NONLOCAL CHARACTER OF THE REDUCED THEORY OF THIN FILMS WITH HIGHER ORDER PERTURBATIONS

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ABSTRACT. In this paper it is shown that, when there is lack of coercivity with respect to some partial derivatives on the underlying field u, then the relaxation of the functional

$$u \mapsto \int_\Omega f(u, Du)\, dx$$

may fail to be local. This result is applied to a singular perturbation model for a membrane energy depending on deformations and out-of-plane bending.

1. INTRODUCTION

It is well-known (see, e.g., [10] and [15]) that, if a continuous integrand $f : \mathbb{R}\times\mathbb{R}^n \to [0,+\infty)$ satisfies appropriate coercivity and growth conditions, the relaxation of the functional

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(u, Du)\, dx & \text{if } u \in W^{1,q}(\Omega), \\ \infty & \text{otherwise} \end{cases} \tag{1.1}$$

in the weak topology of $L^q(\Omega)$ leads to the functional

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega f^{**}(u, Du)\, dx & \text{if } u \in W^{1,q}(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Here Ω is an open bounded set of $\mathbb{R}^n$, $1 < q < \infty$, and for every fixed $s \in \mathbb{R}$, the function $f^{**}(s, \cdot)$ is the convex envelope of $f(s, \cdot)$. The standard approach to prove results such as these consists in introducing a "localized" version of the functional F, precisely,

$$F(u, A) := \begin{cases} \displaystyle\int_A f(u, Du)\, dx & \text{if } u|_A \in W^{1,q}(A), \\ \infty & \text{otherwise,} \end{cases}$$

for $u \in L^q(\Omega)$ and $A \in \mathcal{A}(\Omega)$, where $\mathcal{A}(\Omega)$ is the family of open subsets of Ω. For every fixed $A \in \mathcal{A}(\Omega)$ we consider the relaxed functional $\mathcal{F}(\cdot, A)$ in the weak topology of $L^q(\Omega)$, we prove that for every $u \in L^q(\Omega)$ the set function $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure μ_u, and in this case $\mathcal{F}$ is said to be a *local functional.* The remaining of the argument is then dedicated to finding a characterization of this Radon measure via the Radon-Nikodym theorem. Specifically, within the context of (1.1) and (1.2), one aims at showing that μ_u is absolutely continuous with respect to $\mathcal{L}^n \lfloor \Omega$ and that

$$\frac{d\mu_u}{d\mathcal{L}^n} = f^{**}(u, Du)$$

$\mathcal{L}^n$ a.e. in Ω. Typical arguments involved in this program rely, in an essential way, on a coercivity hypothesis of the type

$$f(s, \xi) \geq \frac{1}{C}|\xi|^q - C \quad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N$$

for some $C > 0$.

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In this paper we show that weakening this coercivity condition may lead to a nonlocal relaxed functional. Precisely, we will show that, when there is lack of coercivity with respect to *some* partial derivatives of u, then, in general, the set function $\mathcal{F}(u,\cdot)$ is not the restriction to $\mathcal{A}(\Omega)$ of a Radon measure. See Theorem 2.1 below for the precise statement.

In the wider context of Γ-convergence, examples of this nonlocality phenomenon appear, in chronological order, in [11], [17], [21], [4], [9], [7], [8]. A common feature of all these examples is that the functionals considered are equi-coercive but the coefficients become singular near suitable one-dimensional sets. Recently Camar-Eddine and Seppecher in [13], [14] have determined a large class of nonlocal functionals that can be approximated by equi-coercive local functionals. To the best of our knowledge, the only example of nonlocality in a relaxation problem is described in [1]. This example is completely different from the one of the present paper, since the functional considered there allows a control of all derivatives and the pathology is given by the discontinuity of the function u for which $\mathcal{F}(u,\cdot)$ is not a measure. For another example we refer to [22].

Besides its intrinsic theoretical interest, the motivation for the relaxation problem considered in this paper is drawn from the study of the asymptotic behavior of an elastic thin film penalized by a van der Wals type interfacial energy. In [5], Bhattacharya and James added the interfacial energy $\kappa\int_{\omega\times\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)}|D^2u|^2\,dx$ to the potential energy $\int_{\omega\times\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)}W(Du)\,dx$ of a thin elastic domain. Here $\omega\subset\mathbb{R}^2$ is an open set, D^2u denotes the third order tensor of the second partial derivatives of the deformation vector $u:\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\to\mathbb{R}^3$, and $\kappa>0$. After rescaling the resulting energy onto a fixed domain of thickness 1 through a $\frac{1}{\varepsilon}$-dilation of the transverse variable, they obtained in the limit a 2-dimensional energy density that depends on the deformation u of the mid-surface and on the Cosserat vector b, which describes transverse shear and normal compression in the thickness.

The case in which the fixed parameter κ is replaced by ε^γ with $\gamma>0$ was addressed by Shu in [23]. The analysis in [23] is restricted to the mid-surface of the film, in the sense that the Cosserat vector is a priori minimized out of the computed energy.

A recent paper [18] (see also [19]) is devoted to the study of the Γ-limit, in an appropriate topology, of the rescaled functional

$$\int_{\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}\left[W\left(D_pu\,\middle|\,\frac{1}{\varepsilon}D_3u\right)+\varepsilon^\gamma\left(|D_p^2u|^2+\frac{1}{\varepsilon^2}|D_{p3}u|^2+\frac{1}{\varepsilon^4}|D_{33}u|^2\right)\right]dx\,, \tag{1.3}$$

keeping track of the cross-sectional behavior. Here D_pu is the gradient of u in the plane (x_1,x_2), defined as the 3×2 matrix whose columns are D_1u and D_2u. The symbol $\left(D_pu\,\middle|\,\frac{1}{\varepsilon}D_3u\right)$ denotes the 3×3 matrix whose first two columns are those of D_pu and whose third column is $\frac{1}{\varepsilon}D_3u$. The symbol D_p^2u is the third order tensor whose components are the second order partial derivatives in x_1 and x_2 of the components of u, and $D_{p3}u$ is the 3×2 matrix whose columns are $D_{13}u$ and $D_{23}u$.

In the case $\gamma<2$ it was proved in [18] that the Γ-limit is a local functional. On the other hand, in the cases $\gamma=2$ and $\gamma>2$ the authors were unable to obtain a local integral representation for the Γ-limit and they conjectured that the resulting energy should be nonlocal.

In this paper we show that indeed in the case $\gamma=2$ the Γ-limit is a nonlocal functional (see Theorem 4.1 below). The basic idea is that when $\gamma=2$ and $W=W\left(\frac{1}{\varepsilon}D_3u\right)$ the Γ-limit of the family of functionals (1.3) exhibits the same pathologies of the relaxed energy of the noncoercive functional

$$\int_{\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}\left[W(b)+|D_3b|^2\right]dx\,, \tag{1.4}$$

which is of the type studied in Section 2.

We remark that the nonlocality property proved in Theorems 2.1 and 4.1 depend heavily on the presence of some of the derivatives of the underlying fields and on the absence of others. Thus, this approach cannot be used to prove nonlocality in the case $\gamma > 2$. Indeed, in this case it was shown in [19] that the Γ-limit coincides with the one found in [6] in the case $\gamma = \infty$, that is, when no interfacial energy is added, and so no derivatives of the Cosserat vector b are involved. To our knowledge, the locality of the Γ-limit in the case $\gamma > 2$ remains open.

This paper is organized as follows. In Section 2 we prove that for noncoercive functionals of the type (1.4), the relaxed functional with respect to the weak topology of $L^2(\Omega)$ may be nonlocal.

In Section 3 we give an abstract representation result for the Γ-limit of the family of functionals (1.3) when $\gamma = 2$ and $\gamma > 2$. The approach is in the spirit of [12] (see also [10]). In Section 4 we prove that in general for $\gamma = 2$ the Γ-limit of the family of functionals (1.3) is nonlocal.

Throughout the paper, constants may change from expression to expression.

2. The Relaxation Problem

In this section we study a relaxation problem in the weak topology of $L^2(\Omega)$ for an integral functional of the type (1.4) depending on a scalar field and on only one of its partial derivatives. This functional is nonconvex due to the presence of a two-well potential acting on the field. We will show that the lack of control with respect to some partial derivatives of the underlying field leads to a nonlocal relaxed functional.

Let $g : \mathbb{R} \to [0, +\infty)$ be a function such that

$$g \text{ is continuous in } \mathbb{R}, \tag{2.1}$$

$$g(s) \geq a_0 s^2 - b_0 \quad \text{for all } s \in \mathbb{R}, \tag{2.2}$$

$$g(1) = g(-1) = 0 < g(s) \quad \text{for } s \neq \pm 1, \tag{2.3}$$

for some constants $a_0 > 0$ and $b_0 \geq 0$. Let Ω be an open set of $\mathbb{R}^n$, with $n \geq 2$, of the form $\Omega = \omega \times I$, where ω is a nonempty bounded open subset of $\mathbb{R}^{n-1}$ and $I := \left(-\frac{1}{2}, \frac{1}{2}\right)$. A generic point $x \in \Omega$ will be written as $x := (y, x_n)$, where $y = (x_1, \ldots, x_{n-1}) \in \omega$ and $x_n \in I$. The set of all open subsets of Ω is denoted by $\mathcal{A}(\Omega)$. For every $v \in L^2(\Omega)$ and every $A \in \mathcal{A}(\Omega)$ we define

$$G(v, A) := \begin{cases} \displaystyle\int_A g(v)\, dx + \int_A |D_n v|^2\, dx & \text{if } D_n v \in L^2(A), \\ \infty & \text{otherwise,} \end{cases} \tag{2.4}$$

where D_n is the partial derivative with respect to x_n.

For every $A \in \mathcal{A}(\Omega)$ let $\overline{G}(\cdot, A)$ be the lower semicontinuous envelope of $G(\cdot, A)$ in the weak topology of $L^2(\Omega)$, i.e., $\overline{G}(\cdot, A)$ is the largest weakly lower semicontinuous functional on $L^2(\Omega)$ below $G(\cdot, A)$.

The main result of this section is the following theorem.

Theorem 2.1. *Assume that g satisfies hypotheses (2.1), (2.2), and (2.3). Let $\Omega := \omega \times I$ be as above and let $v_0(x) := x_n$, $x \in \Omega$. Then $\overline{G}(v_0, \cdot)$ is not a measure on $\mathcal{A}(\Omega)$, in the sense that there is no measure μ defined on the σ-algebra of the Borel subsets of Ω such that $\overline{G}(v_0, A) = \mu(A)$ for every $A \in \mathcal{A}(\Omega)$.*

The proof of this theorem uses the following result.

Lemma 2.2. *Let $w_k \in H^1(I)$ and $z \in L^2(I)$ be such that*

$$w_k' \to z \quad \text{strongly in } L^2(I), \tag{2.5}$$

$$\lim_{k \to \infty} \mathcal{L}^1(\{t \in I : |w_k(t)| > M\}) = 0, \tag{2.6}$$

for some constant $M > 0$. Then there exist a subsequence $\{w_{k_j}\}$ and a function $w \in H^1(I)$, with $w' = z$ $\mathcal{L}^1$ a.e. in I, such that $w_{k_j} \to w$ strongly in $H^1(I)$.

Proof. It is enough to show that the sequence $\{w_k\}$ is bounded in $L^\infty(I)$. Since $w_k \in H^1(I)$, without loss of generality we may assume that each w_k is absolutely continuous. Let $k_1 \in \mathbb{N}$ be so large that $\mathcal{L}^1(\{t \in I : |w_k(t)| > M\}) < \frac{1}{2}$ for all $k \geq k_1$. Then for all $k \geq k_1$ there exists $t_k \in I$ such that $|w_k(t_k)| \leq M$. By the fundamental theorem of calculus

$$w_k(t) = w_k(t_k) + \int_{t_k}^{t} w_k'(s)\, ds\,,$$

hence $|w_k(t)| \leq M + \|w_k'\|_{L^1(I)}$ for all $t \in I$. The conclusion follows. □

We now turn to the proof of Theorem 2.1.

