Lower semicontinuity and relaxation of signed functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity

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Abstract

A lower semicontinuity and relaxation result with respect to weak-$*$ convergence of measures is derived for functionals of the form

$$\mu \in \mathcal{M}(\Omega;\mathbb{R}^d) \to \int_\Omega f(\mu^a(x))\,dx + \int_\Omega f^\infty\left(\frac{d\mu^s}{d|\mu^s|}(x)\right)\,d|\mu^s|(x),$$

where admissible sequences $\{\mu_n\}$ are such that $\{\mathcal{A}\mu_n\}$ converges to zero strongly in $W^{-1,q}_{\text{loc}}(\Omega)$ and $\mathcal{A}$ is a partial differential operator with constant rank. The integrand f has linear growth and L^∞-bounds from below are not assumed.

1 Introduction

In this work we start by deriving a lower semicontinuity result with respect to weak-$*$ convergence of $\mathcal{A}$-free measures for the functional

$$\mathcal{F}(\mu) = \int_\Omega f(\mu^a)\,dx + \int_\Omega f^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right)\,d|\mu^s|, \quad \mu \in \mathcal{M}(\Omega;\mathbb{R}^d), \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $\mathcal{M}(\Omega;\mathbb{R}^d)$ stands for the set of finite $\mathbb{R}^d$-valued Radon measures over Ω, $\mu = \mu^a\mathcal{L}^N + \mu^s$ is the Radon-Nikodým decomposition of μ with respect to the Lebesgue measure $\mathcal{L}^N$. Here and in what follows, the integrand $f : \mathbb{R}^d \to \mathbb{R}$ is assumed to be $\mathcal{A}$-quasiconvex (see Section 2 for other notations and preliminary definitions), where $\mathcal{A}$ is a linear first order partial differential operator of the form

$$\mathcal{A} := \sum_{i=1}^N A^{(i)}\frac{\partial}{\partial x_i}, \quad A^{(i)} \in \mathbb{M}^{M\times d}(\mathbb{R}), \quad M \in \mathbb{N}, \tag{1.2}$$

that we assume throughout to satisfy Murat's condition of *constant rank* (see Murat [15] and Fonseca & Müller [10]) i.e., there exists $c \in \mathbb{N}$ such that

$$\text{rank}\left(\sum_{i=1}^N A^{(i)}\xi_i\right) = c \quad \text{for all } \xi = (\xi_1, ..., \xi_N) \in \mathbb{S}^{N-1}.$$

In addition we assume f to be Lipschitz continuous and we remark that this condition implies f to satisfy a linear growth condition at infinity of the type

$$|f(v)| \le K(1 + |v|) \tag{1.3}$$

for all $v \in \mathbb{R}^d$ and for some $K > 0$. As usual (see Remark 3.1) we denote by f^∞ the *recession function* of f, which for our problem is defined as

$$f^\infty(\xi) := \limsup_{t\to\infty}\frac{f(t\xi)}{t}. \tag{1.4}$$

As already proved by Fonseca & Müller [10] $\mathcal{A}$-quasiconvexity with respect to the last variable turns out to be a necessary and sufficient condition for the lower semicontinuity of

$$(u,v) \to \int_\Omega f(x,u(x),v(x))\,dx$$

for positive normal integrands f with linear growth among sequences (u_n, v_n) such that $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^1 and $\mathcal{A}v_n = 0$. In Fonseca, Leoni & Müller [9] this result was partially extended by considering weak-$*$ convergence in the sense of measures (in the variable v). Precisely the authors considered a funtional of the form

$$v \to \int_\Omega f(x,v(x))\,dx$$

and, in particular, it was proved that

$$\int_\Omega f\left(x,\mu^a(x)\right)\,dx \leq \lim_{n\to\infty}\int_\Omega f(x,v_n(x))\,dx \tag{1.5}$$

for any sequence $v_n \subset L^1(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ and such that $v_n \rightharpoonup \mu$ in the sense of measures, under the assumptions that f is a Borel measurable positive function with linear growth, Lipschitz continuous and $\mathcal{A}$-quasiconvex in the last variable, and satisfying an appropriate continuity condition on the first variable (see Theorem 1.4 in [9]). Note that in (1.5) the term μ^s has not been considered.

Here we extend this last result for a larger class of integrands where L^∞-bounds from below are not assumed and to functionals taking into account the singular part of the limit measure μ. Namely, we prove the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $f : \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$-quasiconvex and Lipschitz continuous. Let $\{\mu_n\} \subset \mathcal{M}(\Omega;\mathbb{R}^d)$ be such that $\mu_n \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\overline{\Omega};\mathbb{R}^d)$, $\mathcal{A}\mu_n \in W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$, $1 < q < \frac{N}{N-1}$, $\mathcal{A}\mu_n \overset{W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$ and $|\mu_n| \overset{*}{\rightharpoonup} \Lambda \in \mathcal{M}(\overline{\Omega})$ with $\Lambda(\partial\Omega) = 0$. Then*

$$\mathcal{F}(\mu) \leq \liminf_{n\to\infty} \mathcal{F}(\mu_n) \tag{1.6}$$

where $\mathcal{F}$ is the functional in (1.1) with f^∞ defined by (1.4).

Note that lower semicontinuity may fail if $\Lambda(\partial\Omega) \neq 0$ (see Example 3.3).

The proof of Theorem 1.1 is reduced to the case of sequences of C^∞-functions by a regularization argument and an upper semicontinuous result based on Reshetnyak Continuity Theorem (see Section 3 and Proposition 3.2). To show Proposition 3.2 with a regular sequence of functions $\{u_n\}$ we start, following ideas of Kristensen & Rindler [13], by estimating from below the limit of the sequence of local energies $\lambda_n(A) := \int_A f(u_n)\,dx$. Contrary to the case for positive integrands, this step is essential to write the limit energy of λ_n, λ, exclusively in terms of μ. The result then follows from pointwise estimates on the Radon-Nikodým Derivatives of λ obtained by the usual blow-up argument (introduced in Fonseca & Müller [11]). The main difficulty here arises in the treatment of the singular part $\frac{d\lambda}{d|\mu^s|}$ since we do not know how to characterize the blow-up limit. This difficulty is overcomed by an appropriate average process that allows us to get the estimate for this singular part.

The motivation for this work relies on a characterization of Young measures generated by uniformly bounded and $\mathcal{A}$-free sequences of measures through the duality with an appropriate set of functions with linear growth (work in progress).

In the particular case where $\mu = Du$ for $u \in BV$ (i.e. $\mathcal{A} = \mathrm{curl}$) Theorem 1.1 has been derived by Kristensen & Rindler [13]. In this context the notion of $\mathcal{A}$-quasiconvexity reduces to that of quasiconvexity (which implies Lipchitz continuity).

The second objective of the present paper is to give a relaxation result for the functional (1.1) in the context of $\mathcal{A}$-quasiconvexity. Namely, in the next theorem we show that the functional $\mathcal{G}$ defined by

$$\mathcal{G}(\mu) := \inf \Big\{ \liminf_{n\to\infty} \mathcal{F}(\mu_n) : \mu_n \overset{*}{\rightharpoonup} \mu,\ \mathcal{A}\mu_n \in W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^M),\ \mathcal{A}\mu_n \overset{W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0, \\ |\mu_n| \overset{*}{\rightharpoonup} \Lambda \text{ with } \Lambda(\partial\Omega) = 0 \Big\}.$$

admits an integral representation.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous. Then for $\mu \in \mathcal{M}(\bar{\Omega};\mathbb{R}^d) \cap \ker\mathcal{A}$ such that $|\mu|(\partial\Omega) = 0$ we have that*

$$\mathcal{G}(\mu) = \int_\Omega Q_{\mathcal{A}} f(\mu^a(x))\, dx + \int_\Omega \big(Q_{\mathcal{A}} f\big)^\infty \Big(\frac{d\mu^s}{d|\mu^s|}\Big)\, d|\mu^s|.$$

where $Q_{\mathcal{A}} f$ denotes the quasiconvex envelope of f and $\big(Q_{\mathcal{A}} f\big)^\infty$ denotes its recession function.

In the proof of Theorem 1.2 the lower bound is a immediate consequence of Theorem 1.1, while the upper bound is based on a regularization procedure together with an approximation by piecewise constant functions, that follows naturally from the definition of $\mathcal{A}$-quasiconvexity.

We finish this introduction by referring to Braides, Fonseca & Leoni [6] for other relaxation results in the context of $\mathcal{A}$-quasiconvexity (for $p > 1$) and to Kristensen & Rindler [13] for relaxation for signed functionals in the context of gradients (i.e, as mentioned before $\mu = Du$ for some $u \in BV$).

The overall plan of this work in the ensuing sections will be as follows: *Section 2* collects the main definitions and auxiliary results used in the proof of Theorem 1.1 that can be found in *Section 3*. In Section 4 we present the proof of Theorem 1.2.

2 Preliminary results

In this section we recall the main results used in our analysis. We start by fixing some notations.

2.1 General Notations

Throughout the text we will use the following notations:

- $\Omega \subset \mathbb{R}^N$, $N \geq 1$, will denote an open bounded set;
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ denote, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$;
- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$;
- Q denotes the open unit cube centered at the origin with one side orthogonal to e_N, where e_N denotes the N^{th}-element of the canonical basis of $\mathbb{R}^N$;
- $Q(x,\delta)$ denotes the open cube centered at x with side length $\delta > 0$ and with one side orthogonal to e_N;
- B stands for the unit open ball centered at the origin;
- $B(x,\delta)$ denotes the ball centered at x with radius $\delta > 0$;
- $\mathbb{M}^{M\times d}(\mathbb{R})$ stand for the set of $M \times d$ real matrices;

- $C^\infty_{\rm per}(Q;\mathbb{R}^d)$ is the space of all Q-periodic functions in $C^\infty(\mathbb{R}^N;\mathbb{R}^d)$;
- $L^q_{\rm per}(Q;\mathbb{R}^d)$ is the space of all Q-periodic functions in $L^q_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$;
- $\mathcal{D}^{'}(\Omega;\mathbb{R}^M)$ denotes the space of distributions in Ω with values in $\mathbb{R}^M$.
- C represents a generic positive constant, which may vary from expression to expression;
- $\lim\limits_{n,m} := \lim\limits_{n\to\infty}\,\lim\limits_{m\to\infty}$.

