$ is convex in $\mathbb{R}^N$, and for all $(x_0,u_0) \in \Omega\times\mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(x_0,u_0,\xi) - f(x,u,\xi) \le \varepsilon(1 + f(x,u,\xi)) \tag{1.7}$$

for all $(x,u) \in \Omega\times\mathbb{R}$ with $|x - x_0| + |u - u_0| \le \delta$ and for all $\xi \in \mathbb{R}^N$. Let $u \in BV_{\mathrm{loc}}(\Omega;\mathbb{R})$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$. Then

$$H(u,\Omega) \le \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\,dx.$$

The main tool in the proof of Theorem 1.1 is the *blow–up method* introduced by Fonseca and Müller [17], [18], where we reduce the domain Ω to a ball and the target function u becomes a piecewise affine function. Since affine functions are locally bounded, in the scalar case we may replace the truncation used in [17], [18], in a vectorial setting, and which required a degenerate coercivity condition, by a considerably simpler argument.

Theorem 1.1 improves Serrin's Theorem A, not only because continuity of the target function u is assumed in Theorem A(ii), and is not needed here, but also because condition (1.7) is significantly weaker than (ii), as the following result illustrates:

Corollary 1.2. *Let $g : \mathbb{R}^N \to [0,\infty)$ be a convex function, and let $h : \Omega\times\mathbb{R} \to [0,\infty)$ be a lower semicontinuous function. If $u \in BV_{\mathrm{loc}}(\Omega;\mathbb{R})$ and $\{u_n\} \subset W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ converges to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$, then*

$$\int_\Omega h(x,u)g(\nabla u)\,dx \le \liminf_{n\to\infty} \int_\Omega h(x,u_n)g(\nabla u_n)\,dx.$$

This result seems to be new in this generality. Note that conditions (1.3) and (1.4) in Theorem C of Dal Maso imply the validity of (1.7), while $f(x,u,\xi) = h(x,u)g(\xi)$ as in Corollary 1.2 satisfies (1.7), but not, in general, (1.4), $(1.6)_1$, and (i), (ii) of Theorem A.

Conditions of the type (1.7) appeared already in the papers of Fonseca and Müller [17], [18], Dal Maso and Sbordone [10], Fusco and Hutchinson [20]. All these results deal with the vectorial case and require some type of coercivity conditions.

In the special case where

$$h = h(x) := \chi_A(x) \qquad \text{for some measurable set } A \subset \Omega,$$

then $h(x)g(\xi)$ satisfies (1.7) if and only if $\mathcal{L}^N(\partial A) = 0$ (i.e. if $\chi_A(x)$ has a lower semicontinuous representative), and thus we recover the condition obtained by Gangbo [21]. Corollary 1.2 attests to the sharpness of condition (1.7). Indeed, when $N = 1$ and Ω is bounded, Fusco [19] proved that the functional

$$F(u) := \int_\Omega h(x)|u'(x)|\,dx, \qquad u \in W^{1,1}(\Omega;\mathbb{R}),$$

where $h(x)$ is a bounded, nonnegative measurable function, is lower semicontinuous in $L^1(\Omega;\mathbb{R})$ if and only if $h(x)$ is lower semicontinuous.

Theorem 1.3. *Assume that Ω is bounded, $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ is a Borel integrand, $f(x,u,\cdot)$ is convex in $\mathbb{R}^N$, and there exists a constant $C > 0$ such that*

$$0 \le f(x,u,\xi) \le C(1+|\xi|) \qquad \text{for all } (x,u,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N. \tag{1.8}$$

Then $\mathcal{F}(u,\cdot)$ is the trace of a finite, Radon measure on the open subsets of Ω, and

(i) *if f is Carathéodory or $f(\cdot,\cdot,\xi)$ is upper semicontinuous then $\mathcal{F}(u,\Omega\backslash(S(u)\cup M(u))) \le \int_\Omega f(x,u,\nabla u)\,dx$;*

(ii) *if $f^\infty(\cdot,\cdot,\xi)$ is upper semicontinuous then $\mathcal{F}(u,M(u)) \le \int_\Omega f^\infty(x,u,dC(u))$;*

(iii) *if $f^\infty(\cdot,u,\xi)$ is upper semicontinuous then $\mathcal{F}(u,S(u)) \le \int_{S(u)\cap\Omega}\left(\int_{u^-(x)}^{u^+(x)} f^\infty(x,s,\nu_u)\,ds\right) d\mathcal{H}^{N-1}(x)$.*

Here, and in what follows, $M(u)$ is the support of the Cantor part of Du. Theorem 1.3 is based on a recent work by Bouchitté, Fonseca and Mascarenhas [7]. We have thus obtained the following relaxation result:

Corollary 1.4. *Under the hypotheses of Theorems 1.1 and 1.3(i), (ii), (iii), we have $\mathcal{F}(u,\Omega) = H(u,\Omega)$ for all $u \in BV(\Omega;\mathbb{R})$.*

If we require (1.8) to be satisfied locally u in compact sets of $\mathbb{R}$, i.e., for every $r > 0$ there exist $C > 0$ and $A \in L^1(\Omega;[0,\infty))$, such that

$$0 \le f(x,u,\xi) \le C|\xi| + A(x)$$

for all $(x,u,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$ with $|u| \le r$, then it can be shown that Corollary 1.4 continues to hold for all $u \in BV(\Omega;\mathbb{R})\cap L^\infty(\Omega;\mathbb{R})$. Thus, when (1.3) and (1.4) are satisfied *for all* $(x_0,u_0) \in \Omega\times\mathbb{R}$ (see also Section 8), Corollary 1.4 improves Theorem C since conditions (1.3) and (1.4) imply condition (1.7), and Corollary 1.4 does not require any coercivity properties.

Next we extend Theorem E to integrands $f = f(x,u,\xi)$ which depend on x. As we pointed out before, there are already several results in this direction, e.g. due to Ambrosio [3] and later extended by De Cicco [11] to BV functions, and where local convergence in L^1 is replaced by weak convergence in BV.

Theorem 1.5. *Assume that $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ is a Borel integrand, $f(x,u,\cdot)$ is convex in $\mathbb{R}^N$, and for all $x_0 \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x_0,u,\xi) - f(x,u,\xi)| \le \varepsilon(1 + f(x,u,\xi)) \tag{1.9}$$

for all $x \in \Omega$ with $|x-x_0| \le \delta$ and for all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Suppose also that $f(x_0,\cdot,0)$ is lower semicontinuous and

$$\limsup_{|\xi|\to 0} \frac{(f(x_0,u,0) - f(x_0,u,\xi))^+}{|\xi|} \in L^1_{\text{loc}}(\mathbb{R};\mathbb{R}).$$

Then $H(u,\Omega) \le \mathcal{F}(u,\Omega)$ for all $u \in BV(\Omega;\mathbb{R})$.

Note that in Theorem 1.5 we do not require $f(x,u,\cdot)$ to be positively homogeneous of degree one as in Theorem D of Dal Maso. The proof of Theorem 1.5 relies on the blow-up method of Fonseca and Müller [17], and on the original proof of De Giorgi, Buttazzo and Dal Maso [13].

Theorem 1.6. *Assume that Ω is bounded and that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is a Borel integrand which satisfies* (1.8)*, with $f(x, u, \cdot)$ convex in $\mathbb{R}^N$, and $f(\cdot, u, \xi)$ continuous in Ω. Then $\mathcal{F}(u, \Omega) \leq H(u, \Omega)$.*

We now turn our attention to the vectorial case, and consider nonnegative integrands

$$f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{dN} \to [0, \infty), \qquad \text{where } d > 1.$$

The situation is considerably more complicated, even when $f(x, u, \cdot)$ is assumed to be convex, rather than quasiconvex, which is the natural assumption when $d > 1$ (see [6], [8], [26]). In his book on Calculus of Variations [26, Theorems 4.1.1, 4.1.2], Morrey extended Serrin's Theorems A and B to the vectorial case. Several years later, Eisen [14] studied the case where $d > 1$ and proved that Lemma 4.14 in [26], which is the core of Theorem B, ceases to be true when $d > 1$, thus placing in doubt the validity of theorem itself. In addition, he constructed counterexamples for Theorems A(ii) and B(iii). Theorem A(ii) seems to fail in the vectorial case due mainly to the truncation techniques of the type used in Lemma 3 of [30] (see also [29, pp. 30–31]) and in our Theorem 1.1, suitable only for the scalar case. On the other hand, Serrin's Theorem A(i) continues to hold in the vectorial case, while the validity of Theorem B(i)–(ii) when $d > 1$ remains open. Note that Eisen's counterexamples were both of the form

$$f = f(u, \xi) = h(u)g(\xi).$$

Thus we cannot hope to fully extend either Theorem E of De Giorgi, Buttazzo and Dal Maso or our Theorems 1.1 and 1.5 to the vectorial case. However, we can prove the following:

Theorem 1.7. *Let f be a nonnegative Borel integrand. Suppose that for all $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ and $\varepsilon > 0$ there exist $\delta > 0$ and a modulus of continuity ρ, with $\rho(s) \leq C(1 + s)$ for $s > 0$ and for some $C > 0$, such that*

$$f(x_0, u_0, \xi) - f(x, u, \xi) \leq \varepsilon(1 + f(x, u, \xi)) + \rho(|u - u_0|) \tag{1.10}$$

for all $x \in \Omega$ with $|x - x_0| \leq \delta$, and for all $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{dN}$. Assume also that either

(a) *$f(x_0, u_0, \cdot)$ is convex in $\mathbb{R}^{dN}$ or*

(b) *$f(x_0, u_0, \cdot)$ is quasiconvex in $\mathbb{R}^{dN}$ and*

$$0 \leq f(x_0, u_0, \xi) \leq C(|\xi|^q + 1) \qquad \text{for all } \xi \in \mathbb{R}^{dN}, \tag{1.11}$$

where $C > 0$ and the exponent $q \geq 1$ may depend on (x_0, u_0). In addition, if $q > 1$ then assume that

$$f(x_0, u_0, \xi) \geq \frac{1}{C}|\xi|^q - C \qquad \text{for all } \xi \in \mathbb{R}^{dN}. \tag{1.12}$$

Let $u \in BV_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ which converges to u in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. Then:

(i) $\displaystyle\int_\Omega f(x, u, \nabla u)\, dx + \int_\Omega f^\infty(x, u, dC(u)) \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx;$

(ii) *if $f = f(x, \xi)$ then* $\displaystyle\int_{S(u)\cap\Omega} f^\infty(x, (u^+(x) - u^-(x)) \otimes \nu_u)\, d\mathcal{H}^{N-1} \leq \liminf_{n\to\infty} \int_\Omega f(x, \nabla u_n)\, dx.$

Theorem 1.7 improves Theorem A(i) of Serrin, since condition (1.10) is significantly weaker than the corresponding (i). Moreover, Theorem 1.7 is also closely related to a recent result of Acerbi, Bouchitté and Fonseca [1] for integrands of the form $f = f(x, \xi)$, convex in ξ, where condition (1.10) is replaced by the growth condition

$$c_1(|\xi|^p - 1) \leq f(x, \xi) \leq C_2(|\xi|^q + 1),$$

with

$$p > q\,\frac{N-1}{N}.$$

In the quasiconvex case we use a result of Ambrosio and Dal Maso [4] for functions $g = g(\xi)$ such that

$$0 \le g(\xi) \le C(1 + |\xi|).$$

This growth condition is of vital importance for their argument to work.

When (1.11) and (1.12) hold then, by a recent result of Kristensen [22], we can approximate $f(x_0, u_0, \xi)$ by an increasing sequence of quasiconvex functions $g_j(\xi)$ which grow at most linearly, and thus we can still use [4] for each g_j. Note that without (1.12) $L^1_{\rm loc}$ lower semicontinuity may fail even for the simplest case when $f = f(\xi)$. This has been shown by Malý [23] for

$$f = f(\xi) = |\det \xi|, \qquad d = N,$$

who constructed a sequence in $W^{1,N}$ which converges to $u(x) = x$ weakly in $W^{1,p}$, where $p < N - 1$, and for which lower semicontinuity fail (see also Fonseca and Malý [16]). Thus, by Theorem 1.7 it follows that $f(\xi) = |\det \xi|$ *cannot be approximated from below by an increasing sequence of functions* $g_j(\xi)$ *which grow at most linearly* (see also example 7.9 in [22] for a different proof). This is in sharp contrast with the convex case, where it is well known that this approximation can always be done (see e.g. Proposition 9.1).

In Theorem 1.7(ii) we have chosen to restrict ourselves to integrands f of the form $f = f(x, \xi)$ because in this case there is a *simple* integral representation formula for the relaxation of F on the jump set $S(u)$, while, when f depends on the full set of variables and $d > 1$, then the formula is rather complicated (see Theorem 1.10 below).

Theorem 1.8. *Theorem 1.7(i) still holds if we replace condition* (1.10) *with the following:*
for all $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ *either* $f(x_0, u_0, \xi) \equiv 0$ *for all* $\xi \in \mathbb{R}^{dN}$, *or for every* $\varepsilon > 0$ *there exist* C_1, C_2, $\delta > 0$ *such that*

$$f(x_0, u_0, \xi) - f(x, u, \xi) \le \varepsilon(1 + f(x, u, \xi)) \tag{1.13}$$

$$f(x, u, \xi) \ge C_1|\xi| - C_2 \tag{1.14}$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$ *with* $|x - x_0| + |u - u_0| \le \delta$ *and for all* $\xi \in \mathbb{R}^{d\,N}$.

Theorem 1.8 was proven by Fonseca and Müller [18], under somewhat stronger hypotheses, and in the case where assumption (b) of Theorem 1.7 holds with $q = 1$. The convex case can be thought of as a natural extension of Theorem A(ii) of Serrin to the vectorial case.

Theorems 1.7 and 1.8 are complemented by the following result:

Theorem 1.9. *Assume that the hypotheses of Theorem 1.3 are verified in the vectorial case, with* $f(x, u, \cdot)$ *quasiconvex in* $\mathbb{R}^{dN}$. *Then Theorem 1.3(i)–(ii) continues to hold. Furthermore, if in Theorem 1.3(iii) we assume that* $f^\infty = f^\infty(x, \xi)$, *then*

$$\mathcal{F}(u, S(u)) \le \int_{S(u)\cap\Omega} f^\infty(x, (u^+(x) - u^-(x)) \otimes \nu_u)\, d\mathcal{H}^{N-1}(x).$$

A similar extension holds for Theorem 1.6.

To obtain an integral representation formula for the relaxation $\mathcal{F}$ over the jump set $S(u)$ in the vectorial case we need a different set of hypotheses.

Theorem 1.10. *Assume that f is a nonnegative Borel integrand which satisfies* (1.8). *Suppose also that for all $x_0 \in \Omega$ and $\varepsilon > 0$ there exist two constants δ, $L > 0$ such that*

$$f^\infty(x_0, u, \xi) - f^\infty(x, u, \xi) \leq \varepsilon(1 + f^\infty(x, u, \xi)) \tag{1.15}$$

for all $x \in \Omega$ with $|x - x_0| \leq \delta$, and for all $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{dN}$; and

$$f^\infty(x, u, \xi) - \frac{f(x, u, t\xi)}{t} \leq \varepsilon \left(1 + \frac{f(x, u, t\xi)}{t}\right) \tag{1.16}$$

for all $x \in \Omega$, with $|x - x_0| \leq \delta$, and for all $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{dN}$ and $t > L$. Then

$$\mathcal{F}(u, S(u)) \geq \int_{S(u)\cap\Omega} h(x, u^+(x), u^-(x), \nu_u)\, d\mathcal{H}^{N-1}(x), \tag{1.17}$$

where

$$h(x_0, \lambda, \theta, \nu) := \inf \left\{ \int_{Q_\nu} f^\infty(x_0, w(y), \nabla w(y))\, dy : w \in W^{1,1}(Q_\nu),\ w|_{\partial Q_\nu(x_0,\varepsilon)} = u_{\lambda,\theta,\nu} \right\} \tag{1.18}$$

and

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y \cdot \nu > 0, \\ \theta & \text{if } y \cdot \nu \leq 0. \end{cases}$$

Furthermore if (1.16) *is replaced by*

$$\left| f^\infty(x, u, \xi) - \frac{f(x, u, t\xi)}{t} \right| \leq \varepsilon \left(1 + \frac{f(x, u, t\xi)}{t}\right) \tag{1.16$'$}$$

and $f^\infty(\cdot, u, \xi)$ is upper semicontinuous, then (1.17) *is an equality.*

Here, and in what follows, $Q_\nu := R_\nu(-\frac{1}{2}, \frac{1}{2})^N$, where R_ν denotes a rotation such that $R_\nu e_N = \nu$, and $Q = (-\frac{1}{2}, \frac{1}{2})^N$. Also, C will denote a generic constant which may vary from line to line.