Proof. We argue by contradiction and assume that there is a measure μ, defined on all Borel subsets of Ω, such that

$$\overline{G}(v_0, A) = \mu(A) \qquad \text{for every } A \in \mathcal{A}(\Omega)\,. \tag{2.7}$$

We claim that in this case

$$\overline{G}(v_0, \Omega) \leq \int_\Omega |D_n v_0|^2\, dx = \mathcal{L}^n(\Omega)\,. \tag{2.8}$$

By (2.1) and (2.3), given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$g(s) < \varepsilon \quad \text{for } |s \pm 1| < \delta\,. \tag{2.9}$$

We cover Ω with a finite number of open sets A_i contained in Ω and such that

$$\operatorname{diam} A_i < \delta \quad \text{and} \quad \textstyle\sum_i \mathcal{L}^n(A_i) < (1+\varepsilon)\,\mathcal{L}^n(\Omega)\,. \tag{2.10}$$

By (2.7),

$$\overline{G}(v_0, \Omega) \leq \textstyle\sum_i \overline{G}(v_0, A_i)\,. \tag{2.11}$$

For each i fix a point $x^{(i)} \in A_i$. Since $x_n^{(i)} \in I$, there exists $\lambda_i \in \left(\frac{1}{4}, \frac{3}{4}\right)$ such that

$$x_n^{(i)} = \lambda_i(-1) + (1-\lambda_i)\,1\,,$$

and so for every $x_n \in I$ we can write

$$v_0(x) = x_n = \lambda_i\left(-1 + x_n - x_n^{(i)}\right) + (1-\lambda_i)\left(1 + x_n - x_n^{(i)}\right)\,. \tag{2.12}$$

Let $\chi^{(i)}$ be the 1-periodic function in $\mathbb{R}$ such that $\chi^{(i)}(t) = 1$ for $t \in [0, \lambda_i]$ and $\chi^{(i)}(t) = 0$ for $t \in (\lambda_i, 1)$, and let $\chi_k^{(i)}(t) := \chi^{(i)}(kt)$. By the Riemann-Lebesgue lemma, $\chi_k^{(i)} \rightharpoonup \lambda_i$ weakly* in $L^\infty(\mathbb{R})$. For $x \in \Omega$ define

$$\begin{aligned} v_k^{(i)}(x) &:= \chi_k^{(i)}(x_1)\left(-1 + x_n - x_n^{(i)}\right) + \left(1 - \chi_k^{(i)}(x_1)\right)\left(1 + x_n - x_n^{(i)}\right) \\ &= \chi_k^{(i)}(x_1)(-1) + \left(1 - \chi_k^{(i)}(x_1)\right)1 + \left(x_n - x_n^{(i)}\right)\,. \end{aligned} \tag{2.13}$$

By (2.10) we have $\left|x_n - x_n^{(i)}\right| < \delta$ for $x \in A_i$, and so, by (2.9) and (2.13),

$$g\left(v_k^{(i)}(x)\right) < \varepsilon \quad \text{for every } x \in A_i\,. \tag{2.14}$$

Moreover, $D_n v_k^{(i)}(x) = 1$ for every $x \in A_i$. Together with (2.4) and (2.14), this implies that

$$G\left(v_k^{(i)}, A_i\right) \leq (1+\varepsilon)\,\mathcal{L}^n(A_i)\,.$$

In view of (2.12) and (2.13), it follows that $v_k^{(i)} \rightharpoonup v_0$ weakly* in $L^\infty(\Omega)$. In particular, $v_k^{(i)} \rightharpoonup v_0$ weakly in $L^2(\Omega)$, and so

$$\overline{G}(v_0, A_i) \leq \liminf_{k\to\infty} G\left(v_k^{(i)}, A_i\right) \leq (1+\varepsilon)\,\mathcal{L}^n(A_i)\,,$$

and using (2.10) and (2.11) we deduce that

$$\overline{G}(v_0, \Omega) \leq (1+\varepsilon) \sum_i \mathcal{L}^n(A_i) \leq (1+\varepsilon)^2 \mathcal{L}^n(\Omega) .$$

Since $\varepsilon > 0$ is arbitrary, this proves (2.8).

On the other hand, since the functional

$$v \mapsto \int_\Omega |D_n v|^2 \, dx \quad \text{is weakly lower semicontinuous in } L^2(\Omega) \tag{2.15}$$

and below $G(\cdot, \Omega)$, we conclude that

$$\overline{G}(v_0, \Omega) = \mathcal{L}^n(\Omega) .$$

By the lower bound in (2.2), and by a general property of the relaxation of coercive functionals (see Propositions 8.1 and 8.10 in [15] or Proposition 3.16 in [18]), there exists a sequence $\{v_k\}$ converging to v_0 weakly in $L^2(\Omega)$ such that

$$\lim_{k\to\infty} \left(\int_\Omega g(v_k) \, dx + \int_\Omega |D_n v_k|^2 \, dx \right) = \lim_{k\to\infty} G(v_k, \Omega) = \overline{G}(v_0, \Omega) = \mathcal{L}^n(\Omega) .$$

Using (2.3) and (2.15), we obtain

$$\lim_{k\to\infty} \int_\Omega g(v_k) \, dx = 0 , \tag{2.16}$$

$$\lim_{k\to\infty} \int_\Omega |D_n v_k|^2 \, dx = \int_\Omega |D_n v_0|^2 \, dx . \tag{2.17}$$

Since $D_n v_k \rightharpoonup D_n v_0$ weakly in $L^2(\Omega)$, the last equality yields that $D_n v_k \to D_n v_0$ strongly in $L^2(\Omega)$. By Fubini's theorem there exists a subsequence, still denoted $\{v_k\}$, such that for $\mathcal{L}^{n-1}$ a.e. $y \in \omega$ we have

$$g(v_k(y, \cdot)) \to 0 \quad \text{strongly in } L^1(I) , \tag{2.18}$$

$$D_n v_k(y, \cdot) \to D_n v_0(y, \cdot) = 1 \quad \text{strongly in } L^2(I) . \tag{2.19}$$

By (2.1)-(2.3) and (2.18), for $\mathcal{L}^{n-1}$ a.e. $y \in \omega$ we have

$$\lim_{k\to\infty} \mathcal{L}^1(\{x_n \in I : |v_k(y, x_n)| > 2\}) = 0 . \tag{2.20}$$

Fix $y \in \omega$ satisfying (2.19) and (2.20). Using Lemma 2.2, we deduce that there exists $w \in H^1(I)$, with $w'(x_n) = 1$ for $\mathcal{L}^1$ a.e. $x_n \in I$, such that, up to a subsequence, $v_k(y, \cdot) \to w$ strongly in $H^1(I)$. From (2.18) it follows that $g(w(x_n)) = 0$ for every $x_n \in I$, and so, by (2.3), $w(x_n) = \pm 1$ for every $x_n \in I$. This contradicts the fact that $w'(x_n) = 1$ for $\mathcal{L}^1$ a.e. $x_n \in I$. □

3. Dimension Reduction

Let $\mathbb{R}^{3\times 3}$ be the space of all 3×3 matrices with real entries and let $W : \mathbb{R}^{3\times 3} \to [0, \infty)$ be a function such that

$$W \text{ is continuous on } \mathbb{R}^{3\times 3} , \tag{3.1}$$

$$\frac{1}{C} |\xi|^q - C \leq W(\xi) \leq C(|\xi|^q + 1) \quad \text{for all } \xi \in \mathbb{R}^{3\times 3} , \tag{3.2}$$

for some constants $q > 1$ and $C > 0$.

Let $\Omega = \omega \times I$ be as in the previous section with $n = 3$, with $\partial\omega$ Lipschitz, let $\mathcal{A}(\omega)$ be the set of all open subsets of ω, and let $\gamma \geq 2$. For $A \in \mathcal{A}(\omega)$, $u \in W^{1,q}(A \times I; \mathbb{R}^3)$ with

$D^2u \in L^2\left(A \times I; \mathbb{R}^{3\times3\times3}\right)$, and $\varepsilon > 0$ consider the functional

$$\mathcal{W}_\varepsilon(u, A) := \int_{A\times I} W\left(D_p u \left| \frac{1}{\varepsilon} D_3 u\right.\right) dx + \varepsilon^\gamma \int_{A\times I} \left(\left|D_p^2 u\right|^2 + \frac{1}{\varepsilon^2}\left|D_{p3}u\right|^2 + \frac{1}{\varepsilon^4}\left|D_{33}u\right|^2\right) dx\,. \tag{3.3}$$

To study the asymptotic behavior of $\mathcal{W}_\varepsilon$ as $\varepsilon \to 0$, it is convenient to introduce the functional

$$\mathcal{F}_\varepsilon : W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right) \times \mathcal{A}(\omega) \to [0, \infty]$$

defined by

$$\mathcal{F}_\varepsilon(u, b, A) := \begin{cases} \mathcal{W}_\varepsilon(u, A) & \text{if } b = \frac{1}{\varepsilon} D_3 u \text{ in } A\times I \text{ and } D^2u \in L^2\left(A\times I; \mathbb{R}^{3\times3\times3}\right), \\ \infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $A \in \mathcal{A}(\omega)$, and let $\varepsilon_k \to 0$. The goal of this section is to establish the existence of the Γ-limit $\mathcal{F}(\cdot, \cdot, A)$ of the sequence $\{\mathcal{F}_{\varepsilon_k}(\cdot, \cdot, A)\}$ in the weak topology of the product $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$, and to prove that it is independent of $\{\varepsilon_k\}$.

As usual, let $\mathcal{F}'(\cdot, \cdot, A)$ and $\mathcal{F}''(\cdot, \cdot, A)$ denote the Γ-limit inferior and superior of $\{\mathcal{F}_{\varepsilon_k}(\cdot, \cdot, A)\}$, respectively (see Definition 4.1 in [15]). By (3.2) and Propositions 8.1 and 8.10 in [15], for every $(u, b) \in W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$ and $A \in \mathcal{A}(\omega)$ we have

$$\mathcal{F}'(u, b, A) = \inf_{\{(u_k, b_k)\}} \left\{ \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, b_k, A) : (u_k, b_k) \rightharpoonup (u, b) \right\},$$

$$\mathcal{F}''(u, b, A) = \inf_{\{(u_k, b_k)\}} \left\{ \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, b_k, A) : (u_k, b_k) \rightharpoonup (u, b) \right\},$$

where $\rightharpoonup$ denotes weak convergence in $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$.

Remark 3.1. It turns out that if $\mathcal{F}'(u, b, A) < \infty$, then $D_3 u = 0$ in $A \times I$, and so we may identify unequivocally $u|_{A\times I}$ with a function in $W^{1,q}\left(A; \mathbb{R}^3\right)$. Indeed, suppose that $\{(u_k, b_k)\}$ converge weakly to (u, b) in $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$ and that

$$\liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, b_k, A) < \infty\,. \tag{3.5}$$

Then

$$\int_{A\times I} |D_3 u|^q \, dx \le \liminf_{k\to\infty} \int_{A\times I} |D_3 u_k|^q \, dx = 0\,,$$

where the last equality results from (3.5) and the coercivity condition in (3.2). Moreover, if $\gamma = 2$, then from the inequality

$$\mathcal{F}_\varepsilon(u, b, A) \ge \begin{cases} \displaystyle\int_{A\times I} |D_3 b|^2 \, dx & \text{if } D^2u \in L^2\left(A \times I; \mathbb{R}^{3\times3\times3}\right) \text{ and } b = \dfrac{1}{\varepsilon} D_3 u \text{ in } A \times I\,, \\ \infty & \text{otherwise} \end{cases}$$

and the fact that $b \mapsto \int_{A\times I} |D_3 b|^2 \, dx$ is weakly lower semicontinuous in $L^q\left(\Omega; \mathbb{R}^3\right)$, we deduce that if $\mathcal{F}'(u, b, A) < \infty$, then $D_3 b \in L^2\left(A \times I; \mathbb{R}^3\right)$ and

$$\mathcal{F}'(u, b, A) \ge \int_{A\times I} |D_3 b|^2 \, dx\,. \tag{3.6}$$

Remark 3.2. Let $\gamma = 2$ and let $\{(u_k, b_k)\} \subset W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega; \mathbb{R}^3)$ converge weakly to (u, b) in $W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega; \mathbb{R}^3)$ and such that

$$\sup_k \mathcal{F}_{\varepsilon_k}(u_k, b_k, A) < \infty\,.$$

Then

$$\sup_k \int_{A\times I} |\varepsilon_k D^2 u_k|^2 \, dx < \infty, \quad \sup_k \int_{A\times I} |D(D_3 u_k)|^2 \, dx < \infty\,.$$

Using Poincaré's inequality and the Rellich–Kondrachov theorem, and reasoning as in Step 1 in Theorem D in [19], it can be shown that

$$\begin{aligned} 0 &= \lim_{k\to\infty}\int_{A\times I}|\varepsilon_k u_k|^2\,dx = \lim_{k\to\infty}\int_{A\times I}|\varepsilon_k Du_k|^2\,dx \\ &= \lim_{k\to\infty}\int_{A\times I}|D_3 u_k|^2\,dx\,. \end{aligned}$$

In what follows, $X\left(\Omega;\mathbb{R}^3\right)$ is the space of functions $b\in L^q\left(\Omega;\mathbb{R}^3\right)$ with $D_3 b\in L^2\left(\Omega;\mathbb{R}^3\right)$, and we consider the isomorphism $b\mapsto\overline{b}$ between $L^q\left(\Omega;\mathbb{R}^3\right)$ and $L^q\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$ given by

$$\overline{b}\,(x_1,x_2)\,(x_3):=b\,(x_1,x_2,x_3)\quad\text{or, equivalently,}\quad\overline{b}\,(y)\,(x_3):=b\,(y,x_3)\;. \tag{3.7}$$

Theorem 3.3. *Assume that W satisfies (3.1) and (3.2) and let $\varepsilon_k\to 0$. Then there exist a subsequence, not relabelled, and a functional*

$$\mathcal{F}:W^{1,q}\left(\Omega;\mathbb{R}^3\right)\times L^q\left(\Omega;\mathbb{R}^3\right)\times\mathcal{A}\left(\omega\right)\to[0,\infty]$$

such that, for every $A\in\mathcal{A}(\omega)$ satisfying the segment property, $\mathcal{F}_{\varepsilon_k}(\cdot,\cdot,A)$ Γ-converge to $\mathcal{F}(\cdot,\cdot,A)$ in the weak topology of the product $W^{1,q}\left(\Omega;\mathbb{R}^3\right)\times L^q\left(\Omega;\mathbb{R}^3\right)$.