2.2 Measure Theory

In this section we recall some notations and well known results in Measure Theory (see e.g Ambrosio, Fusco & Pallara [5], Evans & Gariepy [12] and Fonseca & Leoni [8], as well as the bibliography therein).

Let X be a locally compact metric space and let $C_c(X;\mathbb{R}^d)$, $d \geq 1$, denote the set of continuous functions with compact support on X. We denote by $C_0(X;\mathbb{R}^d)$ the completion of $C_c(X;\mathbb{R}^d)$ with respect to the supremum norm. Let $\mathcal{B}(X)$ be the Borel σ-algebra of X. By the Riesz-Representation Theorem the dual of the Banach space $C_0(X;\mathbb{R}^d)$, denoted by $\mathcal{M}(X;\mathbb{R}^d)$, is the space of finite $\mathbb{R}^d$-valued Radon measures $\mu : \mathcal{B}(X) \to \mathbb{R}^d$ under the pairing

$$< \mu, \varphi >:= \int_X \varphi\, d\mu \equiv \sum_{i=1}^d \int_X \varphi_i\, d\mu_i$$

where $\varphi = (\varphi_1, ..., \varphi_d)$ and $\mu = (\mu_1, ..., \mu_d)$. The space $\mathcal{M}(X;\mathbb{R}^d)$ will be endowed with the weak*-topology deriving from this duality. In particular a sequence $\{\mu_n\} \subset \mathcal{M}(X;\mathbb{R}^d)$ is said to weak*-converge to $\mu \in \mathcal{M}(X;\mathbb{R}^d)$ (indicated by $\mu_n \overset{\star}{\rightharpoonup} \mu$) if for all $\varphi \in C_0(X;\mathbb{R}^d)$

$$\lim_{n\to\infty} \int_X \varphi\, d\mu_n = \int_X \varphi\, d\mu.$$

If $d = 1$ we write by simplicity $\mathcal{M}(X)$ and we denote by $\mathcal{M}^+(X)$ its subset of positive measures.

Given $\mu \in \mathcal{M}(X;\mathbb{R}^d)$ let $|\mu|$ denote its *total variation* and let supp μ denote its *support*.

The following result can be found in Fonseca & Leoni [8, Corollary 1.204].

Proposition 2.1. *Let $\mu_n \in \mathcal{M}(X)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(X)$ and $|\mu_n| \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(X)$. If $A \subset X$ is open, $\bar{A}$ compact and $\nu(\partial A) = 0$ then*

$$\mu_n(A) \to \mu(A).$$

We recall that a measure μ is said to be *absolutely continuous* with respect to a positive measure ν, written $\mu << \nu$, if for every $E \in \mathcal{B}(X)$ the following implication holds:

$$\nu(E) = 0 \;\Rightarrow\; \mu(E) = 0.$$

Two positive measures μ and ν are said to be *mutually singular*, written $\mu \perp \nu$, if there exists $E \in \mathcal{B}(X)$ such that $\nu(E) = 0$ and $\mu(X \setminus E) = 0$. For general vector-valued measures μ and ν we say that $\mu \perp \nu$ if $|\mu| \perp |\nu|$.

Theorem 2.2 (Lebesgue-Radon-Nikodým Theorem)**.** *Let $\mu \in \mathcal{M}^+(X)$ and $\nu \in \mathcal{M}(X;\mathbb{R}^d)$. Then*

(i) there exists two $\mathbb{R}^d$-valued measures ν_a and ν_s such that

$$\nu = \nu_a + \nu_s \tag{2.1}$$

with $\nu_a << \mu$ and $\nu_s \perp \mu$. Moreover, the decomposition (2.1) is unique, that is, if $\nu = \bar{\nu}_a + \bar{\nu}_s$ for some measures $\bar{\nu}_a, \bar{\nu}_s$, with $\bar{\nu}_a << \mu$ and $\bar{\nu}_s \perp \mu$, then $\nu_a = \bar{\nu}_a$ and $\nu_s = \bar{\nu}_s$;

(ii) there is a μ-measurable function $u \in L^1(\Omega;\mathbb{R}^d)$ such that

$$\nu_a(E) = \int_E u\, d\mu$$

for every $E \in \mathcal{B}(\Omega)$. The function u is unique up to a set of μ measure zero.

The decomposition $\nu = \nu_a + \nu_s$ is called the *Lebesgue decomposition* of ν with respect to μ (see [8, Theorem 1.115]) and the function u is called the *Radon-Nikodým derivative* of ν with respect to μ, denoted by $u = d\nu/d\mu$ (see [8, Theorem 1.101]).

The next result is a strong form of Besicovitch derivation Theorem due to Ambrosio and Dal Maso [4] (see also [5, Theorem 2.22 and Theorem 5.52] or [8, Theorem 1.155]).

Theorem 2.3. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $\nu \in \mathcal{M}(\Omega;\mathbb{R}^d)$. Then there exists a Borel set $N \subset \Omega$ with $\mu(N) = 0$ such that for every $x \in$ $(\text{supp } \mu)\backslash N$*

$$\frac{d\nu}{d\mu}(x) = \frac{d\nu_a}{d\mu}(x) = \lim_{\epsilon\to 0} \frac{\nu\big((x+\epsilon D)\cap\Omega\big)}{\mu\big((x+\epsilon D)\cap\Omega\big)} \in \mathbb{R}$$

and

$$\frac{d\nu_s}{d\mu}(x) = \lim_{\epsilon\to 0} \frac{\nu_s\big((x+\epsilon D)\cap\Omega\big)}{\mu\big((x+\epsilon D)\cap\Omega\big)} = 0,$$

where D is any bounded, convex, open set D containing the origin (the exceptional set N is independent of the choice of D).

In the sequel we denote by $W^{-1,q}(\Omega;\mathbb{R}^d)$ the dual space of $W_0^{1,q'}(\Omega;\mathbb{R}^d)$ where q', the conjugate exponent of q, is given by the relation $\frac{1}{q} + \frac{1}{q'} = 1$. We finish this part by recalling that $\mathcal{M}(\Omega;\mathbb{R}^d)$ is compactly imbeded in $W^{-1,q}(\Omega;\mathbb{R}^d)$, $1 < q < \frac{N}{N-1}$, since $W_0^{1,q'}(\Omega;\mathbb{R}^d) \subset\subset C_0(\Omega)$ for $q' > N$.

2.3 A corollary of Reshetnyak's Theorem

The objective of this part is to present a corollary of Reshetnyak Continuity Theorem useful for our main result in Section 3.

Definition 2.4. (The space $E(\Omega;\mathbb{R}^d)$) *Let $E(\Omega;\mathbb{R}^d)$ denote the space of continuous functions $f: \Omega\times\mathbb{R}^d \to \mathbb{R}$ such that the mapping*

$$(x,\xi) \to (1-|\xi|) f\left(x, \frac{\xi}{1-|\xi|}\right), \ x \in \Omega,\ \xi \in B, \tag{2.2}$$

can be extended to a continuous function to the closure $\overline{\Omega\times B}$.

The recession function of an element f of $E(\Omega;\mathbb{R}^d)$ is the continuous extension of (2.2) to the boundary of $\Omega \times B$. Namely we have the following definition.

Definition 2.5. (Recession function) *Let f be a function in $E(\Omega;\mathbb{R}^d)$. Then recession function of f is defined by*

$$f^\infty(x,\xi) = \lim_{\substack{x'\to x\\ \xi'\to\xi\\ t\to\infty}} \frac{f(x',t\xi')}{t}. \tag{2.3}$$

for all $(x,\xi) \in \overline{\Omega\times B}$.

The next lemma is an approximation result by functions in $E(\Omega;\mathbb{R}^d)$ and is due to Alibert and Bouchitté ([3, Lemma 2.3]).

Lemma 2.6. *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function such that*

$$f(x,\xi) \geq -C(1+|\xi|).$$

Then, there exists an nondecreasing sequence $\{f_k\} \subset E(\Omega;\mathbb{R}^d)$ such that

$$\sup_k f_k(x,\xi) = f(x,\xi) \text{ and } \sup_k f_k^\infty(x,\xi) = h_f(x,\xi)$$

where

$$h_f(x,\xi) := \liminf_{\substack{x' \to x \\ \xi' \to \xi \\ t \to \infty}} \frac{f(x', t\xi')}{t}.$$

The version of Reshetnyak's Continuity Theorem we present here can be found in [13, Theorem 5]

Theorem 2.7. (Reshetnyak's Continuity Theorem) *Let $f \in E(\Omega;\mathbb{R}^d)$ and let μ, $\mu_n \in \mathcal{M}(\Omega;\mathbb{R}^d)$ be such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^d)$ and $\langle\mu_n\rangle(\Omega) \to \langle\mu\rangle(\Omega)$, where*

$$\langle\nu\rangle := \sqrt{1+|\nu^a|^2}\mathcal{L}^N + |\nu^s|, \quad \nu = \nu^a\mathcal{L}^N + \nu^s \in \mathcal{M}(\Omega;\mathbb{R}^d).$$

Then

$$\lim_{n\to\infty} \tilde{\mathcal{F}}(\mu_n) = \tilde{\mathcal{F}}(\mu)$$

where

$$\tilde{\mathcal{F}}(\nu) := \int_\Omega f(x,\nu^a(x))\,dx + \int_\Omega f^\infty\left(x, \frac{d\nu^s}{d|\nu^s|}(x)\right)\,d|\nu^s|, \quad \nu \in \mathcal{M}(\Omega;\mathbb{R}^d). \tag{2.4}$$

As a corollary of Lemma 2.6 and Theorem 2.7 we derive an upper semicontinuity result useful in the proof of our main result Theorem 1.1.