It is not difficult to see that conditions $(H2)$ and $(H4)$ in Thm. 4.1.4 of [7] imply (1.16).

In addition to the novelty of the results in this paper, which significantly improve upon classical theorems in the literature, we would like to close this section pointing out some aspects of our approach. One of the main tools exploited in the paper is the *blow–up method* introduced by Fonseca and Müller [17], [18]. This method was first used to deal with quasiconvex integrands, since many of the techniques in convex analysis available for the scalar case could not be easily extended to the vectorial case. It turns out that blow–up arguments in the scalar case, combined with some classical methods for convex integrands, may improve and simplify some important results in the literature. Also, we use the very recent global method of relaxation introduced by Bouchitté, Fonseca and Mascarenhas [7], to show that the relaxed energy density may be written in terms of a Dirichlet problem. Most of the proofs are carried out firstly for f which grow at most linearly in the gradient variable ξ. While this approach is standard in the convex setting, and in the literature there are several results which allow to approximate from below convex functions by an increasing sequence of convex functions which grow at most linearly, it was only very recently that Kristensen brought this idea to the vectorial setting, exploiting his approximation result for quasiconvex functions (see [22]; see also [24]).

In the presentation of the paper, and whenever it was possible, we have tried to treat separately the energies corresponding to the Lebesgue, Cantor, and Jump part of Du, in order to better understand the corresponding scaling and the necessity and sufficiency of our hypotheses. It is interesting to observe that the Lebesgue and Cantor measures may be treated in a similar fashion and, more importantly, under the same hypotheses on the integrand f. On the other hand lower semicontinuity for the jump part requires hypotheses and methods which depart from the above mentioned.

Although the hypotheses on the integrand f are rather mild, they are not by any means minimal. Indeed, it was not our purpose to obtain necessary conditions for lower semicontinuity, but, rather, to find *simple* sufficient assumptions, which would be easy to verify in the applications. It seems, however, that lower semicontinuity of f in the x variable is almost necessary, but it is not clear if it should always be uniform in ξ (at least for functionals which are allowed to vanish). Dal Maso's example (see [9])

$$f(x,\xi) = |(\sin\omega(x), \cos\omega(x)) \cdot \xi|$$

certainly seems to imply that it should. The lower semicontinuity of f in the u variable is not necessary, as proved by Theorem D, but in order to drop it, stronger assumptions on the dependence on x seem to be needed.

§2. Proof of Theorem 1.1.

Throughout this work we will use often truncation arguments, and the result below will be instrumental.

Proposition 2.1. (Truncation) *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ *be a Borel integrand satisfying* (1.8). *Suppose that there exists a sequence* $(\varepsilon_k, \lambda_k, t_k, u_{0k}) \in \mathbb{R}^4$ *such that*

$$\varepsilon_k \to 0^+, \qquad \lambda_k \to \lambda \in [0,\infty), \qquad t_k \to T \in (0,\infty], \qquad u_{0k} \to u_0 \in \mathbb{R},$$

and

$$\lim_{k\to\infty} \frac{1}{t_k} \int_{Q_\nu} f(x_0 + \varepsilon_k y, u_{0k} + \lambda_k w_k(y), t_k \nabla w_k(y))\, dy =: \ell < \infty,$$

where $x_0 \in \Omega$, $\{w_k\} \subset W^{1,1}(Q_\nu;\mathbb{R})$ *converges in* $L^1(Q_\nu;\mathbb{R})$ *to a function* $w_0 \in L^\infty(Q_\nu;\mathbb{R})$.

Let $[\lambda \operatorname{essinf}_{Q_\nu} w_0, \lambda \operatorname{esssup}_{Q_\nu} w_0] \subset (\alpha_1, \alpha_2)$, *for some* α_1, $\alpha_2 \in \mathbb{R}$. *Then there exists a new sequence* $\{v_k\} \subset W^{1,1}(Q_\nu;\mathbb{R})$, *converging to* w_0 *in* $L^1(Q_\nu;\mathbb{R})$, *such that*

$$\lim_{k\to\infty} \frac{1}{t_k} \int_{Q_\nu} f(x_0 + \varepsilon_k y, u_{0k} + \lambda_k v_k(y), t_k \nabla v_k(y))\, dy \le \ell$$

and

$$u_{0k} + \lambda_k v_k(y) \in [u_0 + \alpha_1, u_0 + \alpha_2] \qquad \text{for } \mathcal{L}^N \text{ a.e. } y \in Q_\nu.$$

Remark 2.2. (i) It is easy to check that the conclusion of Proposition 2.1 still holds if we replace Q_ν by any bounded, open, convex subset of $\mathbb{R}^N$ containing the origin.

(ii) Condition (1.8) can be significantly weakened if we specialize the sequences t_k, ε_k and x_0 (see Lemma 8.4).

Proof. Take $0 < 2\varepsilon < \min\{\lambda \operatorname{essinf}_{Q_\nu} w_0 - \alpha_1, \alpha_2 - \lambda \operatorname{esssup}_{Q_\nu} w_0\}$, and let k be so large that $|u_0 - u_{0k}| < \varepsilon/2$. Define

$$\begin{aligned} E_k &:= \{y \in Q_\nu : u_{0k} + \lambda_k w_k(y) \in [u_0+\alpha_1, u_0+\alpha_2]\},\\ E_k^+ &:= \{y \in Q_\nu : u_{0k} + \lambda_k w_k(y) > u_0 + \alpha_2\},\\ E_k^- &:= \{y \in Q_\nu : u_{0k} + \lambda_k w_k(y) < u_0 + \alpha_1\}, \end{aligned} \qquad v_k(y) := \begin{cases} w_k(y) & \text{in } E_k,\\ (u_0 - u_{0k} + \alpha_2)/\lambda_k & \text{in } E_k^+,\\ (u_0 - u_{0k} + \alpha_1)/\lambda_k & \text{in } E_k^-. \end{cases}$$

Then $v_k \in W^{1,1}(Q_\nu;\mathbb{R})$, and for k large enough

$$\alpha_2 > \varepsilon + |u_0 - u_{0k}| + \lambda_k \operatorname*{esssup}_{Q_\nu} w_0, \qquad \alpha_1 + \varepsilon < |u_0 - u_{0k}| + \lambda_k \operatorname*{essinf}_{Q_\nu} w_0, \tag{2.1}$$

with

$$\int_{Q_\nu} |v_k(y) - w_0(y)|dy = \int_{E_k} |w_k(y) - w_0(y)|dy + \int_{E_k^+} \left(\frac{u_0 - u_{0k} + \alpha_2}{\lambda_k} - w_0(y)\right) dy$$
$$+ \int_{E_k^-} \left(w_0(y) - \frac{u_0 - u_{0k} + \alpha_1}{\lambda_k}\right) dy \leq ||w_k - w_0||_{L^1(Q_\nu;\mathbb{R})} \to 0$$

as $k \to \infty$. Moreover

$$\frac{1}{t_k}\int_{Q_\nu} f(x_0 + \varepsilon_k y, u_{0k} + \lambda_k v_k(y), t_k \nabla v_k(y))\, dy = \frac{1}{t_k}\int_{E_k} f(x_0 + \varepsilon_k y, u_{0k} + \lambda_k w_k(y), t_k \nabla w_k(y))\, dy$$
$$+ \frac{1}{t_k}\int_{E_k^+} f(x_0 + \varepsilon_k y, u_0 + \alpha_2, 0)\, dy + \frac{1}{t_k}\int_{E_k^-} f(x_0 + \varepsilon_k y, u_0 - \alpha_1, 0)\, dy.$$

By (1.8) and (2.1)

$$\begin{aligned} 0 &\leq \frac{1}{t_k}\int_{E_k^+} f(x_0 + \varepsilon_k y, u_0 + \alpha_2, 0)\, dy \leq \frac{C}{t_k}\mathcal{L}^N(E_k^+) \\ &\leq \frac{C}{t_k}\mathcal{L}^N(\{y \in Q_\nu : |w_k(y) - w_0(y)| > \varepsilon/\lambda_k\}) \leq \frac{C\lambda_k}{t_k\varepsilon}||w_k - w_0||_{L^1(Q_\nu;\mathbb{R})} \to 0. \end{aligned} \tag{2.2}$$

Similarly, we can show that the integral over E_k^- approaches zero as $k \to \infty$. □

Proof of Theorem 1.1. Without loss of generality we may assume that f is continuous, $f(x, u, \cdot)$ is convex, f satisfies (1.7), (1.8), and for all $(x_0, u_0) \in \Omega \times \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x_0, u, \xi) - f(x, u, \xi) \leq \varepsilon(1 + f(x, u, \xi)) \tag{2.3}$$

for all $(x, u) \in \Omega \times \mathbb{R}$ with $|x - x_0| + |u - u_0| \leq \delta$ and for all $\xi \in \mathbb{R}^N$.

Indeed, suppose that the conclusion of the theorem is true under these additional hypotheses. By applying Proposition 9.3 to the function f, and noting that (1.7) and (9.2) are equivalent if $f \geq 1$, we may find an increasing sequence of nonnegative *continuous* functions f_j convex in ξ, which satisfy (1.7), (1.8), and (2.3), and such that $f(x, u, \xi) + 1 = \sup_j f_j(x, u, \xi)$. Let $\{u_n\} \subset W^{1,1}_{\text{loc}}(\Omega; \mathbb{R})$ converge to $u \in BV_{\text{loc}}(\Omega; \mathbb{R})$ in $L^1_{\text{loc}}(\Omega; \mathbb{R})$, and let $A \Subset \Omega$. For any fixed j

$$\liminf_{n\to\infty}\int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \geq \liminf_{n\to\infty}\int_A f_j(x, u_n(x), \nabla u_n(x))\, dx - \mathcal{L}^N(A) \geq H_j(u, A) - \mathcal{L}^N(A),$$

where H_j is the functional given in (1.2) and corresponding to f_j. If now we let $j \to \infty$, and use Lebesgue Monotone Convergence Theorem and Proposition 9.3, we conclude that

$$\begin{aligned} \liminf_{n\to\infty}\int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx &\geq \int_A f(x, u, \nabla u)\, dx + \int_A f^\infty(x, u, dC(u)) \\ &\quad + \int_{S(u)\cap A}\left(\int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu_u)\, ds\right) d\mathcal{H}^{N-1} \\ &= H(u, A), \end{aligned}$$

where we have used the fact that $(f + 1)^\infty = f^\infty$. The result now follows by letting $A \nearrow \Omega$, and using Lebesgue Monotone Convergence Theorem once again.

Thus, in what follows f is continuous, verifies (1.7), (1.8) and (2.3), $\{u_n\} \subset W^{1,1}_{\text{loc}}(\Omega; \mathbb{R})$ converges to $u \in BV_{\text{loc}}(\Omega; \mathbb{R})$ in $L^1_{\text{loc}}(\Omega; \mathbb{R})$, and, without loss of generality, we may assume that

$$\liminf_{n\to\infty}\int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx = \lim_{n\to\infty}\int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx < \infty. \tag{2.4}$$

Passing to a subsequence, if necessary, we may find a nonnegative Radon measure μ such that

$$f(x,u_n(x),\nabla u_n(x))\,\mathcal{L}^N\lfloor\Omega \overset{\star}{\rightharpoonup} \mu$$

as $n\to\infty$, weakly $\star$ in the sense of measures.

(i) *(Lebesgue part)* We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N}\geq f(x_0,u_n(x_0),\nabla u_n(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0\in\Omega, \tag{2.5}$$

where $Q(x_0,\varepsilon):=x_0+\varepsilon\,Q$. If (2.5) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi\in C_0(\Omega;\mathbb{R})$, $0\leq\varphi\leq1$. We have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)\,dx &\geq \liminf_{n\to\infty}\int_\Omega\varphi\, f(x,u_n,\nabla u_n)\,dx=\int_\Omega\varphi\,d\mu\\ &\geq\int_\Omega\varphi\,\frac{d\mu}{d\mathcal{L}^N}\,dx\geq\int_\Omega\varphi\, f(x,u,\nabla u)\,dx.\end{aligned}$$

By letting $\varphi\to1$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem in what concerns the absolutely continuous part, it suffices to show (2.5).

Fix $x_0\in\Omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N}<\infty, \qquad \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)}|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|dx=0. \tag{2.6}$$

Choosing $\varepsilon_k\to0^+$ such that $\mu(\partial Q(x_0,\varepsilon_k))=0$, then

$$\begin{aligned}\lim_{k\to\infty}\frac{\mu(Q(x_0,\varepsilon_k))}{\varepsilon_k^N}&=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon_k)}f(x,u_n,\nabla u_n)\,dx\\ &=\lim_{k\to\infty}\lim_{n\to\infty}\int_Q f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k w_{n,k}(y),\nabla w_{n,k}(y))\,dy,\end{aligned}$$

where

$$w_{n,k}(y):=\frac{u_n(x_0+\varepsilon_k y)-u(x_0)}{\varepsilon_k}.$$

Clearly $w_{n,k}\in W^{1,1}(Q;\mathbb{R})$, and by (2.6), $\lim_{k\to\infty}\lim_{n\to\infty}||w_{n,k}-w_0||_{L^1(Q;\mathbb{R})}=0$, where $w_0(y):=\nabla u(x_0)y$. By a standard diagonalization argument, we may extract a subsequence $w_k:=w_{n_k,k}$ which converges to w_0 in $L^1(Q;\mathbb{R})$ and such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)=\lim_{k\to\infty}\int_Q f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k w_k(y),\nabla w_k(y))\,dy. \tag{2.7}$$

Fix $\varepsilon>0$ and let δ be provided by (1.7). By Proposition 2.1, with

$$\lambda_k:=\varepsilon_k, \qquad t_k:\equiv1, \qquad u_{0k}:\equiv u(x_0), \qquad -\alpha_1=\alpha_2:=\delta,$$

we may find a new sequence $\{v_k\}\subset W^{1,1}(Q;\mathbb{R})$, still convergent to w_0 in $L^1(Q;\mathbb{R})$, such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)\geq\liminf_{k\to\infty}\int_Q f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k v_k(y),\nabla v_k(y))\,dy,$$

and $|\varepsilon_k v_k(y)| \le \delta$ for $\mathcal{L}^N$ a.e. $y \in Q$. Since $\varepsilon_k \to 0$, by (1.7) we obtain

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \liminf_{k\to\infty} \int_Q f(x_0, u(x_0), \nabla v_k(y))\, dy.$$

We can now apply Serrin's Theorem A(i) to the integrand $g(\xi) := f(x_0, u(x_0), \xi)$ to conclude that

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge f(x_0, u(x_0), \nabla u(x_0)).$$

The result follows by letting $\varepsilon \to 0^+$. □

(ii) *(Cantor part)* The proof for the Cantor part is somewhat similar to the previous one, and we will only indicate the main differences. The inequality (2.5) is now replaced by

$$\frac{d\mu}{d|C(u)|}(x_0) \ge f^\infty\left(x_0, u(x_0), \frac{dC(u)}{d|C(u)|}(x_0)\right) \qquad \text{for } C(u) \text{ a.e. } x_0 \in \Omega,$$

where (see [2])

$$\frac{dC(u)}{d|C(u)|}(x_0) = a_u(x_0)\nu_u(x_0), \tag{2.8}$$

with $a_u(x_0) \in \mathbb{R}$ and $\nu_u(x_0) \in S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. For simplicity of the notation, from now on we will write a and ν to designate $a_u(x_0)$ and $\nu_u(x_0)$, respectively. It is known (see [4], [7], [18]) that for $C(u)$ a.e. $x_0 \in \Omega$ the following hold

$$\begin{gathered}\frac{d\mu}{d|C(u)|}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q_\nu(x_0,\varepsilon))}{|Du|(Q_\nu(x_0,\varepsilon))} < \infty, \qquad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{Q_\nu(x_0,\varepsilon)} |u(x) - u(x_0)|dx = 0,\\ \lim_{\varepsilon\to 0^+} \frac{|Du|(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} = 0, \qquad \lim_{\varepsilon\to 0^+} \frac{|Du|(Q_\nu(x_0,\varepsilon))}{\varepsilon^N} = \infty.\end{gathered} \tag{2.9}$$