If $\gamma>2$, then there exists a lower semicontinuous functional

$$F:\mathbb{R}^{3\times 2}\times L^q\left(I;\mathbb{R}^3\right)\to[0,\infty]\;,$$

with

$$\frac{1}{C}\left(|\xi|^q+\|g\|^q_{L^q(I;\mathbb{R}^3)}\right)-C\le F\left(\xi,g\right)\le C\left(1+|\xi|^q+\|g\|^q_{L^q(I;\mathbb{R}^3)}\right) \tag{3.8}$$

for all $\xi\in\mathbb{R}^{3\times 2}$ and $g\in L^q\left(I;\mathbb{R}^3\right)$, such that

$$\mathcal{F}\left(u,b,A\right)=\int_A F\left(D_p u\left(y\right),\overline{b}\left(y\right)\right)\,dy \tag{3.9}$$

for all $u\in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, $b\in L^q\left(\Omega;\mathbb{R}^3\right)$, and $A\in\mathcal{A}(\omega)$.

If $\gamma=2$, then there exists a lower semicontinuous functional

$$F:\mathbb{R}^{3\times 2}\times\left(L^q\left(I;\mathbb{R}^3\right)\cap W^{1,2}\left(I;\mathbb{R}^3\right)\right)\to[0,\infty]\;,$$

with

$$\begin{aligned} \frac{1}{C}\left(|\xi|^q+\|g\|^q_{L^q(I;\mathbb{R}^3)}\right)+\|g'\|^2_{L^2(I;\mathbb{R}^3)}-C &\le F\left(\xi,g\right) \\ &\le C\left(1+|\xi|^q+\|g\|^q_{L^q(I;\mathbb{R}^3)}\right)+\|g'\|^2_{L^2(I;\mathbb{R}^3)} \end{aligned} \tag{3.10}$$

for all $\xi\in\mathbb{R}^{3\times 2}$ and $g\in L^q\left(I;\mathbb{R}^3\right)\cap W^{1,2}\left(I;\mathbb{R}^3\right)$, such that

$$\mathcal{F}\left(u,b,A\right)=\int_A F\left(D_p u\left(y\right),\overline{b}\left(y\right)\right)\,dy \tag{3.11}$$

for all $u\in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, $b\in X\left(\Omega;\mathbb{R}^3\right)$, and $A\in\mathcal{A}(\omega)$.

Proof. **Step 1:** We prove the first part of the statement. Let $\mathcal{R}(\omega)$ be the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open rectangles in ω with sides parallel to the axes and with vertices with rational coordinates.

By (3.2) there exists $C>0$ such that

$$\mathcal{F}_\varepsilon\left(u,b,A\right)\ge\frac{1}{C}\int_{A\times I}\left(|D_p u|^q+|b|^q\right)\,dx-C\mathcal{L}^2\left(A\right)$$

for every $(u,b)\in W^{1,q}\left(\Omega;\mathbb{R}^3\right)\times L^q\left(\Omega;\mathbb{R}^3\right)$ and $A\in\mathcal{A}(\omega)$. Therefore, by Corollary 8.12 in [15], together with a diagonal argument, there exists a subsequence still denoted $\{\varepsilon_k\}$ such that

$$\mathcal{F}'\left(u,b,A\right)=\mathcal{F}''\left(u,b,A\right)\;\text{for every}\;\left(u,b\right)\in W^{1,q}\left(\Omega;\mathbb{R}^3\right)\times L^q\left(\Omega;\mathbb{R}^3\right)\;,\;A\in\mathcal{R}(\omega)\,. \tag{3.12}$$

We consider the measure theoretic regularized functional

$$\mathcal{F} : W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right) \times \mathcal{A}(\omega) \to [0,\infty]$$

defined by

$$\mathcal{F}(u,b,A) := \sup\left\{\mathcal{F}'(u,b,A') : \, A' \in \mathcal{R}(\omega)\,,\, A' \subset\subset A\right\}. \tag{3.13}$$

In the remainder of this step we prove that for every $A \in \mathcal{A}(\omega)$ satisfying the segment property, the sequence $\{\mathcal{F}_{\varepsilon_k}(\cdot,\cdot,A)\}$ Γ-converges to $\mathcal{F}(\cdot,\cdot,A)$ in the weak topology of the product $W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$.

Substep 1a: Let $A \in \mathcal{A}(\omega)$ satisfying the segment property and let $(u,b) \in W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$ be such that $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $A \times I$ and, if $\gamma = 2$, $D_3 b \in L^2\left(A \times I;\mathbb{R}^3\right)$. We claim that

$$\mathcal{F}''(u,b,A) \le C\left(\int_{A\times I} |D_p u|^q \, dy + \int_{A\times I} |b|^q \, dx\right) + C\mathcal{L}^2(A) \tag{3.14}$$

if $\gamma > 2$, while

$$\mathcal{F}''(u,b,A) \le C\left(\int_{A\times I} |D_p u|^q \, dy + \int_{A\times I} |b|^q \, dx\right) + C\mathcal{L}^2(A) + \int_{A\times I} |D_3 b|^2 \, dx \tag{3.15}$$

if $\gamma = 2$.

We prove this in the case $\gamma = 2$. The case $\gamma > 2$ can be treated in a similar way. Fix u, b, and A as above. By Theorem 5.5 in the Appendix, there exist an extension $u \in W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ and a sequence $\{u_\eta\} \subset W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ such that $D_3 u_\eta = 0$ $\mathcal{L}^3$ a.e. in $A^\eta \times \mathbb{R}$ and $u_\eta \to u$ strongly in $W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ as $\eta \to 0^+$, where

$$A^\eta := \{y \in \omega : \, \mathrm{dist}(y,A) < \eta\}.$$

Moreover, by Theorem 5.6 in the Appendix, there exist an extension $b \in L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$, with $D_3 b \in L^2\left(A \times \mathbb{R};\mathbb{R}^3\right)$, and a sequence $\{b_\eta\} \subset L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$, with $D_3 b_\eta \in L^2\left(A^\eta \times \mathbb{R};\mathbb{R}^3\right)$, such that $b_\eta \to b$ strongly in $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ and $D_3 b_\eta \to D_3 b$ strongly in $L^2\left(A \times I;\mathbb{R}^3\right)$ as $\eta \to 0^+$. Define, for $x \in \mathbb{R}^3$,

$$\begin{aligned} u_{k,\delta,\eta}(x) &:= (u_\eta * \varphi_\delta)(x) + \varepsilon_k \int_0^{x_3} (b_\eta * \varphi_\delta)(y,s)\, ds\,, \\ b_{\delta,\eta}(x) &:= (b_\eta * \varphi_\delta)(x)\,, \end{aligned}$$

where $\varphi \in C_c^\infty\left(\mathbb{R}^3\right)$ is a standard mollifier and $\delta > 0$. Note that if $\delta < \eta$, then, in $A \times \mathbb{R}$,

$$\frac{1}{\varepsilon_k} D_3 u_{k,\delta,\eta} = \frac{1}{\varepsilon_k} D_3 (u_\eta * \varphi_\delta) + b_\eta * \varphi_\delta = b_\eta * \varphi_\delta = b_{\delta,\eta}\,,$$

where we used the facts that $\mathrm{supp}\,\varphi_\delta \subset \overline{B(0,\delta)}$, $D_3 u_\eta = 0$ $\mathcal{L}^3$ a.e. in $A^\eta \times \mathbb{R}$, and thus $D_3(u_\eta * \varphi_\delta) = D_3 u_\eta * \varphi_\delta = 0$ $\mathcal{L}^3$ a.e. in $A \times \mathbb{R}$. Similarly, since $D_3 b \in L^2\left(A^\eta \times \mathbb{R};\mathbb{R}^3\right)$, we have that $D_{33} u_{k,\delta,\eta} = \varepsilon_n D_3 b_\eta * \varphi_\delta$ $\mathcal{L}^3$ a.e. in $A \times \mathbb{R}$.

Reasoning as in the proof of (3.2) in [18], we conclude that

$$\begin{aligned} \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_{k,\delta,\eta}, b_{\delta,\eta}, A) \le C &\left(\int_{A\times I} \left(|D_p(u_\eta * \varphi_\delta)|^q + |b_\eta * \varphi_\delta|^q\right) dx\right) \\ &+ C\mathcal{L}^2(A) + \int_{A\times I} |D_3 b_\eta * \varphi_\delta|^2 \, dx\,. \end{aligned}$$

Using standard properties of mollifiers, together with the facts that $u_\eta \to u$ strongly in $W^{1,q}\left(\Omega;\mathbb{R}^3\right)$, $b_\eta \to b$ strongly in $L^q\left(\Omega;\mathbb{R}^3\right)$, and $D_3 b_\eta \to D_3 b$ strongly in $L^2\left(A \times I;\mathbb{R}^3\right)$

as $\eta \to 0^+$, we obtain that

$$\begin{aligned}&\limsup_{\eta\to 0^+}\limsup_{\delta\to 0}\limsup_{k\to\infty}\mathcal{F}_{\varepsilon_k}\left(u_{k,\delta,\eta},b_{\delta,\eta},A\right)\\ &\leq C\left(\int_A |D_p u|^q\,dy+\int_{A\times I}|b|^q\,dx\right)+C\mathcal{L}^2\left(A\right)+\int_{A\times I}|D_3 b|^2\,dx\,.\end{aligned}$$

Moreover, since

$$\begin{aligned}&\lim_{\eta\to 0^+}\lim_{\delta\to 0}\lim_{k\to\infty}\|u_{k,\delta,\eta}-u\|_{W^{1,q}(\Omega;\mathbb{R}^3)}=0\,,\\ &\lim_{\eta\to 0^+}\lim_{\delta\to 0}\|b_{\delta,\eta}-b\|_{L^q(\Omega;\mathbb{R}^3)}=0\,,\end{aligned}$$

a diagonalization argument yields two subsequences $\{u_k := u_{k,\delta_k,\eta_k}\}$ and $\{b_k := b_{\delta_k,\eta_k}\}$ of $\{u_{k,\delta,\eta}\}$ and $\{b_{\delta,\eta}\}$, respectively, such that $(u_k, b_k) \to (u, b)$ strongly in $W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$ and

$$\begin{aligned}\mathcal{F}''\left(u,b,A\right)&\leq \limsup_{k\to\infty}\mathcal{F}_{\varepsilon_k}\left(u_k,b_k,A\right)\\ &\leq C\left(\int_A |D_p u|^q\,dy+\int_{A\times I}|b|^q\,dx\right)+C\mathcal{L}^2\left(A\right)+\int_{A\times I}|D_3 b|^2\,dx\,.\end{aligned}$$

We remark that the condition that A satisfies the segment property was only used to extend u and b outside A keeping the properties that u does not depend on x_3 and that $D_3 b$ belongs to L^2 in a domain slightly larger than $A \times I$ (see Theorems 5.5 and 5.6 in the Appendix). Thus, (3.15) continues to hold without assuming that A satisfies the segment property if it is known apriori that $(u, b) \in W^{1,q}\left(\omega;\mathbb{R}^3\right) \times X\left(\Omega;\mathbb{R}^3\right)$. Similarly, (3.14) continues to hold if $(u, b) \in W^{1,q}\left(\omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$, without additional hypotheses on A.