Corollary 2.8. *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that*

$$|f(x,\xi)| \leq C(1+|\xi|), \quad \textit{for all } x \in \Omega, \textit{ all } \xi \in \mathbb{R}^d, \textit{ and some } C > 0.$$

Let μ, $\mu_n \in \mathcal{M}(\Omega;\mathbb{R}^d)$ be such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^d)$ and $\langle\mu_n\rangle(\Omega) \to \langle\mu\rangle(\Omega)$. Then

$$\tilde{\mathcal{F}}(\mu) \geq \limsup_{n\to\infty} \tilde{\mathcal{F}}(\mu_n) \tag{2.5}$$

where $\tilde{\mathcal{F}}$ is the functional defined in (2.4) *and where the recession function of f is defined as follows*

$$f^\infty(x,\xi) := \limsup_{\substack{x' \to x \\ \xi' \to \xi \\ t \to \infty}} \frac{f(x', t\xi')}{t}.$$

Proof. By Lemma 2.6 we can find a nondecreasing sequence of continuous functions $f_h \in E(\Omega;\mathbb{R}^d)$, $h \in \mathbb{N}$, such that for all $(x,\xi) \in \Omega \times \mathbb{R}^d$

$$\sup_{h\in\mathbb{N}} f_h(x,\xi) = -f(x,\xi) \quad \text{and} \quad \sup_{h\in\mathbb{N}} f_h^\infty(x,\xi) = h_{-f}(x,\xi) = -f^\infty(x,\xi).$$

For each $h \in \mathbb{N}$ we have that

$$\begin{aligned}
\limsup_{n\to\infty} \tilde{\mathcal{F}}(\mu_n) &= -\liminf_{n\to\infty}\{-\tilde{\mathcal{F}}(\mu_n)\} \\
&\leq -\lim_{n\to\infty}\left[\int_\Omega f_h(x,\mu_n^a(x))\,dx + \int_\Omega f_h^\infty\left(x, \frac{d\mu_n^s}{d|\mu_n^s|}(x)\right)\,d|\mu_n^s|\right] \\
&= -\left[\int_\Omega f_h(x,\mu^a(x))\,dx + \int_\Omega f_h^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right)\,d|\mu^s|\right] \qquad (2.6)
\end{aligned}$$

by Theorem 2.7. Taking the infimum over h in (2.6), inequality (2.5) follows by the Monotone Convergence Theorem.

□

2.4 $\mathcal{A}$-quasiconvexity

We recall here the notion of $\mathcal{A}$-quasiconvexity introduced by Dacorogna [7] and further devoloped by Fonseca & Müller [10], as well as some of its main properties.

Let $\mathcal{A} : \mathcal{D}'(\Omega;\mathbb{R}^d) \to \mathcal{D}'(\Omega;\mathbb{R}^M)$ be the first order linear differential operator defined in (1.2).

Definition 2.9. *($\mathcal{A}$-quasiconvex function) A locally bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if*

$$f(v) \leq \int_Q f(v + w(x))\,dx$$

for all $v \in \mathbb{R}^d$ and for all $w \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^d)$ such that $\mathcal{A}w = 0$ in $\mathbb{R}^N$ with $\int_Q w(x)\,dx = 0$.

Remark 2.10. If f has q-growth, i.e. $|f(v)| \leq C(1 + |v|^q)$ for all $v \in \mathbb{R}^d$, then the space of test functions $C^\infty_{\mathrm{per}}(Q;\mathbb{R}^d)$ in Definition 2.9 can be replaced by $L^q_{per}(Q,\mathbb{R}^d)$ (see Remark 3.3.2 in [10]).

Definition 2.11. *($\mathcal{A}$-quasiconvex envelope) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We define the $\mathcal{A}$-quasiconvex envelope of f, $\mathcal{Q}_\mathcal{A} f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$, as*

$$\mathcal{Q}_\mathcal{A} f(v) := \inf\left\{\int_Q f(v + w(x))\,dx : \ w \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^d) \text{ such that } \mathcal{A}w = 0 \text{ in } \mathbb{R}^N \text{ and } \int_Q w(x)\,dx = 0\right\}.$$

Remark 2.12. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function.

i) If f has linear growth at infinity and $\mathcal{Q}_\mathcal{A} f(0) > -\infty$ then $\mathcal{Q}_\mathcal{A} f(v)$ is finite for all $v \in \mathbb{R}^d$. In addition $\mathcal{Q}_\mathcal{A} f$ has also linear growth at infinity.

ii) If f is Lipschitz continuous then $\mathcal{Q}_\mathcal{A} f$ is also Lipschitz continuous.

The next lemma is an adapted version of Lemma 4 in Kristensen & Rindler [13] for $\mathcal{A}$-quasiconvex envelopes.

Lemma 2.13. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with linear growth at infinity such that $\mathcal{Q}_\mathcal{A} f(0) > -\infty$. Given $\gamma > 0$ define $f_\gamma(v) := f(v) + \gamma|v|$ for $v \in \mathbb{R}^d$. Then $\mathcal{Q}_\mathcal{A} f_\gamma(v) \downarrow \mathcal{Q}_\mathcal{A} f(v)$ and $(\mathcal{Q}_\mathcal{A} f_\gamma)^\infty(v) \downarrow (\mathcal{Q}_\mathcal{A} f)^\infty(v)$ pointwise in v as $\gamma \to 0$.*

The following proposition can be found in [10, Lemma 2.14].

Proposition 2.14. *Given $q > 1$, there exists a linear bounded operator $\mathcal{P} : L^q_{per}(Q;\mathbb{R}^d) \to L^q_{per}(Q;\mathbb{R}^d)$ such that $\mathcal{A}(\mathcal{P}u) = 0$. Moreover we have the following estimate*

$$||u - \mathcal{P}u||_{L^q} \leq C\|\mathcal{A}u\|_{W^{-1,q}}$$

for every $u \in L^q_{per}(Q;\mathbb{R}^d)$ with $\int_Q u = 0$.

The following lower semicontinuity result is used in the proof of Theorem 1.1.

Lemma 2.15. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{A}$-quasiconvex and Lipschitz continuous function. Let $a \in \mathbb{R}^d$ and $\{u_n\} \subset L^q_{per}(Q;\mathbb{R}^d)$ be a sequence such that $u_n \overset{*}{\rightharpoonup} a\mathcal{L}^N$ in $\mathcal{M}(Q;\mathbb{R}^d)$ and $|u_n| \overset{*}{\rightharpoonup} \Lambda$ in $\mathcal{M}^+(\overline{Q})$, with $\Lambda(\partial Q) = 0$, and $\mathcal{A}u_n \to 0$ in $W^{-1,q}(Q;\mathbb{R}^M)$ for some $1 < q < \frac{N}{N-1}$. Then*

$$\liminf_{n\to\infty} \int_Q f(u_n)\,dx \geq f(a).$$

Proof. Choose $\varphi_m \in C_c^\infty(Q;[0,1])$ satisfying the condition $\varphi_m = 1$ on $Q\left(0; 1 - \frac{1}{m}\right)$ and define $\{w_{m,n}\} \subset L^q_{per}(Q;\mathbb{R}^d)$ by $w_{m,n} = \varphi_m(u_n - a)$. Writting

$$\mathcal{A}(w_{m,n}) = (\mathcal{A}\varphi_m)(u_n - a) + \varphi_m \mathcal{A}u_n$$

we can conclude that

$$\lim_{n\to+\infty} \int_Q w_{m,n}(x)\,dx = 0 \qquad \text{and} \qquad \mathcal{A}(w_{m,n}) \underset{n\to\infty}{\overset{W^{-1,q}(Q;\mathbb{R}^M)}{\longrightarrow}} 0 \tag{2.7}$$

since $u_n \overset{*}{\rightharpoonup} a$ in $\mathcal{M}(Q;\mathbb{R}^d)$ implies that $u_n \to a$ in $W^{-1,q}(Q;\mathbb{R}^M)$. Define now the sequence $\{z_{m,n}\} \subset L^q_{per}(Q;\mathbb{R}^d)$ by

$$z_{m,n} := \mathcal{P}\left(w_{m,n} - \int_Q w_{m,n}\,dx\right).$$

Then, by Lipschitz continuity, $\mathcal{A}$-quasiconvexity (see Remark 2.10) and Proposition 2.14 we have that

$$\begin{aligned}
\int_Q f(u_n)\,dx &= \int_Q f(u_n - a + a)\,dx \\
&\geq \int_Q f(w_{m,n} + a)\,dx - L\int_Q |1-\varphi_m||u_n - a|\,dx \\
&\geq \int_Q f\left(w_{m,n} - \int_Q w_{m,n} + a\right)dx - L\int_Q |1-\varphi_m||u_n - a|\,dx \\
&\quad -L\left|\int_Q w_{m,n}\,dx\right| \\
&\geq \int_Q f(z_{m,n} + a) - L\int_Q |1-\varphi_m||u_n - a|\,dx - L\left|\int_Q w_{m,n}\,dx\right| \\
&\quad -L\int_Q \left|w_{m,n} - \int_Q w_{m,n}\,dx - z_{m,n}\right|dx \\
&\geq f(a) - L\int_Q |1-\varphi_m||u_n - a|\,dx - L\left|\int_Q w_{m,n}\,dx\right| \\
&\quad -CL\|\mathcal{A}w_{m,n}\|_{W^{-1,q}}.
\end{aligned}$$

Taking first the limit as $n \to \infty$ and using the definition of $w_{m,n}$ and (2.7), we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_Q f(u_n)\,dx &\geq f(a) - L\Lambda\left(\overline{Q}\setminus Q\left(0, 1 - \frac{1}{m}\right)\right) \\
&\quad -L|a|\left(1 - \left(1 - \frac{1}{m}\right)^N\right).
\end{aligned}$$

The result now follows letting $m \to \infty$ since by hypothesis $\Lambda(\partial Q) = 0$. $\square$

Remark 2.16. *Lemma 2.15 can also be applied to any cube $P \subset \mathbb{R}^N$.*

2.5 Regularization of measures

The aim of this part is to recall the definition of the regularization of a measure by means of its convolution with a standard mollifier as well as to gather its main properties.