Fix $x_0 \in \Omega$ so that (2.9) holds. By Lemma 3.9 in [7] (see also Theorem 2.3 in [4]) there exist $\varepsilon_k \to 0^+$ such that $\mu(\partial Q_\nu(x_0, \varepsilon_k)) = 0$, and a non decreasing function $\Psi : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ such that the followings hold:

$$\begin{aligned}&\Psi(\tfrac{1}{2} - 0) - \Psi(-\tfrac{1}{2} + 0) = 1, \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi(s)\, ds = 0,\\ &z_k(y) := \frac{u(x_0 + \varepsilon_k y) - \frac{1}{\varepsilon_k^N}\frac{1}{\varepsilon_k^N}\int_{Q_\nu} u(x_0 + \varepsilon_k z)dz}{\lambda_k} \to w_0(y) := \Psi(y\cdot\nu)\, a \quad \text{in } L^1(Q_\nu;\mathbb{R}),\\ &\lim_{k\to\infty} |Dz_k|(Q_\nu) = |Dw_0|(Q_\nu),\end{aligned} \tag{2.10}$$

where, by (2.9), $\lambda_k := |Du|(Q_\nu(x_0,\varepsilon_k))/\varepsilon_k^{N-1} \to 0$ and $t_k := \lambda_k/\varepsilon_k \to \infty$ as $k \to \infty$. Then

$$\begin{aligned}\frac{d\mu}{d|C(u)|}(x_0) = \lim_{k\to\infty} \frac{\mu(Q_\nu(x_0,\varepsilon_k))}{|Du|(Q_\nu(x_0,\varepsilon_k))} &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{|Du|(Q_\nu(x_0,\varepsilon_k))}\int_{Q_\nu(x_0,\varepsilon_k)} f(x, u_n, \nabla u_n)\, dx\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{t_k}\int_{Q_\nu} f(x_0 + \varepsilon_k y, u_{0n,k} + \lambda_k w_{n,k}(y), t_k \nabla w_{n,k}(y))\, dy,\end{aligned}$$

where

$$w_{n,k}(y) := \frac{u_n(x_0 + \varepsilon_k y) - u_{0n,k}}{\lambda_k}, \qquad u_{0n,k} := \frac{1}{\varepsilon_k^N}\int_{Q_\nu} u_n(x_0 + \varepsilon_k z)dz.$$

Clearly $w_{n,k} \in W^{1,1}(Q_\nu;\mathbb{R})$, and by (2.10), $(2.9)_2$, and the fact that $\{u_n\}$ converges to u in L^1,

$$\lim_{k\to\infty}\lim_{n\to\infty} ||w_{n,k} - w_0||_{L^1(Q_\nu;\mathbb{R})} = 0, \qquad \lim_{k\to\infty}\lim_{n\to\infty} u_{0n,k} = u(x_0).$$

By a standard diagonalization argument, we may extract two subsequences $\{w_k := w_{n_k,k}\}$, $\{u_{0k} := u_{0n_k,k}\}$, which converge to w_0 in $L^1(Q_\nu;\mathbb{R})$ and $u(x_0)$, respectively, such that

$$\frac{d\mu}{d|C(u)|}(x_0) = \lim_{k\to\infty} \frac{1}{t_k} \int_{Q_\nu} f(x_0 + \varepsilon_k y, u_{0k} + \lambda_k w_k(y), t_k \nabla w_k(y))\, dy. \tag{2.11}$$

We can now continue as in part (i), using Proposition 2.1 and then (1.7) to conclude that

$$(1+\varepsilon)\frac{d\mu}{d|C(u)|}(x_0) + \varepsilon \geq \liminf_{k\to\infty} \frac{1}{t_k} \int_{Q_\nu} f(x_0, u(x_0), t_k \nabla v_k(y))\, dy. \tag{2.12}$$

Due to the presence of the sequence t_k, we cannot apply directly Serrin's Theorem A(i) to the integrand $g(\xi) := f(x_0, u(x_0), \xi)$ as we did in part(i). Although the adaptations to the present setting are quite straightforward, here we present an alternative proof which can be extended to the vectorial case and to quasiconvex functions. Assume, for simplicity, that $\nu = e_N$ and construct a sequence of smooth functions $v_h(y) = \bar{v}_h(y_N)$ such that

$$||v_h - w_0||_{L^1(Q;\mathbb{R})} \leq \frac{1}{h} \qquad \text{and} \qquad |\nabla v_h|(Q) - |Dw_0|(Q) \to 0.$$

as $h \to \infty$. Since v_h depends only on y_N, its trace on ∂Q agrees with the trace of a function

$$A_h\, y + p(y), \qquad A_h := (\bar{v}_h(1/2) - \bar{v}_h(-1/2)) \otimes e_N = \nabla v_h(Q),$$

where p is Q–periodic. Choose open sets Ω_i, $i = 1, 2, 3$, such that $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset Q$. Since g satisfies (1.8), by Lemma 2.5 of Ambrosio and Dal Maso [4] we may find a new sequence

$$v_{h,k}(y) = \varphi(y) w_k(y) + (1 - \varphi(y)) v_h(y)$$

such that

$$\frac{1}{t_k}\int_{\Omega_3} g(t_k \nabla w_k)\, dy + \frac{1}{t_k}\int_{Q\setminus\Omega_1} g(t_k \nabla v_h)\, dy + \frac{4C}{\delta}\int_Q |w_k - v_h|\, dy + \frac{1}{k} \geq \frac{1}{t_k}\int_Q g(t_k \nabla v_{h,k})\, dy$$

where φ is a cut–off function such that $\varphi \equiv 1$ in a neighborhood of Ω_2, $\varphi \equiv 0$ in a neighborhood of $\mathbb{R}^N \setminus \Omega_3$, and $\delta <$ dist $(\Omega_2, \partial\Omega_3)$. By virtue of the quasiconvexity of g, together with the growth (1.8) (recall that in the scalar case quasiconvexity is equivalent to convexity), we obtain

$$\frac{1}{t_k}\int_Q g(t_k \nabla w_k)\, dy + C\left(\frac{\mathcal{L}^N(Q\setminus\Omega_1)}{t_k} + |\nabla v_h|(Q\setminus\Omega_1)\right) + \frac{4C}{\delta}\int_Q |w_k - v_h|\, dy + \frac{1}{k} \geq \frac{1}{t_k} g(t_k \nabla v_h(Q)).$$

Letting $k \to \infty$ gives

$$\lim_{k\to\infty} \frac{1}{t_k}\int_Q g(t_k \nabla w_k)\, dy + C|\nabla v_h|(Q\setminus\Omega_1) + \frac{4C}{\delta}\int_Q |w_0 - v_h|\, dy \geq g^\infty(\nabla v_h(Q));$$

hence, taking the limit as $h \to \infty$, we conclude that

$$\lim_{k\to\infty} \frac{1}{t_k}\int_Q g(t_k \nabla w_k)\, dy + C|Dw_0|(Q\setminus\Omega_1) \geq g^\infty(\nabla w_0(Q)),$$

where we used the continuity of g^∞ (see Proposition 9.1), and the fact that $|\nabla v_n|(Q) \to |Dw_0|(Q)$. If we now let $\Omega_1 \nearrow Q$ we get

$$\lim_{k\to\infty} \frac{1}{t_k}\int_Q g(t_k \nabla w_k)\, dy \geq g^\infty(a\,\nu);$$

thus,

$$(1+\varepsilon)\frac{d\mu}{d|C(u)|}(x_0)+\varepsilon \geq f^\infty\left(x_0, u(x_0), \frac{dC(u)}{d|C(u)|}(x_0)\right).$$

It suffices now to let $\varepsilon \to 0^+$. □

(iii) *(Jump part)* To complete the proof of Theorem 1.1 it remains to show that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu)\, ds \qquad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x_0 \in S(u),$$

where $\nu = \nu_u(x_0)$ is the normal to $S(u)$. It is known that (see [7]) for $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$

$$\begin{gathered}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{\varepsilon\to 0^+}\frac{\mu(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} < \infty, \\ \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q_\nu^+(x_0,\varepsilon)}|u(x)-u^+(x_0)|dx = 0, \qquad \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q_\nu^-(x_0,\varepsilon)}|u(x)-u^-(x_0)|dx = 0,\end{gathered} \tag{2.13}$$

where $Q_\nu^+(x_0,\varepsilon) := \{y \in Q_\nu(x_0,\varepsilon) : y\cdot\nu > 0\}$ and $Q_\nu^-(x_0,\varepsilon) := \{y \in Q_\nu(x_0,\varepsilon) : y\cdot\nu < 0\}$. Fix $x_0 \in S(u)$ such that (2.13) holds, and choose a sequence $\varepsilon_k \to 0^+$ with $\mu(\partial Q_\nu(x_0,\varepsilon_k)) = 0$. Then

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &= \lim_{k\to\infty}\frac{\mu(Q_\nu(x_0,\varepsilon_k))}{\varepsilon_k^{N-1}} = \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\varepsilon_k^{N-1}}\int_{Q_\nu(x_0,\varepsilon_k)} f(x,u_n,\nabla u_n)\,dx \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\nu}\varepsilon_k f\left(x_0+\varepsilon_k y, w_{n,k}(y), \frac{1}{\varepsilon_k}\nabla w_{n,k}(y)\right)\,dy,\end{aligned}$$

where $w_{n,k}(y) := u_n(x_0+\varepsilon_k y)$. Clearly $w_{n,k} \in W^{1,1}(Q_\nu;\mathbb{R})$, and by (2.13) together with the fact that u_n converges to u in L^1,

$$\lim_{k\to\infty}\lim_{n\to\infty}||w_{n,k}-w_0||_{L^1(Q_\nu;\mathbb{R})} = 0, \qquad \text{where} \qquad w_0(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu \leq 0.\end{cases}$$

As before, by a standard diagonalization argument we may extract a subsequence $\{w_k := w_{n_k,k}\}$ converging to w_0 in $L^1(Q_\nu;\mathbb{R})$, with

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{k\to\infty}\int_{Q_\nu}\varepsilon_k f\left(x_0+\varepsilon_k y, w_k(y), \frac{1}{\varepsilon_k}\nabla w_k(y)\right)\,dy. \tag{2.14}$$

Fix $\varepsilon > 0$. By (2.3) for each $u_1 \in [u^-(x_0), u^+(x_0)]$ there is $\delta_{u_1} > 0$ such that

$$f(x_0,u,\xi) - f(x,u,\xi) \leq \varepsilon(1+f(x,u,\xi))$$

for all $|x-x_0| \leq \delta_{u_1}$, $|u-u_1| \leq \delta_{u_1}$, and for all $\xi \in \mathbb{R}^N$. Since

$$\bigcup_{u_1\in[u^-(x_0),u^+(x_0)]} B(u_1,\delta_{u_1}) \supset [u^-(x_0),u^+(x_0)],$$

we may find a finite subcovering

$$\bigcup_{i=1}^M B(u_i,\delta_i) \supset [u^-(x_0),u^+(x_0)].$$

Set $\delta := \min\{\delta_1,\dots,\delta_M,\delta^+,\delta^-\}$, where $\delta^\pm$ are provided by (2.3) corresponding to the points $(x_0,u^\pm(x_0))$, respectively. Then

$$f(x_0,u,\xi) - f(x,u,\xi) \leq \varepsilon(1+f(x,u,\xi)) \tag{2.15}$$

for all $|x - x_0| \le \delta$, $u \in [u^-(x_0) - \delta, u^+(x_0) + \delta]$, and for all $\xi \in \mathbb{R}^N$.

By Proposition 2.1, with

$$\lambda_k :\equiv 1, \qquad t_k := 1/\varepsilon_k, \qquad u_{0k} :\equiv 0, \qquad \alpha_1 = u^-(x_0) - \delta, \qquad \alpha_2 := u^+(x_0) + \delta,$$

there exists a new sequence $v_k \in W^{1,1}(Q_\nu;\mathbb{R})$, convergent to w_0 in $L^1(Q_\nu;\mathbb{R})$, such that $v_k(y) \in [u^-(x_0) - \alpha_1, u^+(x_0) + \alpha_2]$, and by (2.14) and (2.15) we have

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \ge \liminf_{k\to\infty} \int_{Q_\nu} \varepsilon_k f\left(x_0, v_k(y), \frac{1}{\varepsilon_k}\nabla v_k(y)\right)\, dy.$$

Since $h(u,\xi) := f(x_0,u,\xi)$ is continuous, by an approximation result due to Ambrosio [3] we can write

$$h(u,\xi) = \sup_{i\in\mathbb{N}}[a_i(u) + b_i(u)\cdot\xi]^+$$

where the functions $a_i : \mathbb{R} \to \mathbb{R}$ and $b_i : \mathbb{R} \to \mathbb{R}^N$ are bounded and continuous. It is not difficult to see that

$$h^\infty(u,\xi) = \sup_i [b_i(u)\cdot\xi]^+.$$

Therefore (see [12, Lemma 6])

$$\int_{u^-(x_0)}^{u^+(x_0)} h^\infty(s,\nu)\, ds = \sup_{j\in\mathbb{N}} \sup\left\{\sum_{i=1}^{j} \int_{u^-(x_0)}^{u^+(x_0)} \psi_i(s)[b_i(s)\cdot\nu]^+\, ds : \psi_i \in C_0^\infty((u^-(x_0)\text{'}u^+(x_0));[0,1]), \sum_{i=1}^{j}\psi \le 1\right\}. \tag{2.16}$$

Fix $j \in \mathbb{N}$, $\psi_1,\dots,\psi_j$, as in (2.16), and let $\varphi \in C_0^\infty(Q_\nu;[0,1])$. We have

$$\begin{aligned}(1+\varepsilon)\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\ge \liminf_{k\to\infty} \sum_{i=1}^{j} \int_{Q_\nu} \varphi(y)\varepsilon_k\psi_i(v_k(y))\left[a_i(v_k(y)) + \frac{1}{\varepsilon_k} b_i(v_k(y))\cdot\nabla v_k(y)\right]^+ dy \\ &\ge \liminf_{k\to\infty} \sum_{i=1}^{j} \int_{Q_\nu} \varphi(y)\psi_i(v_k(y))\ [b_i(v_k(y))\cdot\nabla v_k(y)]^+\, dy,\end{aligned}$$

where we have used the inequality $(\alpha+\beta)^+ \ge (\beta)^+ - |\alpha|$ for α, $\beta \in \mathbb{R}$, Lebesgue Dominated Convergence Theorem, and the fact that

$$\varepsilon_k|a_i(v_k(y))| \le ||a_i||_{L^\infty(\mathbb{R})}\,\varepsilon_k \to 0.$$

By a result of De Cicco [12, Theorem 1], we have

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \ge \sum_{i=1}^{j} \int_{S(w_0)} \varphi(y) \int_{u^-(x_0)}^{u^+(x_0)} \psi_i(s)\,[b_i(s)\cdot\nu]^+\, ds\, d\mathcal{H}^{N-1}(y),$$

and taking $\varphi \nearrow 1$, we obtain

$$\begin{aligned}(1+\varepsilon)\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\ge \sum_{i=1}^{j} \int_{S(w_0)} \int_{u^-(x_0)}^{u^+(x_0)} \psi_i(s)\,[b_i(s)\cdot\nu]^+\, ds\, d\mathcal{H}^{N-1}(y) \\ &= \sum_{i=1}^{j} \int_{u^-(x_0)}^{u^+(x_0)} \psi_i(s)\,[b_i(s)\cdot\nu]^+\, ds.\end{aligned}$$

In view of (2.16), the proof is concluded by taking the supremum over all $j \in \mathbb{N}$, $\psi_1,\dots,\psi_j$, and letting $\varepsilon \to 0^+$. □

§3. Proof of Theorem 1.3.