Substep 1b: We claim that

$$\mathcal{F}''(u,b,A_1)\leq \mathcal{F}''(u,b,A_2)+\mathcal{F}''(u,b,A_1\setminus\overline{A_3}) \tag{3.16}$$

for every $(u, b) \in W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$ and A_1, A_2, $A_3 \in \mathcal{A}(\omega)$, with $A_3 \subset\subset A_2 \subset\subset A_1$. We prove this in the case $\gamma = 2$. The case $\gamma > 2$ can be treated in a similar way. Without loss of generality, we may assume that the right-hand side of the previous inequality is finite, since otherwise there is nothing to prove. Fix $\eta > 0$ and find $\{u_k\}$, $\{v_k\} \subset W^{1,q}(\Omega;\mathbb{R}^3)$ converging weakly to u in $W^{1,q}(\Omega;\mathbb{R}^3)$ and $\{b_k\}$, $\{z_k\} \subset L^q(\Omega;\mathbb{R}^3)$ converging weakly to b in $L^q(\Omega;\mathbb{R}^3)$ such that

$$\limsup_{k\to\infty}\mathcal{F}_{\varepsilon_k}\left(u_k,b_k,A_1\setminus\overline{A_3}\right)\leq \mathcal{F}''\left(u,b,A_1\setminus\overline{A_3}\right)+\eta<\infty\,, \tag{3.17}$$

$$\limsup_{k\to\infty}\mathcal{F}_{\varepsilon_k}\left(v_k,z_k,A_2\right)\leq \mathcal{F}''\left(u,b,A_2\right)+\eta<\infty\,. \tag{3.18}$$

Then by (3.3) and (3.4) for all k sufficiently large, say $k \geq k_1$, we have that $D^2 u_k \in L^2\left(\left(A_1 \setminus \overline{A_3}\right) \times I;\mathbb{R}^{3\times3\times3}\right)$, $b_k = \frac{1}{\varepsilon_k} D_3 u_k$ in $A_1 \setminus \overline{A_3}$, $D^2 v_k \in L^2\left(A_2 \times I;\mathbb{R}^{3\times3\times3}\right)$, and $z_k = \frac{1}{\varepsilon_k} D_3 v_k$ in A_2. For every $A \in \mathcal{A}(\omega)$, $v \in W^{2,2}(A \times I;\mathbb{R}^3)$, for every Borel set $E \subset A$, and for every $k \geq k_1$ define

$$\begin{aligned}\mathcal{H}_k(v,E):=&\int_{E\times I}\left(1+|D_p v|^q+\frac{1}{\varepsilon_k^q}|D_3 v|^q\right)dx\\ &+\int_{E\times I}\left(\varepsilon_k^2|D_p^2 v|^2+|D_{p3}v|^2+\frac{1}{\varepsilon_k^2}|D_{33}v|^2\right)dx\,.\end{aligned}$$

By (3.2),

$$M=M\left(\eta\right):=\sup_{k\geq k_1}\left(\mathcal{H}_k(u_k,A_2\setminus\overline{A_3})+\mathcal{H}_k(v_k,A_2\setminus\overline{A_3})\right)<\infty\,.$$

Let $\delta > 0$ be so small that

$$A_{3,\delta}:=\{y\in A_2:\ \operatorname{dist}(y,A_3)<\delta\}\subset\subset A_2\,.$$

Fix $m \in \mathbb{N}$ and write

$$A_{3,\delta} \setminus A_3 = \bigcup_{i=1}^{m} L_{i,m} \,,$$

where

$$L_{i,m} := \left\{ y \in A_{3,\delta} : \ \frac{(i-1)\,\delta}{m} < \operatorname{dist}(y, A_3) \leq \frac{i\delta}{m} \right\} .$$

Since for every $k \geq k_1$,

$$\sum_{i=1}^{m} \left(\mathcal{H}_k(u_k, L_{i,m}) + \mathcal{H}_k(v_k, L_{i,m}) \right) \leq M \,,$$

there exists $i_{m,k} \in \{1, \cdots, m\}$ such that

$$\mathcal{H}_k(u_k, L_{i_{m,k},m}) + \mathcal{H}_k(v_k, L_{i_{m,k},m}) \leq \frac{M}{m} .$$

Consider a smooth cut-off function $\varphi_{m,k} \in C_0^\infty(A_2; [0,1])$ such that $\{0 < \varphi_{m,k} < 1\} \subset L_{i_{m,k},m}$, $\varphi_{m,k}(y) = 1$ if $\operatorname{dist}(y, A_3) \leq \frac{i_{m,k}-1}{m}\delta$, $\varphi_{m,k}(y) = 0$ if $\operatorname{dist}(y, A_3) \geq \frac{i_{m,k}}{m}\delta$, and

$$\|D_p \varphi_{m,k}\|_{L^\infty(\omega;\mathbb{R}^2)} \leq Cm \,, \quad \|D_p^2 \varphi_{m,k}\|_{L^\infty(\omega;\mathbb{R}^{2\times 2})} \leq Cm^2 .$$

For $x \in \Omega$ define

$$\tilde{u}_{m,k}(x) := (1 - \varphi_{m,k}(y)) u_k(x) + \varphi_{m,k}(y) v_k(x)$$

and

$$\tilde{b}_{m,k}(x) := \begin{cases} \frac{1}{\varepsilon_k} D_3 \tilde{u}_{m,k}(x) & \text{if } x \in A_1 \,, \\ b_k(x) & \text{otherwise.} \end{cases}$$

Then $\tilde{u}_{m,k} \rightharpoonup u$ weakly in $W^{1,q}(\Omega;\mathbb{R}^3)$ as $k \to \infty$ and since $\varphi_{m,k}$ does not depend on x_3, we also have that $\tilde{b}_{m,k} = \frac{1}{\varepsilon_k} D_3 \tilde{u}_{m,k} \rightharpoonup b$ weakly in $L^q\left(A_1;\mathbb{R}^3\right)$, and, in turn, $\tilde{b}_{m,k} \rightharpoonup b$ weakly in $L^q\left(\Omega;\mathbb{R}^3\right)$ as $k \to \infty$. Hence,

$$\begin{aligned} \mathcal{F}_{\varepsilon_k}\left(\tilde{u}_{m,k}, \tilde{b}_{m,k}, A_1\right) &\leq \mathcal{F}_{\varepsilon_k}\left(u_k, b_k, A_1 \setminus \overline{A_3}\right) + \mathcal{F}_{\varepsilon_k}\left(v_k, z_k, A_2\right) \\ &+ C\left(\mathcal{H}_k(u_k, L_{i_{m,k},m}) + \mathcal{H}_k(v_k, L_{i_{m,k},m})\right) \\ &+ Cm^q \int_{L_{i_{m,k},m}\times I} |u_k - v_k|^q \, dx + C\varepsilon_k^2 m^4 \int_{L_{i_{m,k},m}\times I} |u_k - v_k|^2 \, dx \\ &+ C\varepsilon_k^2 m^2 \int_{L_{i_{m,k},m}\times I} |D_p u_k - D_p v_k|^2 \, dx + Cm^2 \int_{L_{i_{m,k},m}\times I} |D_3 u_k - D_3 v_k|^2 \, dx . \end{aligned}$$

By Remark 3.2, we have that the last four terms on the right-hand side of the previous inequality converge to zero as $k \to \infty$ for m fixed, and thus

$$\begin{aligned} \mathcal{F}''(u, b, A_1) &\leq \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}\left(\tilde{u}_{m,k}, \tilde{b}_{m,k}, A_1\right) \\ &\leq \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}\left(u_k, b_k, A_1 \setminus \overline{A_3}\right) + \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}\left(v_k, z_k, A_2\right) + \frac{M}{m} \\ &\leq \mathcal{F}''\left(u, b, A_1 \setminus \overline{A_3}\right) + \mathcal{F}''\left(u, b, A_2\right) + 2\eta + \frac{M}{m} . \end{aligned}$$

Letting $m \to \infty$ followed by $\eta \to 0$, we conclude that (3.16) holds.

Substep 1c: We claim that for every $A \in \mathcal{A}(\omega)$ satisfying the segment property, $\mathcal{F}(\cdot,\cdot,A)$ is the Γ-limit of $\{\mathcal{F}_{\varepsilon_k}(\cdot,\cdot,A)\}$. It suffices to show

$$\mathcal{F}(u, b, A) = \mathcal{F}'(u, b, A) = \mathcal{F}''(u, b, A) \tag{3.19}$$

for $u \in W^{1,q}\left(\Omega;\mathbb{R}^3\right)$ and $b \in L^q\left(\Omega;\mathbb{R}^3\right)$.

We remark that, as the proof below will show, the requirement that A satisfies the segment property is used only to apply Theorems 5.5 and 5.6 in the Appendix. Therefore, (3.19) holds for an arbitrary set $A \in \mathcal{A}(\omega)$, provided that $(u,b) \in W^{1,q}(\omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^3)$ (respectively, $(u,b) \in W^{1,q}(\omega;\mathbb{R}^3) \times X(\Omega;\mathbb{R}^3)$ if $\gamma = 2$).

We prove (3.19) in the case $\gamma = 2$. The case $\gamma > 2$ can be treated in a similar way. Let $(u,b) \in W^{1,q}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^3)$.

Since $\mathcal{F} \leq \mathcal{F}' \leq \mathcal{F}''$, it suffices to show that $\mathcal{F}''(u,b,A) \leq \mathcal{F}(u,b,A)$. Without loss of generality, we may assume that $\mathcal{F}(u,b,A) < \infty$, since otherwise there is nothing to prove. Then by Remark 3.1 we may identify $u|_{A\times I}$ with a function in $W^{1,q}(A;\mathbb{R}^3)$ and, using also (3.6) and (3.13), we have that $D_3 b \in L^2(A\times I;\mathbb{R}^3)$. For every Borel set $B \subset A$ define

$$\mathcal{H}(u,b,B) := C\left(\int_B |D_p u|^q \, dy + \int_{B\times I} |b|^q \, dx\right) + C\mathcal{L}^2(B) + \int_{B\times I} |D_3 b|^2 \, dx\,,$$

where $C > 0$ is such that (see Substep 1a)

$$\mathcal{F}''(u,b,A) \leq \mathcal{H}(u,b,A)\ .$$

Since $\mathcal{H}(u,b,\cdot)$ is a measure and $\mathcal{H}(u,b,A) < \infty$, given $\varepsilon > 0$, let $K \subset A$ be a compact set such that

$$\mathcal{H}(u,b,A\setminus K) \leq \varepsilon\,.$$

Choose $A_2 \in \mathcal{R}(\omega)$, $A_3 \in \mathcal{A}(\omega)$, with $K \subset A_3 \subset\subset A_2 \subset\subset A_1 := A$. In view of (3.12) and (3.16) we have

$$\begin{aligned}\mathcal{F}''(u,b,A) &\leq \mathcal{F}''(u,b,A_2) + \mathcal{F}''(u,b,A\setminus\overline{A_3}) \leq \mathcal{F}''(u,b,A_2) + \mathcal{F}''(u,b,A\setminus K)\\ &= \mathcal{F}'(u,b,A_2) + \mathcal{F}''(u,b,A\setminus K)\\ &\leq \mathcal{F}(u,b,A) + \mathcal{H}(u,b,A\setminus K) \leq \mathcal{F}(u,b,A) + \varepsilon\,,\end{aligned}$$

and the result follows by letting $\varepsilon \to 0$.

The remainder of the proof is dedicated to proving (3.9) for $\gamma > 2$ and (3.11) for $\gamma = 2$. We recall that we identify functions in $W^{1,q}(\omega;\mathbb{R}^3)$ with functions in $W^{1,q}(\Omega;\mathbb{R}^3)$ that do not depend on x_3.

Step 2: In this step we prove some properties of $\mathcal{F}$ that will be used in the sequel.

(i) By (3.2), (3.3), (3.4), and (3.13), when $\gamma > 2$ for every $(u,b) \in W^{1,q}(\omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^3)$ and $A \in \mathcal{A}(\omega)$,

$$\mathcal{F}(u,b,A) \geq \frac{1}{C}\int_{A\times I} (|D_p u|^q + |b|^q)\, dx - C\mathcal{L}^2(A)\ , \tag{3.20}$$

while by Substep 1a,

$$\mathcal{F}(u,b,A) \leq C\left(\int_A |D_p u|^q \, dy + \int_{A\times I} |b|^q \, dx\right) + C\mathcal{L}^2(A)\ . \tag{3.21}$$

Similarly, when $\gamma = 2$ for every $(u,b) \in W^{1,q}(\omega;\mathbb{R}^3) \times X(\Omega;\mathbb{R}^3)$ and $A \in \mathcal{A}(\omega)$,

$$\mathcal{F}(u,b,A) \geq \frac{1}{C}\int_{A\times I} (|D_p u|^q + |b|^q)\, dx - C\mathcal{L}^2(A) + \int_{A\times I} |D_3 b|^2 \, dx\,, \tag{3.22}$$

while

$$\mathcal{F}(u,b,A) \leq C\left(\int_A |D_p u|^q \, dy + \int_{A\times I} |b|^q \, dx\right) + C\mathcal{L}^2(A) + \int_{A\times I} |D_3 b|^2 \, dx\,. \tag{3.23}$$

(ii) Fix $u \in W^{1,q}(\omega;\mathbb{R}^3)$ and $b \in L^q(\Omega;\mathbb{R}^3)$ (respectively, $b \in X(\Omega;\mathbb{R}^3)$ if $\gamma = 2$). We claim that the set function $\mathcal{F}(u,b,\cdot)$ is the restriction to $\mathcal{A}(\omega)$ of a Radon measure, absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$. We apply Corollary 5.2 in the Appendix. It is easy to see that if A_1, $A_2 \in \mathcal{A}(\omega)$ with $A_1 \cap A_2 = \emptyset$, then

$$\mathcal{F}'(u,b,A_1) + \mathcal{F}'(u,b,A_2) \leq \mathcal{F}'(u,b,A_1\cup A_2)\ , \tag{3.24}$$

while if $\overline{A_1} \cap \overline{A_2} = \emptyset$, then

$$\mathcal{F}''(u,b,A_1\cup A_2) \leq \mathcal{F}''(u,b,A_1) + \mathcal{F}''(u,b,A_2)\ . \tag{3.25}$$

To prove the second inequality we used the fact that since $\operatorname{dist}\left(\overline{A_1}, \overline{A_2}\right) > 0$, using appropriate cut-off functions it is possible to glue realizing sequences $\{(u_k, b_k)\}$ and $\{(v_k, z_k)\}$ for $\mathcal{F}''(u, b, A_1)$ and $\mathcal{F}''(u, b, A_2)$, respectively, in order to obtain a sequence admissible for $\mathcal{F}''(u, b, A_1 \cup A_2)$ and such restricted to A_1 and A_2 it coincides with $\{(u_k, b_k)\}$ and $\{(v_k, z_k)\}$, respectively.