Let $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ with $\operatorname{supp}\rho \subset \overline{B}$ and $\int_{\mathbb{R}^N} \rho\left(x\right) dx = 1$. For every $\varepsilon > 0$ let us define the mollifier

$$\rho_\varepsilon\left(x\right) := \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N. \tag{2.8}$$

Note that $\operatorname{supp}\rho_\varepsilon \subset \overline{B\left(0,\varepsilon\right)}$. Given $\mu \in \mathcal{M}\left(\overline{\Omega};\mathbb{R}^d\right)$ we may think μ as an element of $\mathcal{M}\left(\mathbb{R}^N;\mathbb{R}^d\right)$ with support contained in $\overline{\Omega}$. We define $u_\varepsilon : \mathbb{R}^N \to \mathbb{R}^d$ by

$$u_\varepsilon\left(x\right) := \left(\mu * \rho_\varepsilon\right)\left(x\right) = \int_{\mathbb{R}^N} \rho_\varepsilon\left(x-y\right) d\mu\left(y\right), \quad x \in \mathbb{R}^N \tag{2.9}$$

and for every Borel set $E \subset \Omega$ we denote

$$B_\varepsilon(E) := \{x \in \mathbb{R}^N : \operatorname{dist}\left(x, E\right) < \varepsilon\}.$$

Proposition 2.17. *Let $\mu \in \mathcal{M}\left(\overline{\Omega};\mathbb{R}^d\right)$ and u_ε be given as in* (2.9)*. Then the following statements hold:*

(i) The function $u_\varepsilon \in C^\infty\left(\mathbb{R}^N;\mathbb{R}^d\right)$ and $\operatorname{supp} u_\varepsilon \subset \overline{B_\varepsilon(\Omega)}$. Moreover $D^\alpha(\mu * \rho_\varepsilon) = D^\alpha\mu * \rho_\varepsilon$ for $\alpha \in \mathbb{N}^N$ and the inequality

$$\int_E |\mu * \rho_\varepsilon|(x)\,dx \leq |\mu|(B_\varepsilon(E)) \tag{2.10}$$

holds whenever $E \subset \Omega$ is a Borel set.

(ii) The measures $\mu_\varepsilon := u_\varepsilon\mathcal{L}^N$ and $|\mu_\varepsilon|$ weak- converge in $\mathbb{R}^N$ to μ and $|\mu|$, respectively, as $\varepsilon \to 0$.*

(iii) If $|\mu|(\partial\Omega) = 0$ then $\langle\mu_\varepsilon\rangle(\Omega) \to \langle\mu\rangle(\Omega)$ as $\varepsilon \to 0$.

(iv) If $\mathcal{A}\mu \in W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$, $1 \leq q < \infty$, then $\mathcal{A}u_n \underset{n\to\infty}{\overset{W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow}} \mathcal{A}\mu$.

Proof. The assertions (i)-(ii) follow Theorem 2.2 in Ambrosio & Fusco & Pallara [5].

Proof of (iii). Let $\hat{\mu} := (\mu, \mathcal{L}^N)$. As $\hat{\mu} * \rho_\varepsilon \overset{*}{\rightharpoonup} \hat{\mu}$, we have

$$\liminf |\hat{\mu} * \rho_\varepsilon|(\Omega) \geq |\hat{\mu}|(\Omega).$$

On the other hand as $|\hat{\mu} * \rho_\varepsilon| \overset{*}{\rightharpoonup} |\hat{\mu}|$ and $|\mu|(\partial\Omega) = 0$ we have that

$$\limsup |\hat{\mu} * \rho_\varepsilon|(\Omega) \leq \limsup |\hat{\mu} * \rho_\varepsilon|(\overline{\Omega}) \leq |\hat{\mu}|(\overline{\Omega}) = |\hat{\mu}|(\Omega).$$

Now the result follows from the equalities $\langle\mu_\varepsilon\rangle(\Omega) = |\hat{\mu} * \rho_\varepsilon|(\Omega)$ and $\langle\mu\rangle(\Omega) = |\hat{\mu}|(\Omega)$.

Proof of (iv). We have that $\mathcal{A}u_n = \mathcal{A}\mu * \rho_{\varepsilon_n}$. Given $U \subset\subset \Omega$ let us see that

$$\mathcal{A}u_n \underset{n\to\infty}{\overset{W^{-1,q}(U;\mathbb{R}^M)}{\longrightarrow}} \mathcal{A}\mu.$$

Let V with $U \subset\subset V \subset\subset \Omega$. As $\mathcal{A}\mu \in W^{-1,q}(V;\mathbb{R}^M)$, there exist $T_i \in L^q(V;\mathbb{R}^M)$, $i = 0, ..., N$, such that

$$\mathcal{A}\mu = T_0 + \sum_{i=1}^N \frac{\partial T_i}{\partial x_i}$$

(see Adams [1]). Given $\varphi \in C_c^\infty(U;\mathbb{R}^M)$

$$\begin{aligned}\langle \mathcal{A}u_n - \mathcal{A}\mu, \varphi\rangle &= \left\langle \rho_{\varepsilon_n} * \left(T_0 + \sum_{i=1}^N \frac{\partial T_i}{\partial x_i}\right) - \left(T_0 + \sum_{i=1}^N \frac{\partial T_i}{\partial x_i}\right), \varphi \right\rangle \\ &= \langle \rho_{\varepsilon_n} * T_0 - T_0, \varphi\rangle - \sum_{i=1}^N \left\langle \rho_{\varepsilon_n} * T_i - T_i, \frac{\partial \varphi}{\partial x_i}\right\rangle\end{aligned}$$

and consequently, by Hölder inequality

$$|\langle \mathcal{A}u_n - \mathcal{A}\mu, \varphi\rangle| \le \sum_{i=0}^N \|\rho_{\varepsilon_n} * T_i - T_i\|_{L^q(U;\mathbb{R}^M)} \|\varphi\|_{W^{1,q'}(U;\mathbb{R}^M)}. \tag{2.11}$$

By density (2.11) holds for any $\varphi \in W_0^{1,q'}(U;\mathbb{R}^M)$ and then as

$$\sum_{i=0}^N \|\rho_{\varepsilon_n} * T_i - T_i\|_{L^q(U;\mathbb{R}^M)} \underset{n\to\infty}{\longrightarrow} 0$$

we conclude that $\mathcal{A}u_n \overset{W^{-1,q}(U;\mathbb{R}^M)}{\underset{n\to\infty}{\longrightarrow}} \mathcal{A}\mu$.

□

3 Lower semicontinuity theorem

The aim of this section is to prove Theorem 1.1. Namely, given $\{\mu_n\} \subset \mathcal{M}(\Omega;\mathbb{R}^d)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega};\mathbb{R}^d)$, $\mathcal{A}\mu_n \in W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$, $\mathcal{A}\mu_n \overset{W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$ and $|\mu_n| \overset{*}{\rightharpoonup} \Lambda$ in $\mathcal{M}(\overline{\Omega})$ with $\Lambda(\partial\Omega) = 0$, then

$$\mathcal{F}(\mu) \le \liminf_{n\to\infty} \mathcal{F}(\mu_n) \tag{3.1}$$

where $\mathcal{F}$ is the functional defined in (1.1), that is

$$\mathcal{F}(\nu) = \int_\Omega f(\nu^a(x))\,dx + \int_\Omega f^\infty\left(\frac{d\nu^s}{d|\nu^s|}(x)\right)\,d|\nu^s|, \quad \nu \in \mathcal{M}(\Omega;\mathbb{R}^d),$$

with $f : \mathbb{R}^d \to \mathbb{R}$ a $\mathcal{A}$-quasiconvex and Lipschitz continuous function with recession function f^∞ given by (1.4).

Remark 3.1. The definition of recession function given in (1.4) is the usual one when integrands are assumed to be quasiconvex. It has the advantage to imply f^∞ to be quasiconvex whenever f is quasiconvex (see Kristensen & Rindler [13]). We note that by a similar argument this last property also holds in the case of $\mathcal{A}$-quasiconvex integrands. If we were in the framework of quasiconvexity our measures would be derivatives of BV-functions, i.e, $\nu = Du$ for some $u \in L^1$, and their singular part ν^s would be rank-one (see Alberti [2]). As quasiconvex functions are convex in rank-one directions, the limsup in definition (1.4) would be in fact a limit in these directions and thus

$$\mathcal{F}(\nu) = \mathcal{F}^-(\nu) := \int_\Omega f(\nu^a(x))\,dx + \int_\Omega \underline{f}^\infty\left(\frac{d\nu^s}{d|\nu^s|}(x)\right)\,d|\nu^s|$$

with

$$\underline{f}^\infty(\xi) = \liminf_{t\to\infty} \frac{f(t\xi)}{t}$$

(see Müller [14] for an example of quasiconvex function where $\underline{f}^{\infty} \neq f^{\infty}$).

In the $\mathcal{A}$-quasiconvex framework we do not know if the singular part of an element $\nu \in \mathcal{M}(\Omega;\mathbb{R}^d)$, ν^s, belongs to the directions along which an $\mathcal{A}$-quasiconvex function is convex, i.e, the caracteristic cone (see Fonseca and Muller [10]). If in (1.4) we would have used liminf instead, which is more natural for lower semicontinuity results (see also Rindler [16]), we would have just able to prove the lower semicontinuity of $\mathcal{F}$ for sequences of L^1-functions, i.e., the case where $\mu_n = u_n \subset L^1$ and $u_n\mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ with $\mu \in \mathcal{M}(\overline{\Omega};\mathbb{R}^d)$.

As it will be proven in Subsection 3.1 inequality (3.1) is a consequence of the following proposition.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $f : \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$-quasiconvex and Lipschitz continuous. Let $u_n \in C^\infty(\mathbb{R}^N;\mathbb{R}^d)$ be such that $|u_n| \overset{*}{\rightharpoonup} \Lambda$ in $\mathcal{M}(\overline{\Omega})$, with $\Lambda(\partial\Omega) = 0$. Then if*

$$u_n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega};\mathbb{R}^d) \quad \text{and} \quad \mathcal{A}u_n \to 0 \text{ in } W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^M)$$

for some $1 < q < \frac{N}{N-1}$, we have that

$$\mathcal{F}(\mu) \leq \liminf_{n\to\infty} \mathcal{F}(u_n\mathcal{L}^N). \tag{3.2}$$

Proof. To show (3.2) we assume w.l.o.g. that $\liminf_{n\to\infty} \mathcal{F}(u_n\mathcal{L}^N) = \lim_{n\to\infty} \mathcal{F}(u_n\mathcal{L}^N)$. In addition we may assume that $\lim_{n\to\infty} \mathcal{F}(u_n\mathcal{L}^N) < \infty$, otherwise there is nothing to prove.