(i)*(Lebesgue part)* Assume first that f is Carathéodory. Consider $\mathcal{F} : BV(\Omega;\mathbb{R}) \times \mathcal{A}(\Omega) \to [0,\infty]$, where $\mathcal{A}(\Omega)$ stands for the family of open subsets of Ω. It can be proved that $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure, $\mathcal{F}(\cdot;A)$ is $L^1(A)$ lower semicontinuous, and

$$0 \le \mathcal{F}(u,A) \le C\left(\mathcal{L}^N(A) + |Du|(A)\right).$$

For a proof we refer to Lemma 4.1.2 of [7] (see also [5], [18]). Let $\mathcal{F}_1(u,A) := \mathcal{F}(u,A) + |Du|(A)$. By Theorem 3.7 of [7] we have

$$\frac{d\mathcal{F}_1(u,\cdot)}{d\mathcal{L}^N}(x_0) = \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) + |\nabla u(x_0)| = f_1(x_0,u(x_0),\nabla u(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega,$$

where

$$f_1(x_0,u_0,\xi) := \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \inf\left\{\mathcal{F}_1(v,Q(x_0,\varepsilon)) : v \in BV(Q(x_0,\varepsilon)),\, v|_{\partial Q(x_0,\varepsilon)} = u_0 + \xi\cdot(x-x_0)\right\}.$$

Thus, the proof of part (i) is completed provided we show that

$$f_1(x_0,u_0,\xi) \le f(x_0,u_0,\xi) + |\xi|$$

for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ and for all $(u_0,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Clearly

$$f_1(x_0,u_0,\xi) \le \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{F}(u_0+\xi\cdot(x-x_0),Q(x_0,\varepsilon))+|\xi| \le \limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f(x,u_0+\xi\cdot(x-x_0),\xi)\,dx+|\xi|.$$

Since f is Carathéodory, by the Scorza-Dragoni Theorem for each $i \in \mathbb{N}$ there exists a compact set $K_i \subset \Omega$, with $\mathcal{L}^N(\Omega\backslash K_i) \le 1/i$, such that $f : K_i \times \mathbb{R}\times\mathbb{R}^N \to [0,\infty]$ is continuous. Let K_i^* be the set of Lebesgue points of χ_{K_i}, and set $\omega := \cup_i^\infty (K_i \cap K_i^*)$. Then

$$\mathcal{L}^N(\Omega\backslash\omega) \le \mathcal{L}^N(\Omega\backslash K_i) \le \frac{1}{i} \to 0 \quad \text{as } i \to \infty.$$

If $x_0 \in \omega$ then $x_0 \in K_i \cap K_i^*$ for some index i. Since $g(x) := f(x,u_0+\xi(x-x_0),\xi)$ is continuous over K_i, given $\delta > 0$ there exists $\eta > 0$ such that $g(x) \le g(x_0) + \delta$ for all $x \in K_i$ with $|x-x_0| \le \eta$. Therefore, by (1.8) we have

$$\begin{aligned} f_1(x_0,u_0,\xi) - |\xi| &\le (f(x_0,u_0,\xi)+\delta)\limsup_{\varepsilon\to 0^+}\frac{\mathcal{L}^N(Q(x_0,\varepsilon)\cap K_i)}{\varepsilon^N} + C(1+|\xi|)\limsup_{\varepsilon\to 0^+}\frac{\mathcal{L}^N(Q(x_0,\varepsilon)\backslash K_i)}{\varepsilon^N} \\ &= f(x_0,u_0,\xi)+\delta, \end{aligned}$$

where we have used the fact that x_0 is a Lebesgue point of χ_{K_i}. By letting $\delta \to 0^+$ we obtain the desired inequality. The argument for the case where $f(\cdot,\cdot,\xi)$ is upper semicontinuous is very similar to the one used in Theorem 1.3(ii) below, and therefore we omit the details. □

(ii)*(Cantor part)* By Lemma 3.9 of [7], for $C(u)$ a.e. $x_0 \in \Omega$ there exist a double indexed sequence $\{t_\varepsilon^{(k)}, u_\varepsilon^{(k)}\}$ such that for every k

$$t_\varepsilon^{(k)} \to \infty, \qquad \varepsilon\, t_\varepsilon^{(k)} \to 0^+, \qquad u_\varepsilon^{(k)} \to u(x_0) \quad \text{as } \varepsilon \to 0^+, \tag{3.1}$$

and

$$\begin{aligned} &\frac{d\mathcal{F}_1(u,\cdot)}{d|C(u)|}(x_0) = \frac{d\mathcal{F}(u,\cdot)}{d|C(u)|}(x_0) + |a| \\ &= \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+}\frac{\inf\left\{\mathcal{F}_1(v,Q_\nu^{(k)}(x_0,\varepsilon)) : v \in BV(Q_\nu^{(k)}(x_0,\varepsilon)),\, v|_{\partial Q_\nu^{(k)}(x_0,\varepsilon)} = u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0)\right\}}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}}, \end{aligned}$$

where $\dfrac{dC(u)}{d|C(u)|}(x_0) = a\nu$, $a = a(u,x_0) \in \mathbb{R}$, $\nu = \nu(u,x_0) \in S^{N-1}$, $Q_\nu^{(k)}(x_0,\varepsilon) := x_0 + \varepsilon\, Q_\nu^{(k)}$ with

$$Q_\nu^{(k)} := R_\nu\left(\left(-\frac{k}{2},\frac{k}{2}\right)^{N-1} \times \left(-\frac{1}{2},\frac{1}{2}\right)\right),$$

where R_ν denotes a rotation such that $R_\nu e_N = \nu$. Take $x_0 \in \Omega$, so that all the limits above exist and are finite. Then

$$\begin{aligned}\frac{d\mathcal{F}(u,\cdot)}{d|C(u)|}(x_0) + |a| &\le \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}}\, \mathcal{F}_1(u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), Q_\nu^{(k)}(x_0,\varepsilon)) \\ &\le \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} f(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), t_\varepsilon^{(k)} a\nu)\, dx + |a|.\end{aligned} \tag{3.2}$$

By Proposition 9.1, (9.1), and (1.8),

$$\begin{aligned}\frac{f(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu(x-x_0), t_\varepsilon^{(k)} a\nu)}{t_\varepsilon^{(k)}} &\le f^\infty(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), a\nu) + \frac{f(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), 0)}{t_\varepsilon^{(k)}} \\ &\le f^\infty(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), a\nu) + \frac{C}{t_\varepsilon^{(k)}}.\end{aligned}$$

Therefore, by (3.1) and (3.2),

$$\begin{aligned}\frac{d\mathcal{F}(u,\cdot)}{d|C(u)|}(x_0) &\le \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}\varepsilon^N} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} f^\infty(x, u_\varepsilon^{(k)} + t_\varepsilon^{(k)} a\nu\cdot(x-x_0), a\nu) dx \\ &= \lim_{k\to\infty}\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} f^\infty(x_0+\varepsilon y, u_\varepsilon^{(k)} + \varepsilon\, t_\varepsilon^{(k)} a\nu\cdot y, a\nu)\, dy.\end{aligned}$$

Since the function $f^\infty(\cdot,\cdot,a\nu)$ is upper semicontinuous, given $\delta > 0$ there exists $\eta > 0$ such that $f^\infty(x,u,a\nu) \le f^\infty(x_0,u(x_0),a\nu) + \delta$ for all $|x-x_0| \le \eta$ and $|u-u(x_0)| \le \eta$. By (3.1), for each fixed k if ε is small enough then $x_0 + \varepsilon y \in B(x_0,\eta)$ and $u_\varepsilon^{(k)} + \varepsilon\, t_\varepsilon^{(k)} a\nu\cdot y \in [u(x_0)-\eta, u(x_0)+\eta]$ for all $y \in Q_\nu^{(k)}$. Hence

$$\int_{Q_\nu^{(k)}} f^\infty(x_0+\varepsilon y, u_\varepsilon^{(k)} + \varepsilon\, t_\varepsilon^{(k)} a\nu\cdot y, a\nu)\, dy \le (f^\infty(x_0,u(x_0),a\nu) + \delta)\, k^{N-1}$$

and, in turn,

$$\frac{d\mathcal{F}(u,\cdot)}{d|C(u)|}(x_0) \le f^\infty(x_0,u(x_0),a\nu) + \delta.$$

We now let $\delta \to 0^+$. □

(iii) *(Jump part)* By Theorem 3.7 of [7], for $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$

$$\begin{aligned}\frac{d\mathcal{F}_1(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &= \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) + |u^+(x_0) - u^-(x_0)| \\ &= \limsup_{\varepsilon\to 0^+} \frac{\inf\left\{\mathcal{F}_1(v, Q_\nu(x_0,\varepsilon)) :\, v \in BV(Q_\nu(x_0,\varepsilon)),\, v|_{\partial Q_\nu(x_0,\varepsilon)} = w_0\right\}}{\varepsilon^{N-1}},\end{aligned}$$

where $\nu = \nu_u(x_0)$ is the normal to $S(u)$ and

$$w_0(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu \le 0. \end{cases}$$

Take $x_0 \in S(u)$ so that all the limit above exists and is finite. Then

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) + |u^+(x_0) - u^-(x_0)| \le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_0, Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} + |u^+(x_0) - u^-(x_0)|.$$

In what follows we assume for simplicity that $x_0 = 0$ and $\nu = e_N$, and we set

$$u_{n,\varepsilon}(x_N) := \begin{cases} u^+(x_0) & \text{if } x_N \ge \varepsilon/2n \\ (u^+(x_0) - u^-(x_0))\dfrac{n}{\varepsilon}x_N + \dfrac{u^+(x_0)+u^-(x_0)}{2} & \text{if } -\varepsilon/2n \le x_N \le \varepsilon/2n \\ u^-(x_0) & \text{if } x_N \le -\varepsilon/2n. \end{cases}$$

Clearly $||u_{n,\varepsilon} - w_0||_{L^1(Q_\nu(x_0,\varepsilon))} \to 0$ as $n \to \infty$; thus, by a standard diagonalization argument

$$\begin{aligned} \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\le \lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)} f(x, u_{n,\varepsilon}(x_N), 0, \dots, 0, u'_{n,\varepsilon}(x_N))\, dx \\ &\le \liminf_{k\to\infty} \int_{Q_\nu} \varepsilon_k f\left(x_0 + \varepsilon_k y, v_k(y_N), 0, \dots, 0, \frac{1}{\varepsilon_k} v'_k(y_N)\right) dy, \end{aligned}$$

where

$$v_k(y_N) = \begin{cases} u^+(x_0) & \text{if } y_N \ge 1/2n_k \\ (u^+(x_0) - u^-(x_0))n_k\, y_N + \dfrac{u^+(x_0)+u^-(x_0)}{2} & \text{if } -1/2n_k \le y_N \le 1/2n_k \\ u^-(x_0) & \text{if } y_N \le -1/2n_k \end{cases}$$

and $n_k \to \infty$ as $k \to \infty$. By Proposition 9.1, (9.1), and (1.8), we have

$$\begin{aligned} \varepsilon_k f\left(x_0 + \varepsilon_k y, v_k(y_N), 0, \dots, 0, \frac{1}{\varepsilon_k} v'_k(y_N)\right) &\le f^\infty(x_0 + \varepsilon_k y, v_k(y_N), 0, \dots, 0, v'_k(y_N)) \\ &+ \varepsilon_k f(x_0 + \varepsilon_k y, v_k(y_N), 0) \le f^\infty(x_0 + \varepsilon_k y, v_k(y_N), 0, \dots, 0, v'_k(y_N)) + C\,\varepsilon_k. \end{aligned}$$

Therefore, Fubini's Theorem yields

$$\begin{aligned} \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\le \liminf_{k\to\infty} \int_Q f^\infty(x_0 + \varepsilon_k y, v_k(y_N), 0, \dots, 0, v'_k(y_N))\, dy \\ &= \liminf_{k\to\infty} n_k \int_{Q'} \left(\int_{-1/2n_k}^{1/2n_k} f^\infty(x_0 + \varepsilon_k y, v_k(y_N), (u^+(x_0) - u^-(x_0))\, e_N)\, dy_N \right) dy', \end{aligned} \tag{3.3}$$

where Q' is the unit cube in $\mathbb{R}^{N-1}$, and where we have used the fact that f^∞ is positively homogeneous in ξ. Following [5], we introduce now the Yosida transforms

$$f_\lambda(x,u,\xi) := \sup\{ f^\infty(x',u,\xi) - \lambda|x'-x| : x' \in \Omega \}, \tag{3.4}$$

for $\lambda > 0$. For $\lambda \le \eta$ and by (1.8) it follows that

$$0 \le f^\infty(x,u,\xi) \le f_\eta(x,u,\xi) \le f_\lambda(x,u,\xi) \le C|\xi|. \tag{3.5}$$

We claim that

$$\lim_{\lambda\to\infty} f_\lambda(x,u,\xi) = f^\infty(x,u,\xi). \tag{3.6}$$

Indeed, let $\lambda > 1$ and choose x_λ such that

$$f_\lambda(x,u,\xi) \le f^\infty(x_\lambda,u,\xi) - \lambda|x_\lambda - x| + \frac{1}{\lambda}.$$

By (3.5)

$$f^\infty(x,u,\xi) \le f_\lambda(x,u,\xi) \le f_\lambda(x,u,\xi) + \lambda|x_\lambda - x| \le f^\infty(x_\lambda,u,\xi) + \frac{1}{\lambda}. \tag{3.7}$$

Since the right hand side is bounded by $C|\xi| + 1$ and $f_\lambda \ge 0$, it follows that $x_\lambda \to x$ as $\lambda \to \infty$. If we now let $\lambda \to \infty$ in (3.7), and use the fact that $f^\infty(\cdot,u,\xi)$ is upper semicontinuous, we obtain (3.6).

Next we show that f_λ is Lipschitzian. Fix $\varepsilon > 0$, x, $x_1 \in \Omega$, and find x_ε such that

$$\begin{aligned} f_\lambda(x,u,\xi) &\le f^\infty(x_\varepsilon,u,\xi) - \lambda|x_\varepsilon - x| + \varepsilon \\ &\le f^\infty(x_\varepsilon,u,\xi) - \lambda|x_\varepsilon - x_1| + \lambda|x - x_1| + \varepsilon \le f_\lambda(x_1,u,\xi) + \lambda|x - x_1| + \varepsilon. \end{aligned}$$

If we now let $\varepsilon \to 0$ we obtain

$$f_\lambda(x,u,\xi) - f_\lambda(x_1,u,\xi) \le \lambda|x - x_1|,$$

and, in a similar way,

$$f_\lambda(x_1,u,\xi) - f_\lambda(x,u,\xi) \le \lambda|x - x_1|.$$

We conclude that

$$|f_\lambda(x,u,\xi) - f_\lambda(x_1,u,\xi)| \le \lambda|x - x_1| \tag{3.8}$$

for all $x, x_1 \in \Omega, u \in \mathbb{R}, \xi \in \mathbb{R}^N$.

Fix $\lambda > 0$. By (3.3), (3.5), and the fact that f_λ is Lipschitzian, we have

$$\begin{aligned} \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\le \liminf_{k\to\infty} n_k \int_{Q'} \left(\int_{-1/2n_k}^{1/2n_k} f^\infty(x_0 + \varepsilon_k y, v_k(y_N), (u^+(x_0) - u^-(x_0))\, e_N)\, dy_N \right) dy' \\ &\le \liminf_{k\to\infty} n_k \int_{Q'} \left(\int_{-1/2n_k}^{1/2n_k} f_\lambda(x_0 + \varepsilon_k y, v_k(y_N), (u^+(x_0) - u^-(x_0))\, e_N)\, dy_N \right) dy' \\ &\le \liminf_{k\to\infty} n_k \int_{Q'} \left(\int_{-1/2n_k}^{1/2n_k} f_\lambda(x_0, v_k(y_N), (u^+(x_0) - u^-(x_0))\, e_N) + \lambda \varepsilon_k |y|\, dy_N \right) dy' \\ &= \liminf_{k\to\infty} n_k \int_{-1/2n_k}^{1/2n_k} f_\lambda(x_0, v_k(y_N), (u^+(x_0) - u^-(x_0))\, e_N)\, dy_N. \end{aligned}$$

A simple change of variables now yields

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \le \frac{1}{u^+(x_0) - u^-(x_0)} \int_{u^-(x_0)}^{u^+(x_0)} f_\lambda(x_0, s, (u^+(x_0) - u^-(x_0))\, e_N)\, ds.$$

Letting $\lambda \to \infty$, by (3.5), (3.6), and Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} \frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\le \lim_{\lambda\to\infty} \frac{1}{u^+(x_0) - u^-(x_0)} \int_{u^-(x_0)}^{u^+(x_0)} f_\lambda(x_0, s, (u^+(x_0) - u^-(x_0)) e_N)\, ds \\ &= \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, e_N)\, ds, \end{aligned}$$

where we have used again the fact that f^∞ is positively homogeneous in ξ. □

§4. Proof of Theorem 1.5.