We show that

$$\mathcal{F}(u, b, A_1 \cup A_2) = \mathcal{F}(u, b, A_1) + \mathcal{F}(u, b, A_2) \tag{3.26}$$

whenever A_1, $A_2 \in \mathcal{A}(\omega)$ and $A_1 \cap A_2 = \emptyset$. In view of (3.13) and (3.24), we have that

$$\mathcal{F}(u, b, A_1 \cup A_2) \geq \mathcal{F}(u, b, A_1) + \mathcal{F}(u, b, A_2) .$$

To prove the opposite inequality, let $A' \in \mathcal{R}(\omega)$ be such that $A' \subset\subset A_1 \cup A_2$. Since $A_1 \cap A_2 = \emptyset$, it follows that $A' \cap A_i \in \mathcal{R}(\omega)$, $A' \cap A_i \subset\subset A_i$, $i = 1, 2$, and $\overline{A' \cap A_1} \cap \overline{A' \cap A_2} = \emptyset$, and thus, using (3.13), (3.12), and (3.25), we deduce that

$$\begin{aligned} \mathcal{F}'(u, b, A') = \mathcal{F}''(u, b, A') &\leq \mathcal{F}''(u, b, A' \cap A_1) + \mathcal{F}''(u, b, A' \cap A_2) \\ &\leq \mathcal{F}(u, b, A_1) + \mathcal{F}(u, b, A_2) . \end{aligned}$$

Taking the supremum over all such A' concludes the argument. Hence, Property 1 in Corollary 5.2 holds.

By (3.12), (3.13), and (3.16), we obtain that

$$\mathcal{F}(u, b, A_1) \leq \mathcal{F}(u, b, A_2) + \mathcal{F}\left(u, b, A_1 \setminus \overline{A_3}\right) \tag{3.27}$$

for all A_1, A_2, $A_3 \in \mathcal{A}(\omega)$, with $A_3 \subset\subset A_2 \subset\subset A_1$. Thus, Property 2 in Corollary 5.2 is satisfied, while Property 3 follows from (3.21) and (3.23).

(iii) If $A \in \mathcal{A}(\omega)$, then $\mathcal{F}(u, b, A)$ depends only on the restrictions of (u, b) on $A \times I$, in the sense that if u_1, $u_2 \in W^{1,q}\left(\omega; \mathbb{R}^3\right)$ and b_1, $b_2 \in L^q\left(\Omega; \mathbb{R}^3\right)$ (respectively, b_1, $b_2 \in X\left(\Omega; \mathbb{R}^3\right)$ if $\gamma = 2$) satisfy $(u_1, b_1) = (u_2, b_2)$ $\mathcal{L}^3$ a.e. in $A \times I$, then

$$\mathcal{F}(u_1, b_1, A) = \mathcal{F}(u_2, b_2, A) . \tag{3.28}$$

To prove this, we fix an open set $A' \subset\subset A$. For every sequence $\left\{\left(u_k^{(2)}, b_k^{(2)}\right)\right\}$ converging to (u_2, b_2) in the weak topology of $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$, we can construct a sequence $\left\{\left(u_k^{(1)}, b_k^{(1)}\right)\right\}$ converging weakly to (u_1, b_1) in Ω such that $\left(u_k^{(1)}, b_k^{(1)}\right) = \left(u_k^{(1)}, b_k^{(1)}\right)$ $\mathcal{L}^3$ a.e. on $A' \times I$. This can be done using a cut-off function between A' and A. It follows that $\mathcal{F}'(u_1, b_1, A') \leq \mathcal{F}'(u_2, b_2, A')$. By (3.13), we conclude that $\mathcal{F}(u_1, b_1, A) \leq \mathcal{F}(u_2, b_2, A)$.

(iv) For $z \in \mathbb{R}^2$ we define the translation operator τ_z acting on sets as

$$\tau_z(B) := B + z$$

and on functions as

$$\tau_z(u)(y) := u(y - z) , \quad \tau_z(b)(y, x_3) := b(y - z, x_3) .$$

It is easy to see that $\mathcal{F}$ is translation invariant, in the sense that for every $z \in \mathbb{R}^2$ and $A \in \mathcal{A}(\omega)$, with $\tau_z(A) \subset\subset \omega$, we have

$$\mathcal{F}(\tau_z(u), \tau_z(b), \tau_z(A)) = \mathcal{F}(u, b, A) \tag{3.29}$$

for every $u \in W^{1,q}\left(\omega; \mathbb{R}^3\right)$ and $b \in L^q\left(\Omega; \mathbb{R}^3\right)$ (respectively, $b \in X\left(\Omega; \mathbb{R}^3\right)$ if $\gamma = 2$), where in the left-hand side the functions $\tau_z(u)$ and $\tau_z(b)$ have been modified outside $\tau_z(A)$ in such a way that they belong to $W^{1,q}\left(\omega; \mathbb{R}^3\right)$ and $L^q\left(\Omega; \mathbb{R}^3\right)$ (respectively, $X\left(\Omega; \mathbb{R}^3\right)$ if $\gamma = 2$), respectively. Note that this modification is possible since $\tau_z(A) \subset\subset \omega$ and the left-hand side does not depend on the choice of the modification, due to (3.28).

(v) It is well-known that $\mathcal{F}'(\cdot, \cdot, A)$ is weakly lower semicontinuous in $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$ (see Proposition 6.8 in [15]), and so the functional $\mathcal{F}(\cdot, \cdot, A)$ is itself weakly lower semicontinuous in $W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right)$, being the supremum of weakly lower semicontinuous functionals.

Step 3: Fix $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ (respectively, $b \in X\left(\Omega;\mathbb{R}^3\right)$ if $\gamma = 2$). By Step 2(ii), the set function $\widetilde{\mathcal{F}}(u,b,\cdot)$, defined by

$$\widetilde{\mathcal{F}}(u,b,B) := \inf\left\{\mathcal{F}(u,b,A) : \, A \in \mathcal{A}(\omega)\,,\, B \subset A\right\} \tag{3.30}$$

for every $B \in \mathcal{B}(\omega)$, is a Radon measure and satisfies

$$\widetilde{\mathcal{F}}(u,b,B) \le C\left(\int_B |D_p u|^q\, dy + \int_{B\times I} |b|^q\, dx\right) + C\mathcal{L}^2(B) \tag{3.31}$$

for every $B \in \mathcal{B}(\omega)$ when $\gamma > 2$, and

$$\widetilde{\mathcal{F}}(u,b,B) \le C\left(\int_B |D_p u|^q\, dy + \int_{B\times I} |b|^q\, dx\right) + C\mathcal{L}^2(B) + \int_{B\times I} |D_3 b|^2\, dx \tag{3.32}$$

for $\gamma = 2$.

We claim that if $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $B \in \mathcal{B}(\omega)$, then $\widetilde{\mathcal{F}}(u,b,B)$ depends only on the restriction of b on B, in the sense that if b_1, $b_2 \in L^q\left(\Omega;\mathbb{R}^3\right)$ (respectively, b_1, $b_2 \in X\left(\Omega;\mathbb{R}^3\right)$ if $\gamma = 2$) satisfy $b_1 = b_2$ $\mathcal{L}^3$ a.e. in $B \times I$, then

$$\widetilde{\mathcal{F}}(u,b_1,B) = \widetilde{\mathcal{F}}(u,b_2,B)\,. \tag{3.33}$$

We prove this in the case $\gamma = 2$. Let $\{A_k\} \subset \mathcal{A}(\omega)$ be a decreasing sequence of open sets containing B and such that $\mathcal{L}^2(A_k \setminus B) \to 0$. Define

$$b_2^{(k)} := \begin{cases} b_2 & \text{in } A_k \times I\,, \\ b_1 & \text{in } \Omega \setminus (A_k \times I)\,, \end{cases}$$

and note that $b_2^{(k)} \in X\left(\Omega;\mathbb{R}^3\right)$ (see Proposition 5.3) and $b_2^{(k)} \to b_1$ strongly in $L^q\left(\Omega;\mathbb{R}^3\right)$. Fix an arbitrary open set $A \in \mathcal{A}(\omega)$ containing B, and use the lower semicontinuity of $\mathcal{F}(u,\cdot,A)$ to write

$$\mathcal{F}(u,b_1,A) \le \liminf_{k\to+\infty} \mathcal{F}\left(u,b_2^{(k)},A\right) = \liminf_{k\to+\infty}\left(\mathcal{F}\left(u,b_2^{(k)},A_k\cap A\right) + \widetilde{\mathcal{F}}\left(u,b_2^{(k)},A\setminus A_k\right)\right)\,,$$

where the equality follows from the fact that $\widetilde{\mathcal{F}}(u,b,\cdot)$ is a Radon measure. By (3.28) we have that

$$\mathcal{F}\left(u,b_2^{(k)},A_k\cap A\right) = \mathcal{F}(u,b_2,A_k\cap A) \le \mathcal{F}(u,b_2,A)$$

so that

$$\mathcal{F}(u,b_1,A) \le \mathcal{F}(u,b_2,A) + \liminf_{k\to+\infty} \widetilde{\mathcal{F}}\left(u,b_2^{(k)},A\setminus A_k\right)\,.$$

By (3.32) we have

$$\begin{aligned}\widetilde{\mathcal{F}}\left(u,b_2^{(k)},A\setminus A_k\right) \le C&\left(\int_{A\setminus A_k} |D_p u|^q\, dy + \int_{(A\setminus A_k)\times I} |b_1|^q\, dx\right)\\ &+ C\mathcal{L}^2(A\setminus A_k) + \int_{(A\setminus A_k)\times I} |D_3 b_1|^2\, dx\,,\end{aligned}$$

and so, passing to the limit as $k\to\infty$, we obtain

$$\begin{aligned}\mathcal{F}(u,b_1,A) \le \mathcal{F}(u,b_2,A) + C&\left(\int_{A\setminus B} |D_p u|^q\, dy + \int_{(A\setminus B)\times I} |b_1|^q\, dx\right)\\ &+ C\mathcal{L}^2(A\setminus B) + \int_{(A\setminus B)\times I} |D_3 b_1|^2\, dx\,.\end{aligned}$$

Taking the infimum over all $A \in \mathcal{A}(\omega)$ containing B, we obtain $\widetilde{\mathcal{F}}(u,b_1,B) \le \widetilde{\mathcal{F}}(u,b_2,B)$. By interchanging b_1 and b_2 we conclude the proof of (3.33).

Step 4: For $\lambda > 0$, $\left(u,\overline{b}\right) \in W^{1,q}\left(\omega;\mathbb{R}^3\right) \times L^1\left(\omega; L^q\left(I;\mathbb{R}^3\right)\right)$, and $B \in \mathcal{B}(\omega)$ we introduce the Yosida transform

$$\mathcal{F}^\lambda\left(u,\overline{b},B\right) := \inf\left\{\widetilde{\mathcal{F}}\left(u,a,B\right) + \lambda\int_B \left\|\overline{a}\left(y\right) - \overline{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)} dy : a \in L^q\left(\Omega;\mathbb{R}^3\right)\right\} \quad (3.34)$$

if $\gamma > 2$, and

$$\mathcal{F}^\lambda\left(u,\overline{b},B\right) := \inf\left\{\widetilde{\mathcal{F}}\left(u,a,B\right) + \lambda\int_B \left\|\overline{a}\left(y\right) - \overline{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)} dy : a \in X\left(\Omega;\mathbb{R}^3\right)\right\} \quad (3.35)$$

if $\gamma = 2$, where $\overline{a}$ and $\overline{b}$ are defined as in (3.7).

Substep 4a: We claim that there exists a continuous functional

$$F_\lambda : \mathbb{R}^{3\times 2} \times L^q\left(I;\mathbb{R}^3\right) \to [0,\infty) \, ,$$

with

$$0 \le F_\lambda\left(\xi, g\right) \le C\left(|\xi|^q + 1\right) + \lambda\left\|g\right\|_{L^q(I;\mathbb{R}^3)} \quad (3.36)$$

for every $\xi \in \mathbb{R}^{3\times 2}$ and $g \in L^q\left(I;\mathbb{R}^3\right)$, such that

$$\mathcal{F}^\lambda\left(u,\overline{b},A\right) = \int_A F_\lambda\left(D_p u\left(y\right), \overline{b}\left(y\right)\right) dy \quad (3.37)$$

for all $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, $\overline{b} \in L^1\left(\omega; L^q\left(I;\mathbb{R}^3\right)\right)$, and $A \in \mathcal{A}(\omega)$.

We remark that the monotonicity of $\mathcal{F}^\lambda\left(u,\overline{b},A\right)$ with respect to λ implies the monotonicity of $F_\lambda\left(\xi,g\right)$ with respect to λ, for all $\xi \in \mathbb{R}^{3\times 2}$ and $g \in L^q\left(I;\mathbb{R}^3\right)$.