Given a Borel subset A of Ω we define

$$\mathcal{F}(\nu;A) = \int_A f(\nu^a(x))\,dx + \int_A f^\infty\left(\frac{d\nu^s}{d|\nu^s|}(x)\right)\,d|\nu^s|, \quad \nu \in \mathcal{M}(\Omega;\mathbb{R}^d),$$

and for any $n \in \mathbb{N}$ we set

$$\lambda_n(A) := \mathcal{F}(u_n\mathcal{L}^N;A) = \int_A f(u_n(x))\,dx.$$

Since $\{\lambda_n\}$ is a sequence of bounded Radon measures there exist $\lambda \in \mathcal{M}(\overline{\Omega};\mathbb{R}^d)$ and $\nu \in \mathcal{M}^+(\overline{\Omega})$ such that (up to a subsequence still denoted by $\{\lambda_n\}$)

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \tag{3.3}$$

and

$$|\lambda_n| \underset{n\to\infty}{\overset{\mathcal{M}^+(\overline{\Omega})}{\rightharpoonup}} \nu. \tag{3.4}$$

We remark that by the growth conditions on f (see (1.3)) it follows that

$$\nu \leq \mathcal{L}^N + \Lambda. \tag{3.5}$$

Step 1. Our first goal is to show that

$$\lambda \llcorner \Omega \geq -c_0(\mathcal{L}^N \llcorner \Omega + |\mu|) \tag{3.6}$$

for some positive constant c_0 depending just on the integrand f.

Proof of (3.6). By the inner regular property of Radon measures it suffices to prove (3.6) for every closed cube $P \subset \Omega$. Fixed such a closed cube $P \subset \Omega$, let us see that

$$\lambda(P) \geq -c_0(\mathcal{L}^N + |\mu|)(P). \tag{3.7}$$

For $r > 1$ let P_r denote the open concentric cube of side length r times that of P. Notice that since Ω is open $P_R \subset \Omega$ for some $R > 1$.
As Λ is a positive Radon measure the set

$$\{r \in (1, R) : \Lambda(\partial P_r) > 0\}$$

is at most countable. Therefore we can fix an $r \in (1, R)$ arbitrarily close to 1 such that

$$\Lambda(\partial P_r) = 0 \tag{3.8}$$

and consequently, since $|\mu| \leq \Lambda$,

$$|\mu|(\partial P_r) = 0. \tag{3.9}$$

Let $\varepsilon_n > 0$, $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$, and define

$$v_n(x) := \mu * \rho_{\varepsilon_n}(x), \quad x \in \mathbb{R}^N,$$

where ρ_{ε_n} is as in (2.8). Then by Proposition 2.17

$$v_n \underset{n\to\infty}{\overset{*}{\rightharpoonup}} \mu \tag{3.10}$$

$$|v_n| \underset{n\to\infty}{\overset{*}{\rightharpoonup}} |\mu| \tag{3.11}$$

and since $\mathcal{A}\mu = 0$ we get that

$$\mathcal{A}v_n \underset{n\to\infty}{\overset{W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^d)}{\longrightarrow}} 0. \tag{3.12}$$

By (3.3), the fact that $\nu(\partial P_r) = 0$ (from (3.5) and (3.8)), the Lipschitz continuity of f, (3.11), Lemma 2.15 (see also Remark 2.16), and (3.9) we have that

$$\begin{aligned} \lambda(P_r) &= \lim_{n\to\infty} \int_{P_r} f(u_n)\, dx \\ &\geq \liminf_{n\to\infty} \int_{P_r} f(u_n - v_n)\, dx - L \lim_{n\to\infty} \int_{P_r} |v_n|\, dx \\ &\geq f(0)|P_r| - L|\mu|(P_r). \end{aligned}$$

Therefore inequality (3.7) follows by letting $r \to 1$ with $c_0 = \max\{|f(0)|, L\}$.

Step 2. In this part we prove that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq f(\mu^a(x_0)) \text{ for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega \tag{3.13}$$

and

$$\frac{d\lambda}{d|\mu^s|}(x_0) \geq f^\infty\left(\frac{d\mu^s}{d|\mu^s|}(x_0)\right) \text{ for } |\mu^s|\text{-a.e. } x_0 \in \Omega. \tag{3.14}$$

Proof of (3.13). Let x_0 be such that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{\delta\to 0} \frac{\lambda(Q(x_0;\delta))}{\delta^N} < \infty \tag{3.15}$$

$$\frac{d|\mu^s|}{d\mathcal{L}^N}(x_0) = \lim_{\delta\to 0} \frac{|\mu^s|(Q(x_0;\delta))}{\delta^N}\, dx = 0, \tag{3.16}$$

$$\lim_{\delta\to 0} \fint_{Q(x_0;\delta)} |\mu^a(x) - \mu^a(x_0)|\, dx = 0, \tag{3.17}$$

$$\lim_{\delta\to 0} \fint_{Q(x_0;\delta)} |\Lambda^a(x) - \Lambda^a(x_0)|\, dx = 0, \tag{3.18}$$

$$\frac{d\Lambda^s}{d\mathcal{L}^N}(x_0) = \lim_{\delta\to 0} \frac{\Lambda^s(Q(x_0;\delta))}{\delta^N}\, dx = 0. \tag{3.19}$$

Recall that all the above properties are satisfied for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$. Let $\delta_k \to 0$ be such that $\Lambda(\partial Q(x_0;\delta_k)) = 0$. Then by (3.15), (3.3), (3.5), and a change of variables

$$\begin{aligned}
\frac{d\lambda}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty} \frac{\lambda\,(Q(x_0;\delta_k))}{\delta_k^N} \\
&= \lim_{k,n} \frac{\lambda_n\,(Q(x_0;\delta_k))}{\delta_k^N} \\
&= \lim_{k,n} \fint_{Q(x_0,\delta_k)} f(u_n)\, dx \\
&= \lim_{k,n} \int_Q f(u_n(x_0+\delta_k y))\, dy. \qquad (3.20)
\end{aligned}$$

We claim that for all $\varphi \in C_0(Q;\mathbb{R}^d)$

$$\lim_{k,n} \int_Q u_n(x_0+\delta_k y)\varphi(y)\, dy = \mu^a(x_0) \int_Q \varphi(y)\, dy. \tag{3.21}$$

Indeed, let $\varphi \in C_0(Q;\mathbb{R}^d)$. Then by a change of variables and since $u_n \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\overline{\Omega};\mathbb{R}^d)$ we get that

$$\begin{aligned}
\lim_{n\to\infty} \int_Q u_n(x_0+\delta_k y)\,\varphi(y)\, dy &= \lim_{n\to\infty} \fint_{Q(x_0,\delta_k)} u_n(x)\,\varphi\left(\frac{x-x_0}{\delta_k}\right) dy \\
&= \fint_{Q(x_0,\delta_k)} \varphi\left(\frac{x-x_0}{\delta_k}\right) d\,\mu.
\end{aligned}$$

Hence, decomposing $\mu = \mu^a \mathcal{L}^N + \mu^s$, by (3.16) and (3.17) and a change of variables

$$\begin{aligned}
\lim_{k,n} \int_Q u_n(x_0+\delta_k y)\,\varphi(y)\, dy &= \lim_{k} \fint_{Q(x_0,\delta_k)} \varphi\left(\frac{x-x_0}{\delta_k}\right) \mu^a(x)\, dx \\
&= \mu^a(x_0) \int_Q \varphi(y)\, dy
\end{aligned}$$

which concludes the proof of (3.21). We remark that by a similar argument, using (3.18) and (3.19) it also holds that

$$\lim_{k,n} \int_Q |u_n|(x_0+\delta_k y)\,\varphi(y)\, dy = \Lambda^a(x_0) \int_Q \varphi(y)\, dy. \tag{3.22}$$

By a diagonalization argument from (3.21), (3.22), the fact that $\mathcal{A}u_n \to 0$ in $W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ and (3.20) we can find a subsequence $n = n_k$ such that by letting

$$w_k(y) := u_{n_k}(x_0 + \delta_k y), \quad y \in Q,$$

we have that

$$w_k \overset{*}{\rightharpoonup} \mu^a(x_0)\mathcal{L}^N, \tag{3.23}$$

$$|w_k| \overset{*}{\rightharpoonup} \Lambda^a(x_0)\mathcal{L}^N, \tag{3.24}$$

$$\mathcal{A}w_k \ \underset{k\to\infty}{\overset{W^{-1,q}(Q;\mathbb{R}^d)}{\longrightarrow}} \ 0 \tag{3.25}$$

and

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty}\int_Q f(w_k(y))\,dy, \tag{3.26}$$

from where inequality (3.13) follows by Lemma 2.15.

Proof of (3.14). Let $x_0 \in \operatorname{supp}|\mu^s|$ be such that

$$\frac{d\lambda^s}{d|\mu^s|}(x_0) = \frac{d\lambda}{d|\mu^s|}(x_0) = \lim_{\delta\to 0}\frac{\lambda(Q(x_0;\delta))}{|\mu^s|(Q(x_0;\delta))} < \infty \tag{3.27}$$

and

$$\frac{d\mu^s}{d|\mu^s|}(x_0) = \lim_{\delta\to 0}\frac{\mu^s(Q(x_0;\delta))}{|\mu^s|(Q(x_0;\delta))} < \infty. \tag{3.28}$$

Recall that these properties are satisfied for $|\mu^s|$-a.e. $x_0 \in \Omega$. Let $t_k \underset{k\to\infty}{\longrightarrow} \infty$ be such that

$$f^\infty\left(\frac{d\mu^s}{d|\mu|^s}(x_0)\right) = \lim_{k\to\infty}\frac{f\left(t_k\frac{d\mu^s}{d|\mu|^s}(x_0)\right)}{t_k}, \tag{3.29}$$

and choose $\delta_k \underset{k\to\infty}{\longrightarrow} 0$ such that $\Lambda(\partial Q(x_0;\delta_k)) = 0$ and

$$t_k = \frac{|\mu^s|\big(Q(x_0,\delta_k)\big)}{\delta_k^N} \tag{3.30}$$

(see Appendix A for a detailed description of this step). Then by (3.27) and (3.3),

$$\begin{aligned}
\frac{d\lambda^s}{d|\mu^s|}(x_0) &= \frac{d\lambda}{d|\mu^s|}(x_0)\\
&= \lim_{k\to\infty}\frac{\lambda\big(Q(x_0,\delta_k)\big)}{|\mu^s|\big(Q(x_0,\delta_k)\big)}\\
&= \lim_{k,n}\frac{\lambda_n\big(Q(x_0,\delta_k)\big)}{|\mu^s|\big(Q(x_0,\delta_k)\big)}\\
&= \lim_{k,n}\frac{\int_{Q(x_0,\delta_k)} f(u_n(x))\,dx}{|\mu^s|\big(Q(x_0,\delta_k)\big)}\\
&= \lim_{k,n}\frac{1}{t_k}\int_Q f(u_n(x_0+\delta_k y))\,dy.
\end{aligned}$$