(i) *(Lebesgue part)* We claim that $g(x) := f(x, u(x), \nabla u(x)) \in L^1_{\rm loc}(\Omega; \mathbb{R})$. To show this it is clearly enough to prove that

$$\int_{Q(x_0,\delta)} g(x)\, dx < \infty$$

for any $x_0 \in \Omega$ and $\delta > 0$ sufficiently small. Let $\delta > 0$ correspond to $\varepsilon = 1/2$ in (1.9). Then

$$\infty > \lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \geq \frac{2}{3} \liminf_{n\to\infty} \int_{Q(x_0,\delta)} f(x_0, u_n(x), \nabla u_n(x))\, dx - \frac{1}{3}\,\delta^N.$$

The functional

$$F_0(v, A) := \int_A f(x_0, v, \nabla v)\, dx + \int_A f^\infty(x_0, v, dC(v)) + \int_{S(v)\cap A} \left(\int_{v^-(x)}^{v^+(x)} f^\infty(x_0, s, \nu_u)\, ds \right) d\mathcal{H}^{N-1}$$

satisfies all the conditions of Theorem 2 in [12], and $F_0(u_n, A) = \int_A f(x_0, u_n(x), \nabla u_n(x))\, dx$ since $u_n \in W^{1,1}_{\rm loc}(\Omega; \mathbb{R})$; thus, taking $A := Q(x_0, \delta)$,

$$\infty > \lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \geq \frac{2}{3}\, F_0(u, Q(x_0, \delta)) - \frac{1}{3}\delta^N \geq \frac{1}{3} \int_{Q(x_0,\delta)} g(x)\, dx - \frac{2}{3}\delta^N, \tag{4.1}$$

where we have used (1.9) a second time and H is defined in (1.2). Therefore the claim is proved.

Let $\Omega(u)$ be the set of Lebesgue points of g. Since $g \in L^1_{\rm loc}(\Omega; \mathbb{R})$, we have that $\mathcal{L}^N(\Omega \backslash \Omega(u)) = 0$. We now proceed essentially as in the proof of Theorem 1.1, starting from (2.4) up to (2.6), where without loss of generality, we may also assume that $x_0 \in \Omega(u)$. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that (1.9) holds. Choose $\varepsilon_k \to 0^+$ such that $\mu(\partial Q(x_0, \varepsilon_k)) = 0$, $\varepsilon_k \leq \delta$, and

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x, u_n, \nabla u_n)\, dx \\ &\geq \frac{1}{1+\varepsilon} \liminf_{k\to\infty} \liminf_{n\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x_0, u_n, \nabla u_n)\, dx - \frac{\varepsilon}{1+\varepsilon}. \end{aligned} \tag{4.2}$$

For fixed k, again by Theorem 2 in [12] applied this time to $F_0(\cdot, Q(x_0, \varepsilon_k))$, we obtain

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x_0, u_n(x), \nabla u_n(x))\, dx \geq \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x_0, u(x), \nabla u(x))\, dx$$

(recall that $f^\infty \geq 0$) and, consequently,

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \frac{1}{1+\varepsilon} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x_0, u, \nabla u)\, dx - \frac{\varepsilon}{1+\varepsilon} \\ &\geq \frac{1-\varepsilon}{1+\varepsilon} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(x, u, \nabla u)\, dx - \frac{2\varepsilon}{1+\varepsilon} = \frac{1-\varepsilon}{1+\varepsilon} f(x_0, u(x_0), \nabla u(x_0)) - \frac{2\varepsilon}{1+\varepsilon} \end{aligned}$$

where we have used (1.9) and the fact that $x_0 \in \Omega(u)$. We now let $\varepsilon \to 0^+$. □

(ii) *(Cantor part)* By (4.1) the function $h(x) := f^\infty\left(x, u(x), \frac{dC(u)}{d|C(u)|}(x)\right) \in L^1_{\rm loc}(\Omega; |C(u)|)$; thus, by Lebesgue–Besicovitch Differentiation Theorem,

$$\lim_{\varepsilon\to 0^+} \frac{1}{|C(u)|(Q_\nu(x_0, \varepsilon))} \int_{Q_\nu(x_0,\varepsilon)} h\, d|C(u)| = h(x_0)$$

for $|C(u)|$ a.e. $x_0 \in \Omega$. Moreover, it is known that for $C(u)$ a.e. $x_0 \in \Omega$

$$\lim_{\varepsilon \to 0^+} \frac{|C(u)|(Q_\nu(x_0,\varepsilon))}{|Du|(Q_\nu(x_0,\varepsilon))} = 1;$$

hence,

$$\lim_{\varepsilon \to 0^+} \frac{1}{|Du|(Q_\nu(x_0,\varepsilon))} \int_{Q_\nu(x_0,\varepsilon)} h(x)\, d|C(u)|(x) = h(x_0) \tag{4.3}$$

for $|C(u)|$ a.e. $x_0 \in \Omega$. Let $M_1(u)$ be the set of all points of Ω which satisfy (4.3). Then $|C(u)|(\Omega \backslash M_1(u)) = 0$. We now proceed as in the proof of part (ii) in Theorem 1.1 up to (2.10), with the only difference that we impose the further restriction that $x_0 \in M_1(u)$, to obtain

$$\frac{d\mu}{d|C(u)|}(x_0) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{|Du|(Q_\nu(x_0,\varepsilon_k))} \int_{Q_\nu(x_0,\varepsilon_k)} f(x,u_n,\nabla u_n) dx.$$

The remaining of the proof follows an argument similar to that of part (ii) above after (4.2), except that the integral is now averaged over $Q_\nu(x_0,\varepsilon_k)$, and we use (4.3) at the end. We omit the details. □

(iii) *(Jump part)* It suffices to use arguments similar to those of parts (i) and (ii) above. □

§5. Proof of Theorem 1.6.

(i) *(Lebesgue part)* Fix $n \in \mathbb{N}$. Applying the Scorza–Dragoni Theorem to the function $f : \Omega \times [-n,n] \times \mathbb{R}^N \to [0,\infty)$ for each $i \in \mathbb{N}$ there exists a compact set $K_{i,n} \subset \mathbb{R}$, with $\mathcal{L}^1([-n,n] \backslash K_{i,n}) \leq 1/(i2^n)$, such that $f : \Omega \times K_{i,n} \times \mathbb{R}^N \to [0,\infty)$ is continuous. Let $K^*_{i,n}$ be the set of Lebesgue points of $\chi_{K_{i,n}}$, and set $\omega := \cup_n^\infty \cup_i^\infty (K_{i,n} \cap K^*_{i,n})$. Then

$$\mathcal{L}^1(\mathbb{R} \backslash \omega) \leq \sum_{n=1}^{\infty} \mathcal{L}^1([-n,n] \backslash K_{i,n}) \leq \frac{1}{i} \to 0 \quad \text{as } i \to \infty,$$

and so $|Du|(A) = 0$, where $A := \{x \in \Omega \backslash S(u) : u(x) \in \mathbb{R} \backslash \omega)\}$. Fix $x_0 \in \Omega \backslash S(u)$. If $x_0 \in A$ then, up to a set of $N-$dimensional Lebesgue measure zero, we may assume that $\nabla u(x_0) = 0$, so that, as in the proof of Theorem 1.3(i), we have

$$f_1(x_0,u(x_0),0) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{F}(u(x_0),Q(x_0,\varepsilon)) \leq \limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f(x,u(x_0),0)\, dx = f(x_0,u(x_0),0),$$

where we have used the fact that $f(\cdot,u(x_0),0)$ is continuous. If $x_0 \in (\Omega \backslash S(u)) \backslash A$ and $\nabla u(x_0) = 0$, then we proceed as above. If $x_0 \in (\Omega \backslash S(u)) \backslash A$ and $\nabla u(x_0) \neq 0$, then set

$$\nu := \frac{\nabla u(x_0)}{|\nabla u(x_0)|}, \qquad \xi := \nabla u(x_0), \qquad w_0(x) := u(x_0) + \xi(x - x_0).$$

Find $n \in \mathbb{N}$ such that $||w_0||_{L^\infty(Q_\nu(x_0,1))} \leq n$ and let $i \in \mathbb{N}$ be such that $u(x_0) \in K_{i,n} \cap K^*_{i,n}$. Since $g(x) := f(x,w_0(x),\xi)$ is continuous over $w_0^{-1}(K_{i,n})$, given $\delta > 0$ there exists $\eta > 0$ such that $g(x) \leq g(x_0) + \delta$ for all $x \in w_0^{-1}(K_{i,n})$ with $|x - x_0| \leq \eta$. Therefore, as in proof of Theorem 1.3(i), and by (1.8),

$$\begin{aligned} f_1(x_0,u(x_0),\xi) - |\xi| &= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{F}(u(x_0),Q(x_0,\varepsilon)) \\ &\leq (f(x_0,u(x_0),\xi) + \delta) \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^N(Q_\nu(x_0,\varepsilon) \cap w_0^{-1}(K_{i,n}))}{\varepsilon^N} \\ &\quad + C(1 + |\xi|) \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^N(Q_\nu(x_0,\varepsilon) \backslash w_0^{-1}(K_{i,n}))}{\varepsilon^N} \\ &\leq f(x_0,u(x_0),\xi) + \delta, \end{aligned} \tag{5.1}$$

since, by the coarea formula (see [31, Theorem 2.7.1]), and setting $\eta_\varepsilon(x) := u(x_0) + \varepsilon\xi \cdot x$ for $x \in Q_\nu$, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(Q_\nu(x_0,\varepsilon)\backslash w_0^{-1}(K_{i,n}))}{\varepsilon^N} &= 1 - \liminf_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(Q_\nu(x_0,\varepsilon)\cap w_0^{-1}(K_{i,n}))}{\varepsilon^N} \\ &= 1 - \liminf_{\varepsilon\to 0^+} \frac{1}{|\xi|\,\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)\cap w_0^{-1}(K_{i,n})} |Dw_0(x)|\,dx \\ &= 1 - \liminf_{\varepsilon\to 0^+} \frac{1}{|\xi|\,\varepsilon^N} \int_{K_{i,n}} \mathcal{H}^{N-1}(w_0^{-1}(s)\cap Q_\nu(x_0,\varepsilon))\,ds.\end{aligned}$$

For simplicity, we assume that $x_0 = 0$, $u(x_0) = 0$, $\nu = e_N$, so that

$$w_0^{-1}(s)\cap Q_\nu(x_0,\varepsilon) = \begin{cases} \varepsilon[-\frac{1}{2},\frac{1}{2}]^{N-1}\times\{s\} & \text{if } |s| \le \frac{\varepsilon}{2}|\xi|, \\ \emptyset & \text{if } |s| > \frac{\varepsilon}{2}|\xi|; \end{cases}$$

thus,

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(Q_\nu(x_0,\varepsilon)\backslash w_0^{-1}(K_{i,n}))}{\varepsilon^N} = 1 - \liminf_{\varepsilon\to 0^+} \frac{\mathcal{L}^1([-\frac{\varepsilon}{2}|\xi|,\frac{\varepsilon}{2}|\xi|]\cap K_{i,n})}{|\xi|\varepsilon} = 0$$

where we have used the fact that $u(x_0)$ is a Lebesgue point of $\chi_{K_{i,n}}$. By letting $\delta \to 0^+$ in (5.1) we obtain the desired inequality. □

(ii) *(Cantor part)*

The proof for the Cantor part is very similar to the previous one (see also the proof of Theorem 1.3(ii)), and therefore we omit the details. □

(iii) *(Jump part)* The proof follows the same arguments of the proof of Theorem 1.3(iii). □

§6. Proof of Theorem 1.7.

(i) *(Lebesgue part)* We proceed as in the proof of Theorem 1.1(i) up to (2.7), where instead of using the truncated sequence, we apply condition (1.10) to get

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \liminf_{k\to\infty} \int_Q f(x_0,u(x_0),\nabla w_k(y))\,dy - \int_Q \rho(\varepsilon_k\,|w_k(y)|)\,dy.$$

By Fatou's Lemma, and since ρ is continuous with $\rho(0) = 0$, we have

$$\begin{aligned}C - \limsup_{k\to\infty} \int_Q \rho(\varepsilon_k\,|w_k(y)|)\,dy &= \liminf_{k\to\infty} \int_Q [C(1+\varepsilon_k\,|w_k(y)|) - \rho(\varepsilon_k\,|w_k(y)|)]\,dy \\ &\ge \int_Q \liminf_{k\to\infty} [C(1+\varepsilon_k\,|w_k(y)|) - \rho(\varepsilon_k\,|w_k(y)|)]\,dy = C,\end{aligned}$$

and so

$$\int_Q \rho(\varepsilon_k\,|w_k(y)|)\,dy \to 0 \quad \text{as } k\to\infty.$$

Thus

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \liminf_{k\to\infty} \int_Q f(x_0,u(x_0),\nabla w_k(y))\,dy. \tag{6.1}$$

If $g(\xi) := f(x_0,u(x_0),\xi)$ is convex then we may apply Serrin's Theorem A(i), which continues to hold in the vectorial case. If g is quasiconvex and $q = 1$ in (1.11), then we apply a result of Ambrosio and Dal Maso [4] (see also [18]) to conclude that

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge f(x_0,u(x_0),\nabla u(x_0)).$$

It is now sufficient to let $\varepsilon \to 0^+$ to obtain the desired result. When $q > 1$ and g is quasiconvex in (1.11), then we can apply an approximation result of Kristensen [22, Proposition 1.9] to write

$$f(x_0, u(x_0), \xi) = \sup_j g_j(\xi),$$

where $g_j(\xi)$ is quasiconvex, $g_j(\xi) \le g_{j+1}(\xi)$, and $g_j(\xi) = a_j|\xi| + b_j$ for $|\xi|$ large, say $|\xi| \ge r_j$. From (6.1) and for any fixed j, applying [4] we have

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \liminf_{k\to\infty} \int_Q g_j(\nabla w_k(y))\,dy \ge g_j(\nabla u(x_0)),$$

and then let $j \to \infty$. □

(ii) *(Cantor part)* We proceed as in Theorem 1.1(ii) until (2.11), where (2.8) should now be written as

$$\frac{dC(u)}{d|C(u)|}(x_0) = a_u(x_0) \otimes \nu_u(x_0), \tag{6.2}$$

with $a_u(x_0) \in \mathbb{R}^d$ and $\nu_u(x_0) \in S^{N-1}$, and where we have used Alberti's result [2]. By (2.11) and (1.10)

$$\begin{aligned}(1+\varepsilon)\frac{d\mu}{d|C(u)|}(x_0) + \varepsilon \ge \lim_{k\to\infty} \Big\{ &\frac{1}{t_k}\int_{Q_\nu} f(x_0, u(x_0), t_k \nabla w_k(y))\,dy \\ &- \frac{1}{t_k}\int_{Q_\nu} \rho(|u(x_0) - u_{0k} + \lambda_k w_k(y)|)\,dy \Big\}.\end{aligned} \tag{6.3}$$

As $\rho(s) \le C(1+s)$ for all $s > 0$ and for some $C > 0$, w_k converges to w_0 in $L^1(Q_\nu; \mathbb{R}^d)$, u_{0k} converges to $u(x_0)$, $\lambda_k \to 0$, and $t_k \to \infty$ as $k \to \infty$, we have

$$\frac{1}{t_k}\int_{Q_\nu} \rho(|u(x_0) - u_{0k} + \lambda_k\, w_k(y)|)\,dy \to 0$$

and thus

$$(1+\varepsilon)\frac{d\mu}{d|C(u)|}(x_0) + \varepsilon \ge \lim_{k\to\infty} \frac{1}{t_k}\int_{Q_\nu} f(x_0, u(x_0), t_k \nabla w_k(y))\,dy. \tag{6.5}$$

If g is quasiconvex and $q = 1$ in (1.11) then we can proceed as in Theorem 1.1(ii) starting from (2.12). When $g(\xi) := f(x_0, u(x_0), \xi)$ is convex, or g is quasiconvex $q > 1$ in (1.11), then we use Proposition 9.1 below or Proposition 1.9 of Kristensen [22], to deduce from (6.3) and (6.4):

$$(1+\varepsilon)\frac{d\mu}{d|C(u)|}(x_0) + \varepsilon \ge \lim_{k\to\infty} \frac{1}{t_k}\int_{Q_\nu} g_j(t_k \nabla w_k(y))\,dy.$$

Proceeding as in the case $q = 1$, we obtain

$$\frac{d\mu}{d|C(u)|}(x_0) \ge g_j^\infty(a \otimes \nu).$$

Since the function $h_j(t) := g_j(ta \otimes \nu)$ is convex, by Proposition 9.1, (9.1), we have for $t > 1$

$$\frac{d\mu}{d|C(u)|}(x_0) \ge \frac{g_j(t\,a \otimes \nu)}{t} - \frac{g_j(0)}{t}, \qquad t > 1.$$

As $f(x_0, u(x_0), 0) = \sup_j g_j(0) \le C$, letting $j \to \infty$ yields

$$\frac{d\mu}{d|C(u)|}(x_0) \ge \frac{f(x_0, u(x_0), t\,a \otimes \nu)}{t} - \frac{C}{t}.$$

We now let $t \to \infty$. □

Remark. Note that when (1.12) holds then

$$f^\infty(x_0, u_0, \xi) = \begin{cases} +\infty & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

(iii) *(Jump part)* We proceed as in Theorem 1.1(iii) up to (2.14). By (2.14) and (1.10)

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) + \varepsilon \geq \lim_{k\to\infty} \int_{Q_\nu} \varepsilon_k f\left(x_0, \frac{1}{\varepsilon_k}\nabla w_k(y)\right)\, dy$$

Now we continue exactly as in the proof of the Cantor part in Theorem 1.7, starting from (6.5), with the vector a in place of $u^+(x_0) - u^-(x_0)$. □

§7. Proof of Theorems 1.8–1.10.