Note that (3.9) and (3.11) follow from (3.37) with

$$F\left(\xi,g\right) = \sup_{\lambda>0} F_\lambda\left(\xi,g\right)$$

for $\xi \in \mathbb{R}^{3\times 2}$ and $g \in L^q\left(I;\mathbb{R}^3\right)$, provided

$$\mathcal{F}^\lambda\left(u,\overline{b},A\right) \nearrow \mathcal{F}\left(u,b,A\right) \quad \text{as } \lambda \nearrow +\infty \quad (3.38)$$

for every $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ (respectively, $b \in X\left(\Omega;\mathbb{R}^3\right)$ if $\gamma = 2$) and $A \in \mathcal{A}(\omega)$.

To prove (3.38), we observe that, by the definition of $\mathcal{F}^\lambda$,

$$\sup_{\lambda>0} \mathcal{F}^\lambda\left(u,\overline{b},A\right) \le \mathcal{F}\left(u,b,A\right) \, .$$

Conversely, for every $\lambda > 0$ choose $a_\lambda \in L^q\left(\Omega;\mathbb{R}^3\right)$ such that

$$\mathcal{F}\left(u,b,A\right) \ge \mathcal{F}^\lambda\left(u,\overline{b},A\right) \ge \mathcal{F}\left(u,a_\lambda,A\right) + \lambda\int_A \left\|\overline{a_\lambda}\left(y\right) - \overline{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)} dy - \frac{1}{\lambda} \, .$$

Since $\mathcal{F}\left(u,b,A\right) < \infty$, this implies that $\overline{a_\lambda} \to \overline{b}$ strongly in $L^1\left(A; L^q\left(I;\mathbb{R}^3\right)\right)$ as $\lambda \to +\infty$, and from (3.20) and (3.22), we have that $\{a_\lambda\}$ is bounded in $L^q\left(A\times I;\mathbb{R}^3\right)$. Using (3.28) we can redefine a_λ to be b outside $A \times I$, so that now $\overline{a_\lambda} \to \overline{b}$ strongly in $L^1\left(\omega; L^q\left(I;\mathbb{R}^3\right)\right)$. Passing to a subsequence, not relabelled, the sequence $\{a_\lambda\}$ converges weakly in $L^q\left(\Omega;\mathbb{R}^3\right)$ to some function b^*. To see that $b^* = b$ $\mathcal{L}^3$ a.e. in Ω, it suffices to notice that, equivalently, $\overline{a_\lambda} \rightharpoonup \overline{b^*}$ weakly in $L^q\left(\omega; L^q\left(I;\mathbb{R}^3\right)\right)$, and in turn this implies that $\overline{a_\lambda} \rightharpoonup \overline{b^*}$ weakly in $L^1\left(\omega; L^q\left(I;\mathbb{R}^3\right)\right)$, and so $\overline{b^*} = \overline{b}$. Therefore $a_\lambda \rightharpoonup b$ weakly in $L^q\left(\Omega;\mathbb{R}^3\right)$. In view of the weak lower semicontinuity of $\mathcal{F}\left(u,\cdot,A\right)$ in $L^q\left(\Omega;\mathbb{R}^3\right)$ we conclude that

$$\mathcal{F}\left(u,b,A\right) \le \liminf_{\lambda\to\infty} \mathcal{F}\left(u,a_\lambda,A\right) \le \liminf_{\lambda\to\infty} \mathcal{F}^\lambda\left(u,\overline{b},A\right) \, .$$

This establishes (3.38), which together with (3.37), the monotonicity of $F_\lambda\left(\xi,g\right)$ with respect to λ, and the monotone convergence theorem yields

$$\mathcal{F}\left(u,b,A\right) = \int_A F\left(D_p u\left(y\right), \overline{b}\left(y\right)\right) dy$$

for all $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ (respectively, $b \in X\left(\Omega;\mathbb{R}^3\right)$ if $\gamma = 2$), and $A \in \mathcal{A}(\omega)$.

Note that (3.8) follow from (3.9), (3.20), and (3.21), while (3.10) is a consequence of (3.11), (3.22), and (3.23).

Substep 4b: In order to prove (3.37) and (3.36), we first establish (3.37) for a countable family $\mathcal{G}$ of functions $\bar{b}$. This will require the following properties of the functional $\mathcal{F}^\lambda\left(\cdot,\bar{b},\cdot\right)$ for a fixed $\bar{b} \in L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$.

(i) $\mathcal{F}^\lambda\left(\cdot,\bar{b},\cdot\right)$ satisfies a growth condition of order p, which in view of (3.21) for $\gamma > 2$ and (3.23) for $\gamma = 2$, respectively, here becomes

$$\begin{aligned} 0 \le \mathcal{F}^\lambda\left(u,\bar{b},A\right) &\le \mathcal{F}\left(u,0,A\right) + \lambda \int_A \left\|\bar{b}\right\|_{L^q(I;\mathbb{R}^3)}\,dy \\ &\le C\int_A \left|D_p u\right|^q\,dy + \lambda\int_A \left\|\bar{b}\right\|_{L^q(I;\mathbb{R}^3)}\,dy + C\mathcal{L}^2\left(A\right) \end{aligned} \tag{3.39}$$

for every $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $A \in \mathcal{A}(\omega)$.

(ii) We claim that for every $(u,b) \in W^{1,q}\left(\omega;\mathbb{R}^3\right) \times L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$,

$$\text{the set function } \mathcal{F}^\lambda\left(u,\bar{b},\cdot\right) \text{ is a Radon measure on } \mathcal{B}(\omega). \tag{3.40}$$

It is well-known that if a nonnegative finitely additive set function defined on $\mathcal{B}(\omega)$ is bounded from above by a Radon measure, then it is a Radon measure on $\mathcal{B}(\omega)$. In view of (3.39), it remains to show that for every $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $\bar{b} \in L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$, the set function $\mathcal{F}^\lambda\left(u,\bar{b},\cdot\right)$ is finitely additive, that is

$$B_1 \cap B_2 = \emptyset \quad \Longrightarrow \quad \mathcal{F}^\lambda\left(u,\bar{b},B_1\cup B_2\right) = \mathcal{F}^\lambda\left(u,\bar{b},B_1\right) + \mathcal{F}^\lambda\left(u,\bar{b},B_2\right). \tag{3.41}$$

To see this, fix $\varepsilon > 0$ and by (3.34) find a_1, $a_2 \in L^q\left(\Omega;\mathbb{R}^3\right)$ such that

$$\begin{aligned} \mathcal{F}^\lambda\left(u,\bar{b},B_1\right) &\ge \widetilde{\mathcal{F}}\left(u,a_1,B_1\right) + \lambda\int_{B_1}\left\|\overline{a_2}\left(y\right) - \bar{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy - \frac{\varepsilon}{2}, \\ \mathcal{F}^\lambda\left(u,\bar{b},B_2\right) &\ge \widetilde{\mathcal{F}}\left(u,a_2,B_2\right) + \lambda\int_{B_2}\left\|\overline{a_2}\left(y\right) - \bar{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy - \frac{\varepsilon}{2}. \end{aligned}$$

Define

$$a\left(y\right) := \begin{cases} a_2\left(y\right) & \text{if } y \in B_1, \\ a_1\left(y\right) & \text{elsewhere} \end{cases}$$

and note that $a \in X\left(\Omega;\mathbb{R}^3\right)$ if $\gamma = 2$, by Proposition 5.3. Since $\widetilde{\mathcal{F}}\left(u,a,\cdot\right)$ is a Radon measure, we have

$$\begin{aligned} \mathcal{F}^\lambda\left(u,\bar{b},B_1\cup B_2\right) &\le \widetilde{\mathcal{F}}\left(u,a,B_1\cup B_2\right) + \lambda\int_{B_1\cup B_2}\left\|\bar{a}\left(y\right) - \bar{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy \\ &= \widetilde{\mathcal{F}}\left(u,a_1,B_1\right) + \widetilde{\mathcal{F}}\left(u,a_2,B_2\right) \\ &\quad + \lambda\int_{B_1}\left\|\overline{a_1}\left(y\right) - \bar{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy + \lambda\int_{B_2}\left\|\overline{a_2}\left(y\right) - \bar{b}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy \\ &\le \mathcal{F}^\lambda\left(u,\bar{b},B_1\right) + \mathcal{F}^\lambda\left(u,\bar{b},B_2\right) + \varepsilon, \end{aligned}$$

where we used (3.33), (3.34), and (3.35). By letting $\varepsilon \to 0^+$, we obtain

$$\mathcal{F}^\lambda\left(u,\bar{b},B_1\cup B_2\right) \le \mathcal{F}^\lambda\left(u,\bar{b},B_1\right) + \mathcal{F}^\lambda\left(u,\bar{b},B_2\right).$$

The converse inequality is simpler and we omit its proof.

(iii) $\mathcal{F}^\lambda\left(u,\bar{b},A\right)$ depends only on the restriction of u to A, that is, if $A \in \mathcal{A}(\omega)$, u_1, $u_2 \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$, and $u_1 = u_2$ $\mathcal{L}^2$ a.e. in A, then

$$\mathcal{F}^\lambda\left(u_1,\bar{b},A\right) = \mathcal{F}^\lambda\left(u_2,\bar{b},A\right). \tag{3.42}$$

This follows from (3.28), (3.34), and (3.35).

On the other hand, $\mathcal{F}^\lambda\left(u,\overline{b},B\right)$ depends only on the restriction of $\overline{b}$ to B, that is, if $B\in\mathcal{B}(\omega)$ and $\overline{b}_1$, $\overline{b}_2\in L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$ satisfy $\overline{b}_1=\overline{b}_2$ $\mathcal{L}^2$ a.e. in A, then

$$\mathcal{F}^\lambda\left(u,\overline{b}_1,B\right)=\mathcal{F}^\lambda\left(u_2,\overline{b}_2,B\right)\,. \tag{3.43}$$

This follows from (3.34) and (3.35).
(iv) $\mathcal{F}^\lambda\left(\cdot,\overline{b},A\right)$ is translation invariant for every $A\in\mathcal{A}(\omega)$, i.e., for every $u\in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $c\in\mathbb{R}^3$,

$$\mathcal{F}^\lambda\left(u+c,\overline{b},A\right)=\mathcal{F}^\lambda\left(u,\overline{b},A\right)\,.$$

This is a direct consequence of the translation invariance of $\mathcal{F}\left(\cdot,b,A\right)$, (3.34), and (3.35).
(v) For every $A\in\mathcal{A}(\omega)$, the functional $\mathcal{F}^\lambda\left(\cdot,\overline{b},A\right)$ is sequentially weakly lower semicontinuous in $W^{1,q}\left(\omega;\mathbb{R}^3\right)$. More generally, we will prove that $\mathcal{F}^\lambda\left(\cdot,\cdot,A\right)$ is sequentially weakly lower semicontinuous in $W^{1,q}\left(\omega;\mathbb{R}^3\right)\times L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$. For this purpose, let $u_k\rightharpoonup u$ weakly in $W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $\overline{b}_k\rightharpoonup\overline{b}$ weakly in $L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$, with

$$\liminf_{k\to+\infty}\mathcal{F}^\lambda\left(u_k,\overline{b}_k,A\right)=\lim_{k\to+\infty}\mathcal{F}^\lambda\left(u_k,\overline{b}_k,A\right)<\infty\,.$$

Let $a_k\in L^q\left(\Omega;\mathbb{R}^3\right)$ be such that

$$\mathcal{F}^\lambda\left(u_k,\overline{b}_k,A\right)+\frac{1}{k}\geq\mathcal{F}\left(u_k,a_k,A\right)+\lambda\int_A\left\|\overline{a}_k-\overline{b}_k\right\|_{L^q(I;\mathbb{R}^3)}\,dy\,.$$

By (3.20) and (3.22), $\{a_k\}$ is bounded in $L^q\left(A\times I;\mathbb{R}^3\right)$, and so we may assume that it converges weakly in $L^q\left(A\times I;\mathbb{R}^3\right)$ to some function $a\in L^q\left(A\times I;\mathbb{R}^3\right)$. By extending a_k and a outside $A\times I$ by zero, it follows that $a_k\rightharpoonup a$ weakly in $L^q\left(\Omega;\mathbb{R}^3\right)$. Since the functional

$$\varphi\mapsto\int_A\left\|\overline{\varphi}\left(y\right)\right\|_{L^q(I;\mathbb{R}^3)}\,dy$$

is convex and continuous with respect to the strong topology of $L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$, it is also weakly lower semicontinuous. This property, together with the weak lower semicontinuity of $\mathcal{F}\left(\cdot,\cdot,A\right)$, yields

$$\begin{aligned}\mathcal{F}^\lambda\left(u,\overline{b},A\right)&\leq\mathcal{F}\left(u,a,A\right)+\lambda\int_A\left\|\overline{a}-\overline{b}\right\|_{L^q(I;\mathbb{R}^3)}\,dy\\&\leq\liminf_{k\to+\infty}\left(\mathcal{F}\left(u_k,a_k,A\right)+\lambda\int_A\left\|\overline{a}_k-\overline{b}_k\right\|_{L^q(I;\mathbb{R}^3)}\,dy\right)\leq\liminf_{k\to+\infty}\mathcal{F}^\lambda\left(u_k,\overline{b}_k,A\right)\,.\end{aligned}$$

Substep 4c: We now introduce the family $\mathcal{G}$ mentioned at the end of Substep 4a. Precisely, let $\mathcal{G}$ be a countable dense subset of $L^q\left(I;\mathbb{R}^3\right)$ for $\gamma>2$ and of $L^q\left(I;\mathbb{R}^3\right)\cap W^{1,2}\left(I;\mathbb{R}^3\right)$ if $\gamma=2$, respectively. Fix $g\in\mathcal{G}$ and regard g as a constant element of $L^1\left(\omega;L^q\left(I;\mathbb{R}^3\right)\right)$. In view of Substep 4b we can apply Theorem 4.3.2 in [10] to the functional

$$\mathcal{F}^\lambda\left(\cdot,g,\cdot\right):W^{1,q}\left(\omega;\mathbb{R}^3\right)\times\mathcal{A}(\omega)\to[0,\infty)$$

to obtain a Carathéodory function

$$F_\lambda\left(\cdot,\cdot,g\right):\omega\times\mathbb{R}^{3\times2}\to[0,\infty)\,,$$

with

$$0\leq F_\lambda\left(y,\xi,g\right)\leq C\left(\left|\xi\right|^q+1\right)+\lambda\left\|g\right\|_{L^q(I;\mathbb{R}^3)} \tag{3.44}$$

for $\mathcal{L}^2$ a.e. $y\in\omega$ and for every $\xi\in\mathbb{R}^{3\times2}$, such that

$$\mathcal{F}^\lambda\left(u,g,A\right)=\int_A F_\lambda\left(y,D_pu\left(y\right),g\right)\,dy \tag{3.45}$$

for every $u\in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $A\in\mathcal{A}(\omega)$.