Letting

$$w_{k,n}(y) := \frac{u_n(x_0 + \delta_k y)}{t_k} \quad \text{for } y \in Q$$

and

$$f_k(y) := \frac{f(t_k y)}{t_k} \quad \text{for } y \in Q$$

it follows that

$$\begin{aligned} \frac{d\lambda^s}{d|\mu^s|}(x_0) &= \lim_{k,n} \frac{1}{t_k} \int_Q f(t_k w_{k,n}(y))\, dy \\ &= \lim_{k,n} \int_Q f_k(w_{k,n}(y))\, dy. \end{aligned} \tag{3.31}$$

Note that each f_k inherits the $\mathcal{A}$-quasiconvexity property of f. Let us denote by $\tilde{w}_{k,n}$ the extension Q-periodic to all of $\mathbb{R}^N$ of $w_{k,n}$. For each $k, n, m \in \mathbb{N}$ let us define

$$v_{k,n,m}(y) := \tilde{w}_{k,n}(my), \quad y \in Q, \tag{3.32}$$

and note that by changing variables and the properties of $\{u_n\}$

$$\int_Q |v_{k,n,m}|\, dx = \frac{1}{m^N} \int_{mQ} |\tilde{w}_{k,n}|\, dx = \int_Q |w_{k,n}|\, dx \le C. \tag{3.33}$$

We claim that

$$v_{k,n,m} \underset{m,n\to\infty}{\overset{*}{\rightharpoonup}} \alpha_k \mathcal{L}^N, \qquad \alpha_k := \frac{\mu(Q(x_0,\delta_k))}{|\mu^s|(Q(x_0,\delta_k))} \tag{3.34}$$

and

$$\mathcal{A} v_{k,n,m} \quad \underset{n\to\infty}{\overset{W^{-1,q}(Q;\mathbb{R}^M)}{\longrightarrow}} 0. \tag{3.35}$$

To prove (3.34) let us write for each $m \in \mathbb{N}$

$$Q = \bigcup_{j=1}^{m^N} \left(a_j + \frac{Q}{m} \right), \quad a_j \in \frac{\mathbb{Z}^N}{m}. \tag{3.36}$$

Given $\varphi \in C_0(Q;\mathbb{R}^d)$

$$\begin{aligned} \int_Q v_{k,n,m}(y)\varphi(y)\, dy &= \sum_{j=1}^{m^N} \int_{a_j + \frac{Q}{m}} v_{k,n,m}(y)\varphi(y)\, dy \\ &= \sum_{j=1}^{m^N} \int_{a_j + \frac{Q}{m}} v_{k,n,m}(y)\, (\varphi(y) - \varphi(a_j))\, dy + \sum_{j=1}^{m^N} \varphi(a_j) \int_{a_j + \frac{Q}{m}} v_{k,n,m}(y)\, dy. \end{aligned} \tag{3.37}$$

By changing variables and using (3.30)

$$\sum_{j=1}^{m^N}\varphi(a_j)\int_{a_j+\frac{Q}{m}} v_{k,n,m}(y)\,dy=\sum_{j=1}^{m^N}\frac{\varphi(a_j)}{m^N|\mu^s|(Q(x_0,\delta_k))}\int_{Q(x_0,\delta_k)} u_n(y)\,dy.$$

On the other hand by (3.33)

$$\left|\sum_{j=1}^{m^N}\int_{a_j+\frac{Q}{m}} v_{k,n,m}(y)\left(\varphi(y)-\varphi(a_j)\right)\,dy\right|\le C\varepsilon_\varphi(m)$$

where

$$\varepsilon_\varphi(m)=\max_j \max_{y\in a_j+\frac{\overline{Q}}{m}}|\varphi(y)-\varphi(a_j)|.$$

Note that $\varepsilon_\varphi(m)\underset{m\to\infty}{\longrightarrow}0$. Thus, passing to the limit in (3.37) and using Proposition 2.1 we have that

$$\lim_{m,n}\int_Q v_{k,n,m}(y)\varphi(y)\,dy=\alpha_k\int_Q\varphi(y)\,dy$$

which concludes the proof of (3.34). To prove (3.35) let $\varphi\in C_c^\infty(Q;\mathbb{R}^M)$. Then, by decomposing Q as in (3.36) it follows that

$$\begin{aligned}\langle\mathcal{A}v_{k,n,m},\varphi\rangle &= -\sum_{i=1}^N A^{(i)}\int_Q v_{k,n,m}\frac{\partial\varphi}{\partial x_i}\\ &= -\sum_{i=1}^N\sum_{j=1}^{m^N}A^{(i)}\int_{a_j+\frac{\partial Q}{m}} v_{k,n,m}\varphi\,\nu_i+\sum_{i=1}^N\sum_{j=1}^{m^N}A^{(i)}\int_{a_j+\frac{Q}{m}}\frac{\partial v_{k,n,m}}{\partial x_i}\varphi\end{aligned}$$

where $\nu=(\nu_1,...,\nu_N)$ is the normal vector to ∂Q. Let $\sigma^1_{k,n,m},\sigma^2_{k,n,m}\in W^{-1,q}(Q;\mathbb{R}^M)$ be given by

$$\langle\sigma^1_{k,n,m},\varphi\rangle=-\sum_{i=1}^N\sum_{j=1}^{m^N}A^{(i)}\int_{a_j+\frac{\partial Q}{m}} v_{k,n,m}\varphi\,\nu_i$$

and

$$\langle\sigma^2_{k,n,m},\varphi\rangle=\sum_{i=1}^N\sum_{j=1}^{m^N}A^{(i)}\int_{a_j+\frac{Q}{m}}\frac{\partial v_{k,n,m}}{\partial x_i}\varphi$$

for $\varphi\in W_0^{1,q'}(Q;\mathbb{R}^M)$. If we prove that

$$\sigma^1_{k,n,m}\ \underset{n\to\infty}{\overset{\mathcal{M}(Q;\mathbb{R}^M)}{\rightharpoonup}}\ 0, \tag{3.38}$$

which implies that

$$\sigma^1_{k,n,m}\ \underset{n\to\infty}{\overset{\mathcal{W}^{-1,q}(Q;\mathbb{R}^M)}{\longrightarrow}}\ 0,$$

and, in addition, we show that

$$\sigma^2_{k,n,m}\ \underset{n\to\infty}{\overset{\mathcal{W}^{-1,q}(Q;\mathbb{R}^M)}{\longrightarrow}}\ 0, \tag{3.39}$$

then (3.35) will follow.

Let us see that (3.38) holds. Note that for all $\varphi \in C_0(Q;\mathbb{R}^M)$

$$\begin{aligned} |\langle \sigma^1_{k,n,m}, \varphi\rangle| &\leq C\sum_{j=1}^{m^N}\left(\int_{a_j+\frac{\partial Q}{m}} |v_{k,n,m}|\right)\|\varphi\|_{L^\infty} = Cm\left(\int_{\partial Q}|w_{k,n}|\right)\|\varphi\|_{L^\infty} \\ &= \frac{Cm\delta_k}{|\mu^s|(Q(x_0,\delta_k)}\left(\int_{\partial Q(x_0,\delta_k)}|u_n|\right)\|\varphi\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where in last step we have used the condition $\Lambda(\partial(Q(x_0,\delta_k)) = 0$ and the fact that $|u_n| \overset{*}{\rightharpoonup} \Lambda$ in $\mathcal{M}(\overline{\Omega})$. In a similar way (3.39) follows by changing variables and using the hypothesis that $\mathcal{A}u_n \overset{\mathcal{W}^{-1,q}(Q(x_0,\delta_k);\mathbb{R}^M)}{\underset{n\to\infty}{\longrightarrow}} 0$. Therefore gathering all these steps together, by (3.31), (3.32), a change of variables, (3.34), (3.35) and Lemma 2.15, applied to $v_{k,n,m} \in L^q_{\text{per}}(Q;\mathbb{R}^M)$, we conclude that

$$\begin{aligned} \frac{d\lambda^s}{d|\mu^s|}(x_0) &= \lim_{k,m,n}\int_Q f_k(v_{k,n,m}(y))\,dy \\ &\geq \liminf_k f_k(\alpha_k). \end{aligned}$$

Since f_k is Lipschitz (with the same Lipschitz constant than f) and $\alpha_k \underset{k\to\infty}{\longrightarrow} \frac{d\mu^s}{d|\mu^s|}(x_0)$ (see (3.28)) using (3.29) we have that

$$\liminf_k f_k(\alpha_k) \geq \lim_k \frac{f\left(t_k\frac{d\mu^s}{d|\mu^s|}(x_0)\right)}{t_k} = f^\infty\left(\frac{d\mu^s}{d|\mu^s|}(x_0)\right)$$

from where (3.14) holds.