Proof of Theorem 1.8 (Lebesgue part). We proceed as in Theorem 1.1(i) until (2.7). If $f(x_0, u(x_0), \xi) \equiv 0$ for all ξ then there is nothing to prove. Thus, we assume that (1.13) and (1.14) hold, we fix $\varepsilon > 0$, and let $\delta > 0$ be given by (1.13) and (1.14).

Step 1. We prove first the theorem under the additional hypothesis that there exists $M > 0$ such that

$$0 \leq f(x, u, \xi) \leq M(1 + \xi) \tag{7.1}$$

for all $x \in \Omega$ with $|x - x_0| \leq \delta$, $u \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{dN}$. As in [17, Proposition 2.6], we may find $w_k \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^d)$, with $w_k \to w_0$ in L^1, $w_0(x) := \nabla u(x_0)\, x$, such that by (2.7), (1.14), and for k sufficiently large,

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \int_{Q\cap\{|w_k|\leq\delta/\varepsilon_k\}} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k(y), \nabla w_k(y))\, dy \geq C_1 \int_{Q\cap\{|w_k|\leq\delta/\varepsilon_k\}} |\nabla w_k(y)|\, dy - C_2;$$

thus there exists a constant $K > 0$ such that

$$\int_{Q\cap\{|w_k|\leq\delta/\varepsilon_k\}} |\nabla w_k(y))|\, dy \leq K. \tag{7.2}$$

In order to truncate w_k, fix $s_k > ||w_0||_{L^\infty(Q;\mathbb{R}^d)} + 1$, $L_k > s_k$, and construct a smooth cut–off function $g_k : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$g_k(u) = \begin{cases} u & \text{if } |u| \leq s_k, \\ 0 & \text{if } |u| \geq L_k, \end{cases}$$

with $|g_k(u)| \leq |u|$ and $|Dg_k(u)| \leq C\, L_k/(L_k - s_k)$ for all $u \in \mathbb{R}^d$. Define $v_k(y) := g_k(w_k(y))$, and

$$E_k := \{y \in Q : |w_k(y)| < s_k\}, \quad E_k^+ := \{y \in Q : |w_k(y)| > L_k\}, \quad E_k^- := \{y \in Q : s_k \leq |w_k(y)| \leq L_k\}.$$

Then

$$\begin{aligned} \int_Q |v_k(y) - w_0(y)| dy &= \int_{E_k} |w_k(y) - w_0(y)| dy + \int_{E_k^+} |w_0(y)| dy + \int_{E_k^-} |g_k(w_k(y)) - w_0(y)| dy \\ &\leq ||w_k - w_0||_{L^1(Q;\mathbb{R}^d)} + ||w_0||_{L^\infty(Q;\mathbb{R}^d)} \mathcal{L}^N(E_k^- \cup E_k^+) + \int_{E_k^-} |w_k(y)| dy \\ &\leq 2||w_k - w_0||_{L^1(Q;\mathbb{R}^d)} + 2||w_0||_{L^\infty(Q;\mathbb{R}^d)} \mathcal{L}^N(E_k^- \cup E_k^+) \to 0 \qquad \text{as } k \to \infty, \end{aligned}$$

because

$$0 \le \mathcal{L}^N(E_k^- \cup E_k^+) = \mathcal{L}^N(\{y \in Q : |w_k(y)| \ge s_k\}) \\ \le \mathcal{L}^N(\{y \in Q : |w_k(y) - w_0(y)| \ge 1\}) \le ||w_k - w_0||_{L^1(Q;\mathbb{R}^d)}. \tag{7.3}$$

Moreover

$$\int_Q f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy = \int_{E_k} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k(y), \nabla w_k(y))\, dy \\ + \int_{E_k^+} f(x_0 + \varepsilon_k y, u(x_0), 0)\, dy + \int_{E_k^-} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy. \tag{7.4}$$

We claim that the last two integrals are infinitesimal as $k \to \infty$. Indeed, by (7.1) and (7.3),

$$0 \le \int_{E_k^+} f(x_0 + \varepsilon_k y, u(x_0), 0)\, dy \le M\mathcal{L}^N(E_k^+) \to 0,$$

while, from (7.1) and the coarea formula,

$$\int_{E_k^-} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy \le M \int_{E_k^-} (1 + |\nabla g_k(w_k)\nabla w_k|)\, dy \\ \le M \left(\mathcal{L}^N(E_k^-) + \frac{CL_k}{L_k - s_k} \int_{E_k^-} |\nabla w_k|\, dy \right) \\ = M \left(\mathcal{L}^N(E_k^-) + \frac{CL_k}{L_k - s_k} \int_{s_k}^{L_k} \mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = t\})\, dt \right). \tag{7.5}$$

By Theorem 7.10 of [28] and (7.2), for $\mathcal{L}^1$ a.e. $L \le \delta/\varepsilon_k$ we have

$$\lim_{s \to L} \frac{1}{L - s} \int_s^L \mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = t\})\, dt = \mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = L\}). \tag{7.6}$$

Moreover, by Lemma 2.6 in [17] and (7.2), for any $0 < \alpha < \beta < \delta/\varepsilon_k$ we obtain

$$\operatorname*{essinf}_{L \in (\alpha,\beta)} L\mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = L\}) \le \frac{K}{\log(\beta/\alpha)}.$$

Set $\alpha := \delta/\varepsilon_k^{1/4}$ and $\beta := \delta/\varepsilon_k^{1/2}$, and find $L_k \in (\delta/\varepsilon_k^{1/4}, \delta/\varepsilon_k^{1/2})$ such that (7.6) holds and

$$L_k \mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = L_k\}) \le \frac{2K}{\log(1/\varepsilon_k^{1/4})}.$$

Choose $s_k \ge L_k/2$ so that

$$\frac{L_k}{L_k - s_k} \int_{s_k}^{L_k} \mathcal{H}^{N-1}(\{y \in Q : |w_k(y)| = t\})\, dt \le \frac{2K}{\log\left(1/\varepsilon_k^{1/4}\right)} + \frac{1}{k}.$$

Then the integral in the right hand side of (7.5) approaches zero as $k \to \infty$, and so, from (7.4),

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \liminf_{k \to \infty} \int_Q f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy.$$

Since $\varepsilon_k \to 0$, by (1.13) we obtain

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0)+\varepsilon \geq \liminf_{k\to\infty}\int_Q f(x_0,u(x_0),\nabla v_k(y))\,dy.$$

We can now continue as in the proof of Theorem 1.7(i), and the result is established if f satisfies (7.1).

Step 2. In the general case, let $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ be a cut–off function, with $0 \leq \psi \leq 1$, and such that $\psi \equiv 1$ on $B(u(x_0),\delta/2)$, $\psi \equiv 0$ outside $B(u(x_0),\delta)$. From (2.7)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{k\to\infty}\int_Q \psi(u(x_0)+\varepsilon_k w_k(y))f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y),\nabla w_k(y))\,dy,$$

and by (1.13),

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0)+\varepsilon \geq \liminf_{k\to\infty}\int_Q \psi(u(x_0)+\varepsilon_k w_k(y))f(x_0,u(x_0),\nabla w_k(y))\,dy.$$

If $f(x_0,u(x_0),\cdot)$ is convex (resp. quasiconvex with $q>1$ in (1.11)), we use Proposition 9.1 (resp. Proposition 1.9 of Kristensen [22]) to approximate $f(x_0,u(x_0),\xi)$ by an increasing sequence $g_j(\xi)$ of convex (resp. quasiconvex) functions such that

$$0 \leq g_j(\xi) \leq C_j(|\xi|+1). \tag{7.7}$$

If $f(x_0,u(x_0),\cdot)$ is quasiconvex with $q=1$ in (1.11), we simply take $g_j(\xi) \equiv f(x_0,u(x_0),\xi)$ for all j. For any fixed j

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0)+\varepsilon \geq \liminf_{k\to\infty}\int_Q \psi(u(x_0)+\varepsilon_k w_k(y))g_j(\nabla w_k(y))\,dy,$$

and $f_j(x,u,\xi) := \psi(u)g_j(\xi)$ satisfies (7.1). Moreover, by (7.7), (7.2) continues to hold, provided we replace δ with $\delta/2$. Finally, (1.13) is still satisfied *at the point* $(x_0,u(x_0))$. Therefore we can apply the first part of the proof to get

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \psi(u(x_0))g_j(\nabla u(x_0)) = g_j(\nabla u(x_0)).$$

It suffices to take the supremum in j. □

Proof of Theorem 1.8 (Cantor part). We proceed as in Theorem 1.1(ii) until (2.11). We can now truncate the sequence w_k using an argument similar to that of the Lebesgue part of Theorem 1.8 (note that the only property of $w_0(y)$ which has been used is the fact that it is bounded), and then continue as in the Cantor part of Theorem 1.7, using (1.13) in place of (1.11). We omit the details. □

Proof of Theorem 1.9. The proofs of Theorem 1.3(i)–(ii) and of the first part of Theorem 1.6 continue to hold. We observe that in Theorem 1.3(ii), since $a\otimes\nu$ has rank one, the function $g(t)=f^\infty(x,u,t\,a\otimes\nu)$ is convex and thus we can still use Proposition 9.1, (9.1).

If $f^\infty = f^\infty(x,\xi)$ then the proof of Theorem 1.3(iii) is still valid with some obvious modifications. □

Proof of Theorem 1.10. We proceed as in Theorem 1.1(iii) until (2.14). Fix $\varepsilon>0$ and let k be so large that $\varepsilon_k < \min\{\delta, 1/L\}$, where δ and L are provided by (1.15)–(1.16). Then by (1.16) and (1.15), in this order,

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &= \lim_{k\to\infty}\int_{Q_\nu}\varepsilon_k f\left(x_0+\varepsilon_k y, w_k(y), \frac{1}{\varepsilon_k}\nabla w_k(y)\right)dy\\ &\geq \frac{1}{1+\varepsilon}\liminf_{k\to\infty}\int_{Q_\nu} f^\infty(x_0+\varepsilon_k y, w_k(y),\nabla w_k(y))\,dy - \frac{\varepsilon}{1+\varepsilon}\\ &\geq \frac{1}{(1+\varepsilon)^2}\liminf_{k\to\infty}\int_{Q_\nu} f^\infty(x_0, w_k(y),\nabla w_k(y))\,dy - \frac{\varepsilon}{(1+\varepsilon)^2}-\frac{\varepsilon}{1+\varepsilon}\\ &\geq \frac{1}{(1+\varepsilon)^2}\limsup_{k\to\infty}\int_{Q_\nu} f^\infty(x_0, v_k(y),\nabla v_k(y))\,dy - \frac{\varepsilon}{(1+\varepsilon)^2}-\frac{\varepsilon}{1+\varepsilon},\end{aligned} \tag{7.8}$$

where we have used Lemma 2.6 and Remark 2.7(1) of [7] to obtain a new sequence $v_k \in W^{1,1}(Q_\nu;\mathbb{R}^d)$ which converges to w_0 in $L^1(Q_\nu;\mathbb{R}^d)$ and such that $v|_{\partial Q_\nu} = w_0$. It now follows from (7.8) and the definition of the function h in (1.18) that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \frac{1}{(1+\varepsilon)^2}\, h(x_0, u^+(x_0), u^-(x_0), \nu) - \frac{\varepsilon}{(1+\varepsilon)^2} - \frac{\varepsilon}{1+\varepsilon},$$

and we obtain the first part of the theorem upon letting $\varepsilon \to 0^+$.

We first prove the reverse inequality to (1.17) under the additional coercivity assumption that there exists $C > 0$ such that

$$f(x,u,\xi) \geq C|\xi| \qquad \text{for all } (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{dN}. \tag{7.9}$$

Fix $\varepsilon_0 > 0$, and define

$$u_0(x) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu \leq 0. \end{cases}$$

By Lemma 4.1.3 and (3.17) of [7] for $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$

$$\begin{aligned}
&\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \\
&= \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1}} \inf\left\{\int_{Q_\nu(x_0,\varepsilon)} f(x, v(x), \nabla v(y))dy : v \in W^{1,1}(Q_\nu(x_0,\varepsilon);\mathbb{R}^d), v|_{\partial Q_\nu(x_0,\varepsilon)} = u_0(\cdot - x_0)\right\} \\
&= \limsup_{\varepsilon\to 0^+} \inf\left\{\int_{Q_\nu} \varepsilon\, f\left(x_0+\varepsilon\, y, w(y), \frac{1}{\varepsilon}\nabla w(y)\right)\, dy \,:\, w \in W^{1,1}(Q_\nu;\mathbb{R}^d),\; w|_{\partial Q_\nu} = u_0\right\} \\
&\leq \frac{1}{1-\varepsilon_0} \limsup_{\varepsilon\to 0^+} \inf\left\{\int_{Q_\nu} f^\infty(x_0+\varepsilon\, y, w(y), \nabla w(y))\, dy \,:\, w \in W^{1,1}(Q_\nu;\mathbb{R}^d),\; w|_{\partial Q_\nu} = u_0\right\} + \frac{\varepsilon_0}{1-\varepsilon_0} \\
&\leq \frac{1}{1-\varepsilon_0} \limsup_{\varepsilon\to 0^+} \int_{Q_\nu} f^\infty(x_0+\varepsilon\, y, w_1(y), \nabla w_1(y))\, dy + \frac{\varepsilon_0}{1-\varepsilon_0},
\end{aligned}$$

for any $w_1 \in W^{1,1}(Q_\nu;\mathbb{R}^d)$, with $w_1|_{\partial Q_\nu} = u_0$, and where we have used (1.16)$'$. We now take w_1 in the previous inequality such that

$$\int_{Q_\nu} f^\infty(x_0, w_1(y), \nabla w_1(y))\, dy \leq h(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) + \varepsilon_0. \tag{7.10}$$

By (3.5), (3.6), and Lebesgue Dominated Convergence Theorem,

$$\lim_{\lambda\to\infty} \int_{Q_\nu} f_\lambda(x_0, w_1(y), \nabla w_1(y))\, dy = \int_{Q_\nu} f^\infty(x_0, w_1(y), \nabla w_1(y))\, dy, \tag{7.11}$$

where the Yosida transforms f_λ were introduced in (3.4); thus, for fixed λ sufficiently large, by (7.10), and by (7.11)

$$\int_{Q_\nu} f_\lambda(x_0, w_1(y), \nabla w_1(y))\, dy \leq h(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) + 2\varepsilon_0. \tag{7.12}$$

Consequently, also from (3.5) and (3.8),

$$\begin{aligned}
\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\leq \frac{1}{1-\varepsilon_0} \limsup_{\varepsilon\to 0^+} \int_{Q_\nu} f_\lambda(x_0+\varepsilon\, y, w_1(y), \nabla w_1(y))\, dy + \frac{\varepsilon_0}{1-\varepsilon_0} \\
&\leq \frac{1}{1-\varepsilon_0} \limsup_{\varepsilon\to 0^+} \left(\int_{Q_\nu} f_\lambda(x_0, w_1(y), \nabla w_1(y))\, dy + \varepsilon\,\lambda\right) + \frac{\varepsilon_0}{1-\varepsilon_0} \\
&\leq \frac{1}{1-\varepsilon_0} \left(h(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) + 2\varepsilon_0\right) + \frac{\varepsilon_0}{1-\varepsilon_0}
\end{aligned}$$

by (7.12). Letting $\varepsilon_0 \to 0^+$ in the previous inequality yields the desired result when (7.9) holds.