Since the translation invariance (3.29) is inherited by $\mathcal{F}^\lambda$, we have that

$$\mathcal{F}^\lambda\left(\tau_z\left(u\right),g,\tau_z\left(A\right)\right)=\mathcal{F}^\lambda\left(u,g,A\right)\,,$$

and so, by the explicit expression of $F_\lambda(y,\xi,g)$ given in (4.3.1) in [10], we deduce that F_λ does not depend on y, that is $F_\lambda = F_\lambda(\xi,g)$.

Hence, for every $u \in W^{1,q}(\omega;\mathbb{R}^3)$, $g \in \mathcal{G}$, and $A \in \mathcal{A}(\omega)$,

$$\mathcal{F}^\lambda(u,g,A) = \int_A F_\lambda(D_p u(y), g)\, dy. \tag{3.46}$$

By property (ii) in Substep 4b, the representation (3.46) may be extended to Borel sets in ω, i.e.,

$$\mathcal{F}^\lambda(u,g,B) = \int_B F_\lambda(D_p u(y), g)\, dy \tag{3.47}$$

for every $B \in \mathcal{B}(\omega)$.

Substep 4d: To conclude (3.37), it remains to show that (3.46) may be extended from $g \in \mathcal{G}$ to $\bar{b} \in L^1(\omega; L^q(I;\mathbb{R}^3))$.

In view of (3.44), standard properties of sequentially weakly lower semicontinuous multiple integrals ensure that $F_\lambda(\cdot, g)$ is quasiconvex, and so it satisfies a q-Lipschitz condition, that is, there exists a constant $C_q > 0$, depending only on C and q, such that

$$|F_\lambda(\xi_1,g) - F_\lambda(\xi_2,g)| \leq C_q\left(|\xi_1|^{q-1} + |\xi_2|^{q-1} + \lambda\|g\|_{L^q(I;\mathbb{R}^3)} + 1\right)|\xi_1 - \xi_2| \tag{3.48}$$

for every ξ_1, $\xi_2 \in \mathbb{R}^{3\times 2}$.

Assume that $b \in L^1(\omega; L^q(I;\mathbb{R}^3))$ is of the form

$$\bar{b}(y) = \textstyle\sum_{i=1}^m g_i \chi_{B_i}(y), \quad y \in \omega, \tag{3.49}$$

where $m \in \mathbb{N}$, the family $\{B_i\} \subset \mathcal{B}(\omega)$ is a partition of ω and $g_i \in \mathcal{G}$ for $i = 1,\dots,m$. By property (ii) in Substep 4b and by (3.43), we have that

$$\mathcal{F}^\lambda(u,\bar{b},A) = \textstyle\sum_i \mathcal{F}^\lambda(u,\bar{b},A\cap B_i) = \sum_i \mathcal{F}^\lambda(u,g_i,A\cap B_i),$$

hence by (3.47) we conclude that

$$\mathcal{F}^\lambda(u,\bar{b},A) = \textstyle\sum_i \displaystyle\int_{A\cap B_i} F_\lambda(D_p u(y), g_i)\, dy = \int_A F_\lambda(D_p u(y), \bar{b}(y))\, dy. \tag{3.50}$$

It follows from the definition of $\mathcal{F}^\lambda$ that

$$\left|\mathcal{F}^\lambda(u,\bar{b}_1,A) - \mathcal{F}^\lambda(u,\bar{b}_2,A)\right| \leq \lambda\int_A \left\|\bar{b}_1 - \bar{b}_2\right\|_{L^q(I;\mathbb{R}^3)} dy \tag{3.51}$$

for every $u \in W^{1,q}(\omega;\mathbb{R}^3)$, $\bar{b}_1$, $\bar{b}_2 \in L^1(\omega;L^q(I;\mathbb{R}^3))$, and $A \in \mathcal{A}(\omega)$. It follows from (3.51) and (3.47) that

$$|F_\lambda(\xi,g_1) - F_\lambda(\xi,g_2)| \leq \lambda\|g_1 - g_2\|_{L^q(I;\mathbb{R}^3)} \tag{3.52}$$

for every $\xi \in \mathbb{R}^{3\times 2}$ and for every g_1, $g_2 \in \mathcal{G}$.

By (3.48) and (3.52) and a density argument we can extend F_λ continuously to $\mathbb{R}^{3\times 2} \times L^q(I;\mathbb{R}^3)$ in such a way that (3.48) and (3.52) remain valid. Since both sides of (3.50) are Lipschitz with respect to $\bar{b}$ (see (3.51) and (3.52)), and functions of the type (3.49) are dense in $L^1(\omega;L^q(I;\mathbb{R}^3))$, from (3.50) we obtain

$$\mathcal{F}^\lambda(u,\bar{b},A) = \int_B F_\lambda(D_p u(y), \bar{b}(y))\, dy \tag{3.53}$$

for all $u \in W^{1,q}(\omega;\mathbb{R}^3)$, $\bar{b} \in L^1(\omega;L^q(I;\mathbb{R}^3))$, and $A \in \mathcal{A}(\omega)$. □

4. Example of Nonlocality in Dimension Reduction

Let $\Omega = \omega \times I$ be as in Section 3. Fix a function $g : \mathbb{R}^3 \to [0, +\infty)$ such that

$$g \text{ is continuous in } \mathbb{R}^3 , \tag{4.1}$$

$$g(z) \geq a_0 |z|^2 - b_0 \quad \text{for all } z \in \mathbb{R}^3 , \tag{4.2}$$

$$g(e_1) = g(-e_1) = 0 < g(z) \quad \text{for } z \neq \pm e_1 , \tag{4.3}$$

for some constants $a_0 > 0$ and $b_0 \geq 0$, where $e_1 := (1, 0, 0)$, and define

$$W(\xi) = W(\xi_1 |\xi_2| \xi_3) := |\xi_1|^2 + |\xi_2|^2 + g(\xi_3) , \tag{4.4}$$

where ξ_1, ξ_2, and ξ_3 are the columns of the 3×3 matrix ξ.

For $A \in \mathcal{A}(\Omega)$, $u \in W^{1,2}(A; \mathbb{R}^3)$ with $D^2 u \in L^2(A; \mathbb{R}^{3\times 3\times 3})$, and $\varepsilon > 0$ consider the functional

$$\mathfrak{W}_\varepsilon(u, A) := \int_A \left[W\left(D_p u \left| \frac{1}{\varepsilon} D_3 u \right.\right) + \varepsilon^2 \left| D_p^2 u \right|^2 + |D_{p3} u|^2 + \frac{1}{\varepsilon^2} |D_{33} u|^2 \right] dx .$$

To study the asymptotic behavior of $\mathfrak{W}_\varepsilon$ as $\varepsilon \to 0$, it is convenient to introduce the functional

$$\mathfrak{F}_\varepsilon : W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \times \mathcal{A}(\Omega) \to [0, \infty]$$

defined by

$$\mathfrak{F}_\varepsilon(u, b, A) := \begin{cases} \mathfrak{W}_\varepsilon(u, A) & \text{if } b = \frac{1}{\varepsilon} D_3 u \text{ in } A \text{ and } D^2 u \in L^2(A; \mathbb{R}^{3\times 3\times 3}) , \\ \infty & \text{otherwise.} \end{cases} \tag{4.5}$$

By Proposition 8.10, Corollary 8.12, and Theorem 16.9 in [15] there exists a sequence $\varepsilon_k \to 0$ such that $\mathfrak{F}_{\varepsilon_k}$ $\overline{\Gamma}$-converges to a functional $\mathfrak{F}$ in the sense of Definition 16.2 in [15] with $X := W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)$ endowed with the weak topology. Note that $\mathfrak{F}(u, b, A' \times I) = \mathcal{F}(u, b, A')$ for every $u \in W^{1,2}(\omega; \mathbb{R}^3)$, $b \in L^2(\Omega; \mathbb{R}^3)$, and $A' \in \mathcal{A}(\omega)$. In this section we give an example of an energy density W such that the corresponding $\mathfrak{F}(u, b, \cdot)$ is not a measure on $\mathcal{A}(\Omega)$.

The main result of this section is the following theorem.

Theorem 4.1. *Let $b_0(x) := (x_3, 0, 0)$, $x \in \Omega$. Then $\mathfrak{F}(0, b_0, \cdot)$ is not a measure on $\mathcal{A}(\Omega)$, in the sense that there is no measure μ defined on the σ-algebra of the Borel subsets of Ω such that $\mathfrak{F}(0, b_0, A) = \mu(A)$ for every $A \in \mathcal{A}(\Omega)$.*

Proof. The proof is very similar to the one of Theorem 2.1 and we indicate only the main changes. We argue by contradiction and assume that there is a measure μ, defined on all Borel subsets of Ω, such that

$$\mathfrak{F}(0, b_0, A) = \mu(A) \qquad \text{for every } A \in \mathcal{A}(\Omega) . \tag{4.6}$$

We claim that in this case

$$\mathfrak{F}(0, b_0, \Omega) \leq \int_\Omega |D_3 b_0|^2 \, dx = \mathcal{L}^3(\Omega) . \tag{4.7}$$

By (4.1) and (4.3), given $\eta > 0$, there exists $\delta > 0$ such that

$$g(z) < \eta \quad \text{for } |z \pm e_1| < \delta . \tag{4.8}$$

We cover Ω with a finite number of open sets A_i contained in Ω and such that

$$\operatorname{diam} A_i < \delta \quad \text{and} \quad \textstyle\sum_i \mathcal{L}^3(A_i) < (1 + \eta) \mathcal{L}^3(\Omega) . \tag{4.9}$$

By (4.6),

$$\mathfrak{F}(0, b_0, \Omega) \leq \textstyle\sum_i \mathfrak{F}(0, b_0, A_i) . \tag{4.10}$$

For each i fix a point $x^{(i)} \in A_i$. Since $x_3^{(i)} \in I$, there exists $\lambda_i \in \left(\frac{1}{4}, \frac{3}{4}\right)$ such that

$$x_3^{(i)} = \lambda_i (-1) + (1 - \lambda_i) 1 ,$$

and we can write

$$b_0(x) = x_3 = \lambda_i \left(-1 + x_3 - x_3^{(i)}\right) + (1 - \lambda_i)\left(1 + x_3 - x_3^{(i)}\right) . \tag{4.11}$$

Let $\chi^{(i)}$ be the 1-periodic function in $\mathbb{R}$ such that $\chi^{(i)}(t) = 1$ for $t \in [0, \lambda_i]$ and $\chi^{(i)}(t) = 0$ for $t \in (\lambda_i, 1)$. We introduce a sequence of positive numbers $\alpha_k \to 0$, which will chosen later, and we define $\chi_k^{(i)}(t) := \chi^{(i)}(t/\alpha_k)$. By the Riemann-Lebesgue lemma, $\chi_k^{(i)} \rightharpoonup \lambda_i$ weakly* in $L^\infty(\mathbb{R})$. Define for $x \in \Omega$,

$$\beta_k^{(i)}(x) := \left[\chi_k^{(i)}(x_1)\left(-1 + x_3 - x_3^{(i)}\right) + \left(1 - \chi_k^{(i)}(x_1)\right)\left(1 + x_3 - x_3^{(i)}\right)\right] e_1 . \tag{4.12}$$

By (4.9) we have $\left|x_3 - x_3^{(i)}\right| < \delta$ for $x \in A_i$, and so, by (4.8),

$$g\left(\beta_k^{(i)}(x)\right) < \eta \quad \text{for every } x \in A_i . \tag{4.13}$$

Moreover, $D_3\beta_k^{(i)}(x) = e_1$ for every $x \in A_i$.