Step 3. We finally prove inequality (3.2). Let us denote by λ^s_μ the singular part of λ^s with respect to $|\mu^s|$. Since λ^s_μ is mutually singular with respect to $|\mu| + \mathcal{L}^N$ then by (3.6)

$$\lambda^s_\mu(B) \geq -c_0(\mathcal{L}^N(B) + |\mu|(B)) = 0$$

for all Borel sets $B \subset \operatorname{supp}\lambda^s_\mu$, that is

$$\lambda^s_\mu \geq 0. \tag{3.40}$$

Now by the fact that $\Lambda(\partial\Omega) = 0$ and by (3.40), (3.13) and (3.14) we get that

$$\begin{aligned} \liminf_{n\to\infty}\mathcal{F}(u_n) &= \liminf_{n\to\infty}\mathcal{F}(u_n;\Omega) \\ &= \liminf_{n\to\infty}\lambda_n(\Omega) \\ &\geq \lambda(\Omega) \\ &= \int_\Omega \frac{d\lambda}{d\mathcal{L}^N}\,dx + \int_\Omega \frac{d\lambda^s}{d|\mu^s|}\,d|\mu^s| + \lambda^s_\mu(\Omega) \\ &\geq \int_\Omega f(\mu^a)\,dx + \int_\Omega f^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right)\,d|\mu^s|. \end{aligned}$$

□

3.1 Proof of Theorem 1.1

Given $\{\mu_n\} \subset \mathcal{M}(\Omega;\mathbb{R}^d)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$, $|\mu_n| \overset{*}{\rightharpoonup} \Lambda$ (with $\Lambda(\partial\Omega) = 0$) and $\mathcal{A}\mu_n \in W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^M)$, $\mathcal{A}\mu_n \overset{W^{-1,q}_{\text{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$, and using a regularization procedure (see Proposition 2.17), we can find a sequence of regular functions $v_{m,n} \in C^\infty(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$v_{m,n} \overset{*}{\rightharpoonup} \mu_n,\ \langle v_{m,n}\rangle(\Omega) \to \langle \mu_n\rangle(\Omega) \text{ and } \mathcal{A}v_{m,n} \to \mathcal{A}\mu_n \text{ in } W^{-1,q}_{\rm loc}(\Omega;\mathbb{R}^M)$$

as $m \to \infty$. Thus, by Corollary 2.8, we have that

$$\limsup_{m\to\infty} \mathcal{F}(v_{m,n}) \le \mathcal{F}(\mu_n).$$

By an appropriate diagonalization procedure we can find a sequence $u_n := v_{m_n,n}$ such that

$$\mathcal{F}(u_n) \le \mathcal{F}(\mu_n) + \frac{1}{n},$$

$u_n \overset{*}{\rightharpoonup} \mu$, $|u_n| \overset{*}{\rightharpoonup} \Lambda$ and $\mathcal{A}u_n \to 0$ in $W^{-1,q}_{\rm loc}(\Omega;\mathbb{R}^M)$. Then by Proposition 3.2

$$\mathcal{F}(\mu) \le \liminf_{n\to\infty} \mathcal{F}(u_n) \le \liminf_{n\to\infty} \mathcal{F}(\mu_n).$$

The next example shows that the conclusion of Theorem 1.1 may not hold if the boundary condition $\Lambda(\partial\Omega) = 0$ is dropped.

Example 3.3. Let $\Omega = (0,1)$, $u_n = \chi_{(0,\frac{1}{n})}$ and $\mu_n := Du_n = \delta_{\frac{1}{n}}$. We have that $\mu_n \overset{*}{\rightharpoonup} \delta_0$ and $\operatorname{curl}\mu_n = 0$. Let $f(v) = -v$, $v \in \mathbb{R}$. We note that $f^\infty = f$. Then

$$\liminf_{n\to\infty} \mathcal{F}(\mu_n) = \liminf_{n\to\infty} \int_0^1 f^\infty(1)\, d\delta_{\frac{1}{n}} = f^\infty(1) = -1 < \mathcal{F}(\delta_0) = 0.$$

In this case $\Lambda(\partial\Omega) = \delta_0(\partial\Omega) \neq 0$.

4 Relaxation

In this section we prove Theorem 1.2, that is, we give an integral representation of the relaxation of the functional (1.1) with respect the class of sequences $\{\mu_n\} \subset \mathcal{M}(\Omega;\mathbb{R}^d)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega};\mathbb{R}^d)$, $\mathcal{A}\mu_n \in W^{-1,q}_{\rm loc}(\Omega;\mathbb{R}^M)$, $\mathcal{A}\mu_n \overset{W^{-1,q}_{\rm loc}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$ and $|\mu_n| \overset{*}{\rightharpoonup} \Lambda$ in $\mathcal{M}(\overline{\Omega})$ with $\Lambda(\partial\Omega) = 0$.

Proof of Theorem 1.2. Set

$$\mathcal{H}(\mu) := \int_\Omega Q_{\mathcal{A}} f(\mu^a(x))\, dx + \int_\Omega Q_{\mathcal{A}} f^\infty\Big(\frac{d\mu^s}{d|\mu^s|}\Big)\, d|\mu^s|.$$

From the lower semicontinuity Theorem 1.1 the lower bound $\mathcal{G} \ge \mathcal{H}$ follows immediately. We show now the upper bound, that is, given $\mu \in \mathcal{M}(\bar\Omega;\mathbb{R}^d) \cap \ker\mathcal{A}$ such that $|\mu|(\partial\Omega) = 0$ we have to see that $\mathcal{G}(\mu) \le \mathcal{H}(\mu)$. For this purpose let $\gamma > 0$ and define

$$f_\gamma(v) := f(v) + \gamma|v|,\ v \in \mathbb{R}^d.$$

It is then enough to show that

$$\mathcal{G}(\mu) \le \int_\Omega Q_{\mathcal{A}} f_\gamma(\mu^a)\, dx + \int_\Omega Q^\infty_{\mathcal{A}} f_\gamma\Big(\frac{d\mu^s}{d|\mu^s|}\Big)\, d|\mu^s| \tag{4.1}$$

and to let $\gamma \to 0$ (see Lemma 2.13).

Proof of (4.1). By Lemma 2.17 let $\{u_n\} \subset C^\infty(\mathbb{R}^N;\mathbb{R}^d)$ such that $u_n \overset{*}{\rightharpoonup} \mu$, $\langle u_n\rangle(\Omega) \to \langle\mu\rangle(\Omega)$ and $\mathcal{A}u_n \overset{W^{-1,q}_{\rm loc}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$. By Corollary 2.8 and Remark 2.12 we have

$$\limsup_{n\to\infty} \int_\Omega Q_{\mathcal{A}} f_\gamma(u_n) \le \int_\Omega Q_{\mathcal{A}} f_\gamma(\mu^a) + \int_\Omega Q_{\mathcal{A}} {f_\gamma}^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right). \tag{4.2}$$

We now decompose

$$\Omega = \bigcup_{i=1}^{J_n} Q_{i,n} \cup \Omega_n$$

where $Q_{i,n} = x_i + r_i Q$ $(i = 1, ..., J_n)$ are open and disjoint cubes and Ω_n is disjoint from any $Q_{i,n}$ and such that

$$\Omega_n \subset \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) < 1/n\}. \tag{4.3}$$

Using the fact that the class of piecewise constant functions is (strongly) dense in $L^1(\Omega; \mathbb{R}^d)$, let v_n be a piecewise constant function such that

$$\|u_n - v_n\|_{L^1(\Omega)} \leq \frac{1}{n} \tag{4.4}$$

and $v_n = \zeta_{i,n}$ on $Q_{i,n}$ for some $\zeta_{i,n} \in \mathbb{R}^d$. For each i, n, by Definition 2.9, we can find $w_{i,n} \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ with $\mathcal{A} w_{i,n} = 0$ and $\int_Q w_{i,n}(x)\, dx = 0$ such that

$$\int_Q f_\gamma(\zeta_{i,n} + w_{i,n})\, dx < Q_{\mathcal{A}} f_\gamma(\zeta_{i,n}) + \frac{1}{n}. \tag{4.5}$$

Note that there exist a constant K_n such that

$$|w_{i,n}(x)| \leq K_n, \text{ for all } x \in Q, \ i = 1, ..., J_n.$$

Let $\phi_n \in C^\infty_c(Q; \mathbb{R})$, $\phi_n(x) \in [0,1]$, $x \in Q$, such that $\phi_n = 1$ on $Q\,(0, \tau_n)$ with $\tau_n \to 1$, as $n \to \infty$, and such that

$$K_n |\Omega| (1 - {\tau_n}^N) \leq \frac{1}{n}. \tag{4.6}$$

For each $x \in \Omega$ set

$$v_{n,m}(x) := \begin{cases} u_n(x) + \phi_n\left(\frac{x - x_i}{r_i}\right) w_{i,n}\left(m\left(\frac{x - x_i}{r_i}\right)\right) & \text{if } x \in Q_{i,n} \\ u_n(x) & \text{if } x \in \Omega_n \end{cases}$$

We claim that

$$\int_\Omega |v_{n,m}| \leq C. \tag{4.7}$$

Since $\{u_n\}$ is bounded in L^1 and $\|\phi_n\|_{L^\infty} \leq 1$ to see (4.7) it is enough to prove that

$$\sum_{i=1}^{J_n} \int_{Q_{i,n}} \left| w_{i,n}\left(m\left(\frac{x - x_i}{r_i}\right)\right)\right| dx \leq C.$$

By a change of variables

$$\sum_{i=1}^{J_n} \int_{Q_{i,n}} \left| w_{i,n}\left(m\left(\frac{x - x_i}{r_i}\right)\right)\right| dx = \sum_{i=1}^{J_n} r_i^N \int_Q |w_{i,n}(y)|\, dy. \tag{4.8}$$

We now use (4.5) to bound (4.8). Indeed by Remark 2.12

$$\sum_{i=1}^{J_n} r_i^N Q_A f_\gamma(\zeta_{i,n}) = \sum_{i=1}^{J_n} \int_{Q_{i,n}} Q_{\mathcal{A}} f_\gamma(v_n)\, dx \leq C \int_\Omega (1 + |v_n|)\, dx \leq C$$

so that

$$\sum_{i=1}^{J_n} r_i^N \int_Q f_\gamma(\zeta_{i,n} + w_{i,n})\, dx \le C. \tag{4.9}$$

On the other hand

$$\begin{aligned}\sum_{i=1}^{J_n} r_i^N \int_Q f(\zeta_{i,n} + w_{i,n})\, dx &\ge \sum_{i=1}^{J_n} r_i^N Q_{\mathcal{A}} f(\zeta_{i,n}) \\ &= \sum_{i=1}^{J_n} \int_{Q_{i,n}} Q_{\mathcal{A}} f(v_n)\, dx \\ &\ge -C \int_\Omega (1 + |v_n|)\, dx \\ &\ge -C\end{aligned}$$

that together with (4.9) implies that

$$\sum_{i=1}^{J_n} r_i^N \int_Q |\zeta_{i,n} + w_{i,n}|\, dx \le C.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{J_n} r_i^N \int_Q |w_{i,n}(x)|\, dx &\le \sum_{i=1}^{J_n} r_i^N \int_Q |\zeta_{i,n} + w_{i,n}|\, dx + \sum_{i=1}^{J_n} r_i^N \int_Q |\zeta_{i,n}(x)|\, dx \\ &\le C + \int_\Omega |v_n|\, dx \\ &\le C.\end{aligned}$$

Note that as $\int_Q w_{i,n}(x)\, dx = 0$ then by Riemman-Lebesgue we have that

$$w_{i,n}\left(m\left(\frac{\cdot - x_i}{r_i}\right)\right) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} 0 \tag{4.10}$$

and hence $v_{n,m} \overset{*}{\rightharpoonup} \mu$. In addition

$$w_{i,n}\left(m\left(\frac{x - x_i}{r_i}\right)\right) \mathcal{A}_x \phi_n \left(\frac{x - x_i}{r_i}\right) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} 0$$

and so $\mathcal{A} v_{n,m} \underset{n,m}{\overset{W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow}} 0$.