In the general case, it suffices to consider the family of perturbed energy densities

$$f_\rho(x,u,\xi) := f(x,u,\xi) + \rho\,|\xi|, \qquad \rho > 0.$$

Then, since $\mathcal{F} \le \mathcal{F}_\rho$, it follows that

$$\begin{aligned}\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\le \frac{d\mathcal{F}_\rho(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)\\ &\le \inf\left\{\int_{Q_\nu} f^\infty(x_0,w(y),\nabla w(y))\,dy + \rho\int_{Q_\nu}|\nabla w(y))|\,dy : w \in W^{1,1}(Q_\nu;\mathbb{R}^d),\ w|_{\partial Q_\nu(x_0,\varepsilon)} = u_0\right\}\\ &\le \int_{Q_\nu} f^\infty(x_0,w_1(y),\nabla w_1(y))\,dy + \rho\int_{Q_\nu}|\nabla w_1(y))|\,dy\end{aligned}$$

for any fixed $w_1 \in W^{1,1}(Q_\nu;\mathbb{R}^d)$, such that $w_1|_{\partial Q_\nu(x_0,\varepsilon)} = u_0$. Letting $\rho \to 0^+$ yields

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \le \int_{Q_\nu} f^\infty(x_0,w_1(y),\nabla w_1(y))\,dy,$$

and since w_1 is arbitrary, by taking the infimum over all functions w_1 we obtain

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \le h(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)).$$

□

§8. Further results.

As in Theorems C and D of Dal Maso [9], some of our results continue to hold if the regularity conditions on the integrand f are required everywhere except at most on "small" sets. In order to establish the main result of this section, Theorem 8.5, we prove first three lemmas.

Lemma 8.1. *Let N_0 be a Borel subset of $\Omega \times \mathbb{R}$, with $\mathcal{H}^N(N_0) = 0$, let $u \in BV(\Omega;\mathbb{R})$, and define*

$$A := \{x \in \Omega \backslash (S(u) \cup M(u)) : (x,u(x)) \in N_0\}, \quad B := \{x \in M(u) : (x,u(x)) \in N_0\},$$

where $M(u)$ is the support of $C(u)$. Then

(i) $\mathcal{L}^N(A) = 0$;

(ii) $|C(u)|(B) = 0$.

Proof. (i) If $\mathcal{L}^N(A) > 0$, then by Corollary 1 in Sec. 2.4.1 of [15] we obtain

$$0 = \mathcal{H}^N(N_0) \ge \mathcal{H}^N(\{(x,u(x)) : x \in A\}) \ge \mathcal{H}^N(A) = \mathcal{L}^N(A) > 0,$$

which is clearly a contradiction.

(ii) Let

$$\begin{aligned}G_+ &:= \{(x,s) \in \Omega \times \mathbb{R} : s < u^+(x)\},\\ G_\pm &:= \{(x,s) \in \Omega \times \mathbb{R} : u^-(x) \le s \le u^+(x)\}.\end{aligned}$$

Then $\chi_{G_+} \in BV_{\rm loc}(\Omega \times \mathbb{R})$ (see [25]), and for any Borel set $K \subset \Omega \times \mathbb{R}$ and $D \subset \Omega$ we have

$$|D\chi_{G_+}|(K) = \mathcal{H}^N(K \cap G_\pm), \qquad \int_{D\times\mathbb{R}} |D\chi_{G_+}| = \int_D |\zeta(u)|, \tag{8.1}$$

where $\zeta(u) := (Du, -\mathcal{L}^N)$ (see [9], Lemma 2.2). Take $K := B \times \mathbb{R}$. If $x \in M(u)$ then $u^+(x) = u^-(x)$, and thus

$$|D\chi_{G_+}|(K) = \mathcal{H}^N(\{(x,u(x)) : x \in B\}) \le \mathcal{H}^N(N_0) = 0.$$

In turn, by (8.1), $\int_B |\zeta(u)| = 0$. Since $|\zeta(u)|$ coincides with $|Du|$ on $M(u)$ (recall that $\mathcal{L}^N(M(u)) = 0$) it follows that $|Du|(B) = 0$. □

The following Lemma is a generalization of Theorem 3 in Sec. 2.4.3 of [15]

Lemma 8.2. *Let $h \in L^1_{\rm loc}(\Omega;\mathbb{R})$, let μ be a positive Radon measure, and define*

$$B_0 := \left\{ x_0 \in \Omega : \limsup_{\varepsilon\to 0^+} \frac{1}{\mu(Q_\nu(x_0,\varepsilon))} \int_{Q_\nu(x_0,\varepsilon)} |h(x)|\,dx > 0 \right\}, \tag{8.2}$$

where in the limsup we consider only those $\varepsilon > 0$ such that $\mu(\partial Q_\nu(x,\varepsilon)) = 0$. Then $\mu_s(B_0) = 0$, where $\mu := \frac{d\mu}{d\mathcal{L}^N}\mathcal{L}^N + \mu_s$.

Proof. Without loss of generality, we can assume that Ω is bounded, $h \geq 0$ and $h \in L^1(\Omega;\mathbb{R})$. Given $\eta > 0$ there exists $\delta > 0$ such that

$$\int_U h(x)\,dx < \eta \qquad \text{whenever } \mathcal{L}^N(U) < \delta.$$

Let E be the support of the measure μ_s. Since $\mu_s(X) = \mu_s(X \cap E)$, we consider $B_0' = B_0 \cap E$. Clearly $\mathcal{L}^N(B_0') = 0$, and

$$B_0' = \bigcup_{r\in\mathbb{Q}^+} B_r, \quad \text{where} \quad B_r := \left\{ x_0 \in E : \limsup_{\varepsilon\to 0^+} \frac{1}{\mu(Q_\nu(x_0,\varepsilon))} \int_{Q_\nu(x_0,\varepsilon)} h(x)\,dx > r \right\}.$$

We claim that $\mu(B_r) = 0$, from what will follow that $\mu(B_0') = 0$. Let U be an open set such that $B_r \subset U$ and $\mathcal{L}^N(U) < \delta$. Fix $\rho > 0$ and consider

$$\mathcal{F}_1^\rho := \left\{ Q_\nu(x,\varepsilon) : x \in B_r, 0 < \varepsilon < \rho,\ \mu(\partial Q_\nu(x,\varepsilon)) = 0,\ \overline{Q_\nu(x,\varepsilon)} \subset U,\ \int_{Q_\nu(x,\varepsilon)} h(y)dy > r\mu(Q_\nu(x,\varepsilon)) \right\}$$

and consider the Borel sets

$$U^\rho := \bigcup \{Q_\nu(x,\varepsilon) \,:\, Q_\nu(x,\varepsilon) \in \mathcal{F}_1^\rho\}, \quad U_0 := \bigcap_{\rho>0} U^\rho.$$

Since $B_r \subset U_0$, it suffices to prove that $\mu(U_0) = 0$. Fix a compact set $K \subset U_0$, $\rho_0 > 0$, and let

$$\mathcal{F}_2^{\rho_0} := \left\{ Q_\nu(x,\varepsilon) : x \in B_r,\ 0 < \varepsilon < \rho,\ \mu(\partial Q_\nu(x,\varepsilon)) = 0,\ \overline{Q_\nu(x,\varepsilon)} \subset U^{\rho_0}\backslash K \right\}.$$

Then U^{ρ_0} admits a fine covering

$$U^{\rho_0} := \left(\bigcup_{Q_\nu(x,\varepsilon)\in\mathcal{F}_1^{\rho_0}} Q_\nu(x,\varepsilon) \right) \cup \left(\bigcup_{Q_\nu(x,\varepsilon)\in\mathcal{F}_2^{\rho_0}} Q_\nu(x,\varepsilon) \right),$$

and by Morse's version of Besicovitch's Covering Theorem (see [27], Theorem 5.11) we may find a subcovering of U^{ρ_0} such that

$$U^{\rho_0} := \left(\bigcup_{i\in I} Q_i \right) \cup \left(\bigcup_{j\in I} Q_j \right) \cup N, \qquad K \subset \left(\bigcup_{i\in I} \overline{Q_i} \right) \cup N,$$

I and J are countable, $Q_i \in \mathcal{F}_1^{\rho_0}$, $Q_j \in \mathcal{F}_2^{\rho_0}$, the sets $\overline{Q_i}$ and $\overline{Q_j}$ are mutually disjoint, and $\mu(N) = 0$. Then

$$\eta > \int_U h(x)\,dx \geq \sum_{i\in I} \int_{Q_i} h(x)\,dx \geq r \sum_{i\in I} \mu(Q_i) = r \sum_{i\in I} \mu(\overline{Q_i}) \geq r\mu(K).$$

By letting $\eta \to 0$ we obtain $\mu(K) = 0$, and by the inner regularity of μ we conclude that $\mu(U_0) = 0$. □

Remark 8.3. Since for μ_s a.e. $x_0 \in \Omega$

$$\lim_{\varepsilon \to 0^+} \frac{\mu(Q_\nu(x_0,\varepsilon))}{\varepsilon^N} = \infty,$$

it is clear that if $x_0 \in B_0$ then μ_s a.e. x_0 is not a Lebesgue point for $|h|$, otherwise

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\mu(Q_\nu(x_0,\varepsilon))} \int_{Q_\nu(x_0,\varepsilon)} |h(x)|\,dx = \limsup_{\varepsilon \to 0^+} \frac{\varepsilon^N}{\mu(Q_\nu(x_0,\varepsilon))} |h(x_0)| = 0.$$

Using Lemma 8.2, it is possible in some cases to weaken (1.8) in Proposition 2.1. Indeed, assume that for $\mathcal{L}^1$ a.e. $u \in \mathbb{R}$

$$f(\cdot,u,0) \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}). \tag{8.3}$$

Then there exists a countable set $\mathcal{R}_0 := \{r_j\}_j$, dense in $\mathbb{R}$, such that $f(\cdot,r_j,0) \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$ for all j. Let Ω_j be the set of Lebesgue points of $f(\cdot,r_j,0)$ and set $A_j := \Omega \backslash \Omega_j$. Then

$$\mathcal{L}^N(\cup_{j=1}^\infty A_j) = 0.$$

Let B_j be the set of points corresponding to the set B_0 in Lemma 8.2 when $h := f(\cdot,r_j,0)$ and $\mu := |Du|$. Then

$$|D_s u|(\cup_{j=1}^\infty B_j) = 0.$$

Lemma 8.4. *Proposition 2.1 is still valid provided we replace* (1.8) *by* (8.3)*, and we take*

$$x_0 \in \Omega \backslash \left(\bigcup_{i,j=1}^\infty A_j \cup B_i \right), \qquad t_k := \begin{cases} 1 & \text{if } x_0 \in \bigcap_{j=1}^\infty \Omega_j \\ |Du|(Q_\nu(x_0,\varepsilon_k))/\varepsilon_k^N & \text{otherwise,} \end{cases}$$

where $|Du|(\partial Q_\nu(x_0,\varepsilon_k)) = 0$*,* $t_k \to T \in (0,\infty]$*, and* $u_0 + \alpha_1$*,* $u_0 + \alpha_2 \in \mathcal{R}_0$*.*

Proof. The only change is in (2.2). Considering first the case where $x_0 \in \Omega \backslash (\cup_{j=1}^\infty A_j) = \cap_{j=1}^\infty \Omega_j$, then $t_k = 1$ and (2.2) becomes

$$\begin{aligned} 0 &\le \int_{E_k^+} f(x_0 + \varepsilon_k y, u_0 + \alpha_2, 0)\,dy = \frac{1}{\varepsilon_k^N} \int_{Q_\nu(x_0,\varepsilon_k) \cap (x_0 + \varepsilon_k E_k^+)} f(x, u_0 + \alpha_2, 0)\,dx \\ &\le \frac{1}{\varepsilon_k^N} \int_{Q_\nu(x_0,\varepsilon_k)} |f(x,u_0+\alpha_2,0) - f(x_0,u_0+\alpha_2,0)|\,dx + f(x_0,u_0+\alpha_2,0)\mathcal{L}^N(E_k^+). \end{aligned}$$

Since x_0 is a Lebesgue point for $f(x,u_0+\alpha_2,0)$ (recall that $u_0 + \alpha_2 \in \mathcal{R}_0$ and that $x_0 \in \cap_{j=1}^\infty \Omega_j$) the first integral on the right hand side approaches zero as $k \to \infty$. Moreover $\mathcal{L}^N(E_k^+) \to 0$ as before.

If $x_0 \in \Omega \backslash (\cup_{i=1}^\infty B_i)$ and $x_0 \notin \cap_{j=1}^\infty \Omega_j$ then (2.2) may be estimated as follows:

$$0 \le \lim_{k \to \infty} \frac{1}{t_k} \int_{E_k^+} f(x_0 + \varepsilon_k y, u_0 + \alpha_2, 0)\,dy \le \lim_{k \to \infty} \frac{1}{|Du|(Q(x_0,\varepsilon_k))} \int_{Q(x_0,\varepsilon_k)} f(x,u_0+\alpha_2,0)\,dx = 0,$$

where we have used the fact that $t_k = |Du|(Q_\nu(x_0,\varepsilon_k))/\varepsilon_k^N$ and the definition of B_j as in (8.3), with $h := f(\cdot,r_j,0)$. □

We are now ready to state the main result of this section.

Theorem 8.5. *Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is a Borel integrand, $f(x, u, \cdot)$ is convex in $\mathbb{R}^N$, and f satisfies* (8.3). *Suppose also that* (1.7) *holds for all $(x_0, u_0) \in (\Omega \times \mathbb{R})\backslash N_0$, where N_0 is a Borel subset of $\Omega \times \mathbb{R}$. Let $u \in BV_{\rm loc}(\Omega; \mathbb{R})$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\rm loc}(\Omega; \mathbb{R})$ converging to u in $L^1_{\rm loc}(\Omega; \mathbb{R})$.*

(i) *If either $\mathcal{H}^N(N_0) = 0$ or $N_0 = M_0 \times \mathbb{R}$ with $\mathcal{L}^N(M_0) = 0$ then*

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx.$$

(ii) *If either $\mathcal{H}^N(N_0) = 0$ or $N_0 = M_0 \times \mathbb{R}$ with $\mathcal{H}^{N-1}(M_0) < \infty$ then*

$$\int_\Omega f^\infty(x, u(x), dC(u(x))) \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx.$$

(iii) *If $N_0 = M_0 \times \mathbb{R}$ and either $\mathcal{H}^N(N_0) = 0$ or $\mathcal{H}^{N-1}(M_0) = 0$ and we assume that for all $(x_0, u_0) \in (\Omega \times \mathbb{R})\backslash N_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(x_0, u, \xi) - f(x, u, \xi) \leq \varepsilon(1 + f(x, u, \xi)) \tag{8.4}$$

for all $(x, u) \in \Omega \times \mathbb{R}$ with $|x - x_0| + |u - u_0| \leq \delta$ and for all $\xi \in \mathbb{R}^N$, then

$$\int_{S(u)\cap\Omega} \left(\int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu_u)\, ds \right) d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx.$$

Proof. (i) We proceed as in Theorem 1.1(i) starting from (2.4). If $\mathcal{H}^N(N_0) = 0$ then in (2.6) we take $x_0 \in \Omega\backslash A$, where A is the set given in Lemma 8.1(i), otherwise take $x_0 \in \Omega\backslash M_0$. Using the notation introduced in Lemma 8.1 and thereafter, we may assume, in addition, that x_0 is also a Lebesgue point for all the functions $f(\cdot, r_j, 0)$'s, precisely

$$x_0 \in \Omega\backslash \left(\cup_{j=1}^\infty A_j\right).$$

We can now continue with the same argument as in the proof of the Lebesgue part in Theorem 1.1, except that we invoke Lemma 8.4 instead of Proposition 2.1 to justify the truncation step.