Using (4.7) by a diagonalization process we can obtain a sequence $\{v_{n,m_n}\}$ such that $v_{n,m_n} \overset{*}{\rightharpoonup} \mu$ and $\mathcal{A} v_{n,m_n} \overset{W^{-1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)}{\longrightarrow} 0$.

Let $\tilde{v}_n := v_{n,m_n}$ then by the Lipschitz continuity of f (and hence of f_γ) and (4.4) we get that

$$\begin{aligned}
\int_\Omega f_\gamma(\tilde{v}_n)\,dx &\le \sum_{i=1}^{J_n}\int_{Q_{i,n}} f_\gamma\left(u_n+\phi_n\left(\frac{x-x_i}{r_i}\right)w_{i,n}\left(m_n\left(\frac{x-x_i}{r_i}\right)\right)\right)\,dx\\
&\quad +C\int_{\Omega_n}(1+|u_n|)\,dx\\
&\le \sum_{i=1}^{J_n}\int_{Q_{i,n}} f_\gamma\left(\zeta_{i,n}+w_{i,n}\left(m_n\left(\frac{x-x_i}{r_i}\right)\right)\right)\,dx+\frac{C}{n}\\
&\quad +C\sum_{i=1}^{J_n}\int_{Q_{i,n}}\left(1-\phi_n\left(\frac{x-x_i}{r_i}\right)\right)\left|w_{i,n}\left(m_n\left(\frac{x-x_i}{r_i}\right)\right)\right|\,dx\\
&\quad +C\int_{\Omega_n}(1+|u_n|)\,dx.
\end{aligned}$$

Therefore by changing variables and using the periodicity of $w_{i,n}$

$$\begin{aligned}
\int_\Omega f_\gamma(\tilde{v}_n)\,dx &\le \sum_{i=1}^{J_n} {r_i}^N\int_Q f_\gamma\left(\zeta_{i,n}+w_{i,n}(y)\right)\,dy+\frac{C}{n}\\
&\quad +C\sum_{i=1}^{J_n} r_i^N\int_Q\big(1-\phi_n(y)\big)\,|w_{i,n}(m_n y)|\,dy+C\int_{\Omega_n}(1+|u_n|)\,dx
\end{aligned}$$

Next, by (4.5) and the Lipschitz continuity of $Q_{\mathcal{A}}f_\gamma$, we have that

$$\begin{aligned}
\int_\Omega f_\gamma(\tilde{v}_n)\,dx &\le \sum_{i=1}^{J_n} r_i^N Q_{\mathcal{A}}f_\gamma(\zeta_{i,n})+\frac{C}{n}\\
&\quad +C\sum_{i=1}^{J_n} r_i^N\int_Q\big(1-\phi_n(y)\big)\,|w_{i,n}(m_n y)|\,dy+C\int_{\Omega_n}(1+|u_n|)\,dx\\
&\le \int_\Omega Q_{\mathcal{A}}f_\gamma(v_n)\,dx-\int_{\Omega_n} Q_{\mathcal{A}}f_\gamma(u_n)\,dx+\frac{C}{n}\\
&\quad +C\sum_{i=1}^{J_n} r_i^N\int_Q\big(1-\phi_n(y)\big)\,|w_{i,n}(m_n y)|\,dy+C\int_{\Omega_n}(1+|u_n|)\,dx\\
&\le \int_\Omega Q_{\mathcal{A}}f_\gamma(u_n)\,dx+\frac{C}{n}\\
&\quad +C\sum_{i=1}^{J_n} r_i^N\int_Q\big(1-\phi_n(y)\big)\,|w_{i,n}(m_n y)|\,dy+C\int_{\Omega_n}(1+|u_n|)\,dx \qquad (4.11)
\end{aligned}$$

By (4.6) it follows that

$$\sum_{i=1}^{J_n} r_i^N\int_Q(1-\phi_n(y)\,|w_{i,n}(m_n y)|\,dy\le K_n\sum_{i=1}^{J_n} r_i^N|Q\setminus Q(0,\tau_n)|\le K_n|\Omega|(1-{\tau_n}^N)\le\frac{1}{n}$$

which implies that

$$\limsup_{n\to\infty}\sum_{i=1}^{J_n} r_i^N\int_Q(1-\phi_n(y)\,|w_{i,n}(m_n y)|\,dy=0.$$

Given $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $\mathcal{L}^N(\Omega_n) + \Lambda(\overline{\Omega_n}) < \varepsilon$ for all $n \geq n_0$. Then

$$\limsup_{n\to\infty} \int_{\Omega_n} (1 + |u_n|)\, dx < \varepsilon.$$

Therefore from (4.11) and (4.2) we get that

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega f_\gamma(\tilde{v}_n)\, dx &\leq \limsup_{n\to\infty} \int_\Omega Q_{\mathcal{A}} f_\gamma(u_n)\, dx + \varepsilon \\ &\leq \int_\Omega Q_{\mathcal{A}} f_\gamma(\mu^a) + \int_\Omega Q_{\mathcal{A}} f_\gamma{}^\infty \left(\frac{d\mu^s}{d|\mu^s|}\right) + \varepsilon.\end{aligned}$$

Hence

$$\mathcal{G}(\mu) \leq \int_\Omega Q_{\mathcal{A}} f_\gamma(\mu^a) + \int_\Omega Q_{\mathcal{A}} f_\gamma{}^\infty \left(\frac{d\mu^s}{d|\mu^s|}\right) + \varepsilon.$$

By letting ε go to zero inequality (4.1) finally follows.

A Appendix to the proof of (3.14)

With the notation used in the proof of inequality (3.14) (see (3.27) and (3.28)) let

$$g(\delta) := \frac{|\mu^s|(Q(x_0;\delta))}{\delta^N}$$

for $\delta > 0$ such that $Q(x_0;\delta) \subset \Omega$. Notice that the function $\delta \to |\mu^s|(Q(x_0;\delta))$ is nondecreasing and consequently it has right and left limit at every point. Thus, also the function g has right and left limit at every point, and we have

$$g^-(\delta_0) \leq g^+(\delta_0),$$

for every $\delta_0 > 0$, where $g^-(\delta_0) = \lim_{\delta\to\delta_0^-} g(\delta)$ and $g^+(\delta_0) = \lim_{\delta\to\delta_0^+} g(\delta)$

Lemma A.1. *For every $t > \inf\{g\}$ there exists $\bar{\delta} > 0$ such that*

$$g(\bar{\delta}) = t.$$

In addition g is continuous at $\bar{\delta}$.

Proof. As $g(\delta) \underset{\delta\to 0}{\longrightarrow} \infty$, we can find $\delta_0 > 0$ such that

$$\delta < \delta_0 \implies g(\delta) > t. \tag{A.1}$$

Define

$$\bar{\delta} = \sup\{\delta : \text{ (A.1) holds}\}.$$

Thus

$$g^+(\bar{\delta}) \leq t \text{ and } g^-(\bar{\delta}) \geq t$$

and we conclude that

$$g^-(\bar{\delta}) = g^+(\bar{\delta}) = g(\bar{\delta}) = t.$$

□

Lemma A.2. *Let $\Lambda \in \mathcal{M}^+(\Omega)$. Given $a \in \mathbb{R}^d$ there exists $\{s_k\}$ with $s_k \underset{k\to\infty}{\longrightarrow} \infty$ such that*

$$f^\infty(a) = \lim_{k\to\infty} \frac{f(s_k a)}{s_k} \tag{A.2}$$

and $\Lambda(\partial Q(x_0;\delta_k)) = 0$ for $\{\delta_k\}$ such that $g(\delta_k) = s_k$.

Proof. We start with a sequence $\{\bar{s}_k\}$ ($\bar{s}_k > \inf\{g\}$) verifying the condition (A.2). By Lemma A.1 we consider $\{\bar{\delta}_k\}$ such that $g(\bar{\delta}_k) = \bar{s}_k$. We may not have the condition $\Lambda(\partial Q(x_0;\bar{\delta}_k)) = 0$. As g is continuous at $\bar{\delta}_k$ we can choose δ_k close enough to $\bar{\delta}_k$ such that $\Lambda(\partial Q(x_0;\delta_k) = 0$ and $g(\delta_k) = s_k$ is such that $\{s_k - \bar{s}_k\}$ is bounded. As f is Lipschitz continuous (A.2) holds. Indeed, since $\{s_k - \bar{s}_k\}$ is bounded

$$\frac{\bar{s}_k}{s_k} = 1 + \frac{\bar{s}_k - s_k}{s_k} \underset{k\to\infty}{\longrightarrow} 1.$$

In addition as

$$\frac{f(s_k a)}{s_k} - \frac{f(\bar{s}_k a)}{\bar{s}_k} = \frac{f(s_k a) - f(\bar{s}_k a)}{s_k} + \frac{f(\bar{s}_k a)}{s_k}\Big(1 - \frac{s_k}{\bar{s}_k}\Big),$$

and, by the Lipchitz continuity of f,

$$\frac{|f(\bar{s}_k a) - f(s_k a)|}{s_k} \leq L|a|\frac{|\bar{s}_k - s_k|}{s_k} \underset{k\to\infty}{\longrightarrow} 0$$

and $\{\frac{f(\bar{s}_k a)}{s_k}\}$ is bounded, then

$$\lim_{k\to\infty} \frac{f(s_k a)}{s_k} = \lim_{k\to\infty} \frac{f(\bar{s}_k a)}{\bar{s}_k}.$$

□

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