(ii) If $\mathcal{H}^N(N_0) = 0$ then take $x_0 \in \Omega\backslash B$, where B is the set given in Lemma 8.1(ii), otherwise take $x_0 \in \Omega\backslash M_0$. As before, let B_j be the set introduced in Lemma 8.2 and corresponding to $f(\cdot, r_j, 0) \in L^1_{\rm loc}(\Omega; \mathbb{R})$ (see (8.2)). Since

$$|C(u)|(\cup_{j=1}^\infty B_j) = 0,$$

we may assume that $x_0 \in \Omega\backslash(\cup_{j=1}^\infty B_j)$. Now we continue as in Theorem 1.1(ii), using Lemma 8.4 in place of Proposition 2.1, but now in order to apply Lemma 2.5 of Ambrosio and Dal Maso [4] we first need to approximate $g(\xi) := f(x_0, u(x_0), \xi)$ from below by a nondecreasing sequence of convex functions which grow at most linearly. This can be done by virtue of Proposition 9.1.

(iii) Since $N_0 = M_0 \times \mathbb{R}$ and $\mathcal{H}^N = \mathcal{H}^{N-1} \times \mathcal{L}^1$ on $S(u) \times \mathbb{R}$, it follows that $\mathcal{H}^{N-1}(M_0 \cap S(u)) = 0$. Moreover, by Lemma 8.2,

$$\mathcal{H}^{N-1}\left(S(u) \cap \left(\cup_{j=1}^\infty B_j\right)\right) = 0.$$

Take

$$x_0 \in S(u)\backslash \left(M_0 \cup \bigcup_{j=1}^\infty B_j \right).$$

We pursue the proof of the jump part as in Theorem 1.1(iii), using Lemma 8.4 instead of Proposition 2.1, but now in order to apply the density result of Ambrosio we first need to approximate $h(u, \xi) := f(x_0, u, \xi)$ from below by a nondecreasing sequence of continuous functions which grow at most linearly. For this purpose we invoke Proposition 9.3 below. □

Remark 8.6. (i) The hypothesis placed on part (iii) above, i.e. in the jump part, ensuring that the set N_0 is of the form $N_0 = M_0 \times \mathbb{R}$, is used heavily to apply the compactness argument leading to (2.15).

(ii) Theorems 1.3, 1.7, 1.8 may be improved similarly to Theorem 8.5 versus Theorem 1.1. We leave this to the interested reader.

(iii) As in Theorem D of Dal Maso [9], in the special case where $f(x, u, 0) \equiv 0$ condition (1.9) can be weakened as follows:

Assume that there exists a set $P_0 \subset \mathbb{R}$ with $\mathcal{L}^1(P_0) = 0$ such that for all $x_0 \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_0, u, \xi) - f(x, u, \xi)| \leq \varepsilon(1 + f(x, u, \xi)) \tag{1.9$'$}$$

for all $x \in \Omega$ with $|x - x_0| \leq \delta$ and for all $(u, \xi) \in (\mathbb{R} \backslash P_0) \times \mathbb{R}^N$.

The proof of Theorem 1.5 should now be modified accordingly, using the fact that, if $A := \{x \in \Omega \backslash S(u) : u(x) \in P_0\}$ then $\nabla u \equiv 0$ for $\mathcal{L}^N$ a.e. $x \in A$ and $|C(u)| = 0$ a.e. $x \in A$. We omit the details.

§9. Approximation of convex functions.

Let $g : \mathbb{R}^N \to [0, \infty)$ be a convex function. Then

$$t \mapsto \frac{g(t\,\xi) - g(0)}{t}$$

is increasing, and we define the *recession function*

$$g^\infty(\xi) := \lim_{t \to \infty} \frac{g(t\,\xi)}{t} = \sup_{t > 0} \frac{g(t\,\xi) - g(0)}{t}.$$

Proposition 9.1. *Let $g : \mathbb{R}^N \to [0, \infty)$ be a convex function. Then*

$$\frac{g(t\,\xi)}{t} \leq g^\infty(\xi) + \frac{g(0)}{t} \tag{9.1}$$

for $t > 0$, and there exists an increasing sequence $\{g_j\}_j$ of nonnegative convex functions such that:

(i) $g(\xi) = \sup_j g_j(\xi)$ *for all $\xi \in \mathbb{R}^N$;*

(ii) $g^\infty(\xi) = \sup_j g_j^\infty(\xi)$ *for all $\xi \in \mathbb{R}^N$;*

(iii) *g_j is Lipschitz continuous with Lipschitz constant j;*

(iv) *if $g(\xi) \geq C(|\xi| - 1)$ for some $C > 0$, then g_j satisfies the same growth condition for $j \geq [C] + 1$.*

Proof. Inequality (9.1) follows immediately from the definition of the g^∞. Define

$$g_j(\xi) := \sup_{|\xi^*| \leq j} (\xi^* \cdot \xi - g^*(\xi^*)),$$

where g^* is the Young–Fenchel conjugate of g. Since g is convex, $g = g^{**}$; hence $g(\xi) = \sup_j g_j(\xi)$. Also, g_j are convex and

$$g_j(\xi) \geq -g^*(0) \geq \inf g \geq 0.$$

This proves (i).

Since $g(\xi) \geq g_j(\xi)$ for each j, it follows that $g^\infty(\xi) \geq \sup_j g_j^\infty(\xi)$. Conversely, and by (9.1),

$$\frac{g(t\,\xi)}{t} = \sup_j \frac{g_j(t\,\xi)}{t} = \sup_j \left[\frac{g_j(t\,\xi) - g_j(0)}{t} + \frac{g_j(0)}{t}\right] \leq \sup_j g_j^\infty(\xi) + \sup_j \frac{g_j(0)}{t} = \sup_j g_j^\infty(\xi) + \frac{g(0)}{t}.$$

Letting $t \to \infty$ we conclude that $g^\infty(t) \leq \sup_j g_j^\infty(\xi)$.

Property (iii) is straightforward. We prove (iv). If $g(\xi) \geq C(|\xi|-1)$ then $g^*(\xi^*) \leq [C(|\cdot|-1)]^*(\xi^*)$; hence

$$g_j(\xi) = \sup_{|\xi^*|\leq j} (\xi^*\cdot\xi - g^*(\xi^*)) \geq \sup_{|\xi^*|\leq j} (\xi^*\cdot\xi - [C(|\cdot|-1)]^*(\xi^*)) .$$

Since

$$[C(|\cdot|-1)]^*(\xi^*) = \begin{cases} C & \text{if } |\xi^*|\leq C, \\ \infty & \text{otherwise,} \end{cases}$$

we conclude that if $j \geq [C]+1$ then

$$g_j(\xi) \geq \sup_{|\xi^*|\leq C} (\xi^*\cdot\xi - C) = C(|\xi|-1).$$

□

The last proposition of this section uses a corollary of Lindelöf Theorem which allows us to select a countable collection of functions yielding the supremum function of a noncountable family. For convenience, we include the proof below.

Lemma 9.2. *Let X be a σ-compact metric space, let $\mathcal{G} \subset C(X;\mathbb{R})$, and let $f(x) := \sup_{g\in\mathcal{G}} g(x)$, for all $x \in X$. Then there exists a countable collection $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$, such that*

$$f(x) = \sup_n g_n(x) \quad \textit{for all } x \in X.$$

Proof. It is clear that f is lower semicontinuous. Therefore, for every $x \in X$ there exist $0 < r(x) < 1$ and $g^1_x \in \mathcal{G}$ such that

$$f(y) \geq f(x) - \frac{1}{2} \text{ for all } y \in B(x, r^1(x)), \quad g^1_x(x) \geq f(x) - \frac{1}{2}.$$

Let $\rho^1(x) < r(x)$ be such that

$$|g^1_x(y) - g^1_x(y')| \leq \frac{1}{2} \quad \text{for all } y, y' \in B(x, \rho^1(x)).$$

Since $\{B(x,\rho^1(x))\}_{x\in X}$ is an open covering of X, by Lindelöf Theorem we may extract a countable subcovering $\{B(x^1_n, \rho^1_n)\}$. Recursively, we may find an open covering of X, $\{B(x,\rho^k(x))\}_{x\in X}$, $\rho^k(x) < \frac{1}{2^k}$, and functions $g^k_x \in \mathcal{G}$, such that for all $x \in X$

$$f(y) \geq f(x) - \frac{1}{2^k} \text{ for all } y \in B(x,\rho^k(x)),\ g^k_x(x) \geq f(x) - \frac{1}{2^k},\ |g^k_x(y) - g^k_x(y')| \leq \frac{1}{2^k} \text{ for all } y, y' \in B(x, \rho^k(x)).$$

Again by Lindelöf Theorem, X is covered by a countable family $\{B(x^k_n, \rho^k_n)\}$. We claim that

$$f(x) = \sup_{m,k} g^k_{x^k_m}(x).$$

Let $x \in X$, $k \in \mathbb{N}$, and choose $n \in \mathbb{N}$ such that $x \in B(x^k_n, \rho^k_n)$. Then

$$f(x) \geq g^k_{x^k_n}(x) \geq g^k_{x^k_n}(x^k_n) - \frac{1}{2^k} \geq f(x^k_n) - \frac{2}{2^k} \geq \inf_{B(x;\frac{1}{2^k})} f - \frac{2}{2^k}.$$

As f is lower semicontinuous, $\liminf_{\varepsilon\to 0} \inf_{B(x,\varepsilon)} f = f(x)$, and we conclude that

$$f(x) \leq \liminf_{k\to\infty} \inf_{B(x;\frac{1}{2^k})} f - \frac{2}{2^k} \leq \liminf_{k\to\infty} g^k_{x^k_n}(x) \leq \sup_{m,k} g^k_{x^k_m}(x) \leq f(x).$$

□

Proposition 9.3. ([10]) *Let A be an open set of $\mathbb{R}^N$ and let $h : A\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ be a function convex in the ξ variable, and such that for every $(x_0,u_0)\in A\times\mathbb{R}$ and $\varepsilon>0$ there exists $\delta>0$ such that*

$$h(x,u,\xi) \geq (1-\varepsilon)h(x_0,u_0,\xi) \tag{9.2}$$

for all $(x,u)\in A\times\mathbb{R}$ with $|x-x_0|\leq\delta$, $|u-u_0|\leq\delta$ and for all $\xi\in\mathbb{R}^N$. Then there exists an increasing sequence $\{h_j\}_j$ of nonnegative continuous functions, convex in the ξ variable, satisfying (9.2) and such that:

(i) *$h(x,u,\xi)=\sup_j h_j(x,u,\xi)$ for all $(x,u,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N$;*
(ii) *$h^\infty(x,u,\xi)=\sup_j h_j^\infty(x,u,\xi)$ for all $(x,u,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N$;*
(iii) *$h_j(x,u,\xi)\leq C_j(|\xi|+1)$ for all $(x,u,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N$ and for some $C_j>0$;*
(iv) *for every $(x_0,u_0)\in A\times\mathbb{R}$ and $\varepsilon>0$ there exists $\delta_j>0$ such that*

$$h_j(x,u,\xi)\geq(1-\varepsilon)h_j(x_0,u,\xi), \qquad h_j(x,u,\xi)\geq(1-\varepsilon)h_j(x,u_0,\xi)$$

for all $(x,u)\in A\times\mathbb{R}$ with $|x-x_0|\leq\delta_j$, $|u-u_0|\leq\delta_j$, and for all $\xi\in\mathbb{R}^N$.

Proof. Let $\mathcal{G}$ be the class of all continuous functions $g : A\times\mathbb{R}\times\mathbb{R}^N\to[0,\infty)$, convex in the ξ variable, and such that:

(1) $g(x,u,\xi)\leq h(x,u,\xi)$ for all $(x,u,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N$;
(2) for every $(x_0,u_0)\in A\times\mathbb{R}$ and $\varepsilon>0$ there exists $\delta>0$ such that

$$g(x,u,\xi)\geq(1-\varepsilon)g(x_0,u_0,\xi)$$

and

$$g(x,u,\xi)\geq(1-\varepsilon)g(x_0,u,\xi), \qquad g(x,u,\xi)\geq(1-\varepsilon)g(x,u_0,\xi)$$

for all $(x,u)\in A\times\mathbb{R}$ with $|x-x_0|\leq\delta$, $|u-u_0|\leq\delta$, and for all $\xi\in\mathbb{R}^N$;
(3) there exists $C>0$ such that

$$g(x,u,\xi)\leq C(|\xi|+1) \qquad \text{for all } (x,u,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N.$$

Clearly $\mathcal{G}\neq\emptyset$, as $0\in\mathcal{G}$. Following [10], we claim that

$$h(x_0,u_0,\xi)=\sup_{g\in\mathcal{G}} g(x_0,u_0,\xi) \qquad \text{for all } (x_0,u_0,\xi)\in A\times\mathbb{R}\times\mathbb{R}^N. \tag{9.3}$$

By definition of $\mathcal{G}$, it follows immediately that $h\geq\sup_{g\in\mathcal{G}} g$. Conversely, fix $(x_0,u_0)\in A\times\mathbb{R}$, $\varepsilon>0$, and let δ be such that (9.2) is satisfied. Consider two cut–off functions $\varphi\in C_0^\infty(A)$, $\psi\in C_0^\infty(\mathbb{R})$, with $0\leq\varphi\leq1$, $0\leq\psi\leq1$, $\varphi\equiv1$ on $B(x_0,\delta/2)$, $\varphi\equiv0$ outside $B(x_0,\delta)$, and, similarly, $\psi\equiv1$ on $B(u_0,\delta/2)$, $\psi\equiv0$ outside $B(u_0,\delta)$. We can write

$$h(x_0,u_0,\xi)=\sup_j h_j(\xi),$$

where h_j are convex functions satisfying the properties stated in Proposition 9.1. Consider

$$h_j^\varepsilon(x,u,\xi):=(1-\varepsilon)\varphi(x)\psi(u)h_j(\xi).$$

Clearly $h_j^\varepsilon\in\mathcal{G}$ (in particular property (1) follows from (9.2)). Letting $j\to\infty$ we get

$$(1-\varepsilon)h(x_0,u_0,\xi)=\sup_j h_j^\varepsilon(x_0,u_0,\xi)\leq\sup_{g\in\mathcal{G}} g(x_0,u_0,\xi);$$

hence the claim follows by letting $\varepsilon\to0^+$.

By Lemma 9.2 and (9.3) there exist a sequence h_j in $\mathcal{G}$ such that $h(x,u,\xi) = \sup_j h_j(x,u,\xi)$ for all (x,u,ξ) in $A \times \mathbb{R} \times \mathbb{R}^N$. Due to the stability properties of the class $\mathcal{G}$, we can assume the sequence $\{h_j\}_j$ increasing. Indeed it is easy to see that if $g_1, g_2 \in \mathcal{G}$ then $g_1 \vee g_2 \in \mathcal{G}$ (while in general $g_1 \wedge g_2 \notin \mathcal{G}$, since we may loose convexity). This proves (i). Clearly, properties (iii) and (iv) follow immediately from the definition of $\mathcal{G}$.

(ii) follows easily from Proposition 9.1, (9.1), and the fact that clearly $h^\infty(x,u,\xi) \geq \sup_j h_j^\infty(x,u,\xi)$. Indeed,

$$\frac{h(x,u,t\xi)}{t} = \sup_j \left[\frac{h_j(x,u,t\xi) - h_j(x,u,0)}{t} + \frac{h_j(x,u,0)}{t}\right] \leq \sup_j h_j^\infty(x,u,\xi) + \frac{h(x,u,0)}{t},$$

so letting $t \to \infty$ we obtain (ii). □

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