Let $\rho \in C_c^\infty((-1,1))$ be such that $\int_{\mathbb{R}} \rho(t)\, dt = 1$ and $\rho \geq 0$. We introduce another sequence of positive numbers $\sigma_k \to 0$, which will chosen later, and we set $\rho_k(t) := \rho(t/\sigma_k)/\sigma_k$ and $\varphi_k^{(i)} := \rho_k * \chi_k^{(i)}$. For every i and for $x \in \Omega$ define

$$u_k^{(i)}(x) := \left[\varepsilon_k \varphi_k^{(i)}(x_1) \int_0^{x_3} \left(-1 + t - x_3^{(i)}\right) dt + \varepsilon_k \left(1 - \varphi_k^{(i)}(x_1)\right) \int_0^{x_3} \left(1 + t - x_3^{(i)}\right) dt\right] e_1,$$

$$b_k^{(i)}(x) := \frac{1}{\varepsilon_k} D_3 u_k^{(i)}(x) = \left[\varphi_k^{(i)}(x_1)\left(-1 + x_3 - x_3^{(i)}\right) + \left(1 - \varphi_k^{(i)}(x_1)\right)\left(1 + x_3 - x_3^{(i)}\right)\right] e_1 .$$

We choose $\alpha_k \to 0$ and $\sigma_k \to 0$ so that $\sigma_k/\alpha_k \to 0$ and $\varepsilon_k/\sigma_k \to 0$. Since $\left|\varphi_k^{(i)}\right| \leq c$, $\left|\left(\varphi_k^{(i)}\right)'\right| \leq c/\sigma_k$, $\left|\left(\varphi_k^{(i)}\right)''\right| \leq c/\sigma_k$, and $\mathcal{L}^1\left(\left\{\varphi_k^{(i)} \neq \chi_k^{(i)}\right\} \cap I\right) \leq c\sigma_k/\alpha_k$, we have that $u_k^{(i)} \to 0$ strongly in $L^\infty\left(\Omega; \mathbb{R}^3\right)$, $D_p u_k^{(i)} \to 0$ strongly in $L^2\left(\Omega; \mathbb{R}^{3\times 2}\right)$, $\varphi_k^{(i)} \rightharpoonup \lambda_i$ weakly* in $L^\infty(I)$, $b_k^{(i)} \rightharpoonup b_0$ weakly in $L^2\left(\Omega; \mathbb{R}^3\right)$, $\varepsilon_k D_p^2 u_k^{(i)} \to 0$ strongly in $L^2\left(\Omega; \mathbb{R}^{3\times 2\times 2}\right)$, $D_{p3} u_k^{(i)} \to 0$ strongly in $L^2\left(\Omega; \mathbb{R}^{3\times 2}\right)$, and $\frac{1}{\varepsilon_k} D_{33} u_k^{(i)} = D_3 b_k^{(i)} = e_1$.

By (4.4) and (4.13) and the fact that

$$\lim_{k\to\infty} \left| \int_{A_i} g\left(\beta_k^{(i)}(x)\right) dx - \int_{A_i} g\left(b_k^{(i)}(x)\right) dx \right| = 0,$$

we have

$$\mathfrak{F}_{\varepsilon_k}\left(u_k^{(i)}, b_k^{(i)}, A_i\right) \leq (1 + \eta)\, \mathcal{L}^3(A_i) + o(1) ,$$

and so by $\overline{\Gamma}$-convergence

$$\mathfrak{F}(0, b_0, A_i) \leq (1 + \eta)\, \mathcal{L}^3(A_i) ,$$

and using (4.9) and (4.10) we deduce

$$\mathfrak{F}(0, b_0, \Omega) \leq (1 + \eta) \textstyle\sum_i \mathcal{L}^3(A_i) \leq (1 + \eta)^2\, \mathcal{L}^3(\Omega) .$$

Letting $\eta \to 0$, we obtain (4.7).

On the other hand, since the functional

$$b \mapsto \int_\Omega |D_3 b|^2\, dx \quad \text{is weakly lower semicontinuous in } L^2\left(\Omega; \mathbb{R}^3\right) \tag{4.14}$$

and below $\mathfrak{F}(\cdot, \cdot, \Omega)$, we conclude that

$$\mathfrak{F}(0, b_0, \Omega) = \mathcal{L}^3(\Omega) .$$

By the lower bound in (2.2) and by a general property of the Γ-limit of coercive functionals (see Propositions 8.1 and 8.10 in [15]), there exists a sequence $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ converging to 0 weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that $b_k = \frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b_0$ weakly in $L^2(\Omega;\mathbb{R}^3)$ and

$$\lim_{k\to\infty} \int_\Omega \left[W\left(D_p u_k \,|b_k\right) + \varepsilon_k^2 \left|D_p^2 u_k\right|^2 + \left|D_{p3} u_k\right|^2 + \left|D_3 b_k\right|^2 \right] dx$$
$$= \lim_{k\to\infty} \mathfrak{F}_{\varepsilon_k}(u_k, b_k, \Omega) = \mathfrak{F}(0, b_0, \Omega) = \mathcal{L}^3(\Omega) .$$

Using (4.3), (4.4), and (4.14), we obtain

$$\lim_{k\to\infty} \int_\Omega g(b_k)\, dx = 0 ,$$
$$\lim_{k\to\infty} \int_\Omega \left|D_3 b_k\right|^2 dx = \int_\Omega \left|D_3 b_0\right|^2 dx = \mathcal{L}^3(\Omega) .$$

We can now continue exactly as in the proof of Theorem 2.1 (see (2.16) and (2.17)) to obtain a contradiction. ☐

5. Appendix

In this appendix we present some auxiliary results used in the proof of Theorem 3.3. The theorem may be found in [20].

Theorem 5.1 (De Giorgi–Letta). *Let (X,d) be a metric space, let $\mathcal{A}(X)$ be the collection of all open subsets of X, and let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X. Assume that $\rho : \mathcal{A}(X) \to [0,\infty]$ is an increasing set function such that*

- (i) $\rho(\emptyset) = 0$;
- (ii) *(subadditivity)* $\rho(A_1 \cup A_2) \le \rho(A_1) + \rho(A_2)$ *for all* A_1, $A_2 \in \mathcal{A}(X)$;
- (iii) *(inner regularity)* $\rho(A) = \sup\{\rho(A_1) : A_1 \subset\subset A\}$ *for every* $A \in \mathcal{A}(X)$;
- (iv) *(superadditivity)* $\rho(A_1 \cup A_2) \ge \rho(A_1) + \rho(A_2)$ *for all* A_1, $A_2 \in \mathcal{A}(X)$, *with* $A_1 \cap A_2 = \emptyset$.

Then the extension of ρ to $\mathcal{B}(X)$, defined for every $B \in \mathcal{B}(X)$ by

$$\widetilde{\rho}(B) = \inf\{\rho(A) : A \in \mathcal{A}(X),\, B \subset A\} , \tag{5.1}$$

is a measure on $\mathcal{B}(X)$.

The following corollary is an adaptation of a similar result in [3].

Corollary 5.2. *Let (X,d) be a locally compact metric space such that every open set $A \subset X$ is σ-compact. Assume that $\rho : \mathcal{A}(X) \to [0,\infty)$ is an increasing set function such that*

1. *(additivity on disjoint sets)* $\rho(A_1 \cup A_2) = \rho(A_1) + \rho(A_2)$ *for all* A_1, $A_2 \in \mathcal{A}(X)$, *with* $A_1 \cap A_2 = \emptyset$;
2. *for all* A, B, $C \in \mathcal{A}(X)$, *with* $C \subset\subset B \subset\subset A$ *we have*
$$\rho(A) \le \rho(B) + \rho(A \setminus \overline{C});$$
3. *there exists a measure* $\mu : \mathcal{B}(X) \to [0,\infty)$ *such that*
$$\rho(A) \le \mu(A) < +\infty$$
for every $A \in \mathcal{A}(X)$.

Then ρ is the restriction to $\mathcal{A}(X)$ of a measure defined on $\mathcal{B}(X)$.

Proof. We start by checking the validity of the hypotheses of the De Giorgi-Letta theorem. Property (i) follows from (3), while (iv) follows from (1).

We prove inner regularity. Fix $A \in \mathcal{A}(X)$, $\varepsilon > 0$, and find a compact set $K \subset A$ such that

$$\mu(A \setminus K) < \varepsilon .$$

By Theorem A.12 in [20] there exists an open set $C \subset\subset A$ such that $K \subset C \subset A$. Hence,

$$\rho(A \setminus \overline{C}) \le \mu\left(A \setminus \overline{C}\right) < \varepsilon .$$

Choose $B \in \mathcal{A}(X)$ such that $C \subset\subset B \subset\subset A$, and apply property (2) to obtain

$$\begin{aligned}\rho(A) &\le \rho(B) + \rho(A \setminus \overline{C}) \le \rho(B) + \varepsilon \\ &\le \sup\left\{\rho\left(A'\right) : \, A' \subset\subset A\right\} + \varepsilon .\end{aligned}$$

Letting $\varepsilon \to 0^+$ and in view of the monotonicity of ρ, we conclude that (iii) holds.

Finally, to show subadditivity, fix $A, B \in \mathcal{A}(X)$ with $A \setminus B \neq \emptyset \neq B \setminus A$. For every $t \in (0,1)$ define

$$\begin{aligned}A_t &:= \left\{x \in A \cup B : \, t \operatorname{dist}(x, A \setminus B) < (1-t) \operatorname{dist}(x, B \setminus A)\right\} , \\ B_t &:= \left\{x \in A \cup B : \, t \operatorname{dist}(x, A \setminus B) > (1-t) \operatorname{dist}(x, B \setminus A)\right\} , \\ S_t &:= \left\{x \in A \cup B : \, t \operatorname{dist}(x, A \setminus B) = (1-t) \operatorname{dist}(x, B \setminus A)\right\} .\end{aligned}$$

Since the sets S_t are pairwise disjoint, there exists $t_0 \in (0,1)$ such that $\mu\left(S_{t_0}\right) = 0$. Then we may find A_0', $B_0' \in \mathcal{A}(X)$ with $A_0' \subset\subset A_{t_0}$ and $B_0' \subset\subset B_{t_0}$ such that

$$\mu((A \cup B) \setminus \overline{A_0' \cup B_0'}) = \mu((A_{t_0} \cup B_{t_0}) \setminus \overline{A_0' \cup B_0'}) < \varepsilon ,$$

and so by (3),

$$\rho((A \cup B) \setminus \overline{A_0' \cup B_0'}) \le \mu((A \cup B) \setminus \overline{A_0' \cup B_0'}) < \varepsilon .$$

Applying property (2) to $A \cup B$, $A_0'' \cup B_0''$, $A_0' \cup B_0'$, where A_0'', $B_0'' \in \mathcal{A}(X)$ with $A_0' \subset\subset A_0'' \subset\subset A_{t_0}$ and $B_0' \subset\subset B_0'' \subset\subset B_{t_0}$, we obtain

$$\begin{aligned}\rho(A \cup B) &\le \rho(A_0'' \cup B_0'') + \rho((A \cup B) \setminus \overline{A_0' \cup B_0'}) \\ &\le \rho(A_0'') + \rho(B_0'') + \varepsilon \le \rho(A) + \rho(B) + \varepsilon ,\end{aligned}$$

where we used (1) and the fact that A_0'' and B_0'' are disjoint, together with the fact that ρ is increasing.

In view of the De Giorgi-Letta theorem it follows that ρ is the restriction to $\mathcal{A}(X)$ of a measure defined on $\mathcal{B}(X)$. ☐

The following proposition provides a useful integration-by-parts formula.

Proposition 5.3. *Let $\Omega := \omega \times I$, where $\omega \subset \mathbb{R}^2$ is an open set, and let $b \in L^q(\Omega)$ be such that $D_3 b \in L^2(\Omega)$. Then*

$$\int_{B \times I} \varphi D_3 b \, dx = -\int_{B \times I} b D_3 \varphi \, dx \tag{5.2}$$

for every Borel set $B \subset \omega$ and for every $\varphi \in C^1\left(\overline{\Omega}\right)$ with $\varphi = 0$ in a neighborhood of $\omega \times \partial I$.

Proof. Let φ be as in the statement. We first prove (5.2) when B is an open set. In this case we construct a sequence $\{\psi_k\} \subset C_c^\infty(B; [0,1])$ such that $\psi_k(x) \nearrow 1$ for every $x \in B$ as $k \to \infty$. Since $\psi_k \varphi \in C_c^1(\Omega)$, we have

$$\int_\Omega \psi_k \varphi D_3 b \, dx = -\int_\Omega b D_3 \left(\psi_k \varphi\right) \, dx = -\int_\Omega b \psi_k D_3 \varphi \, dx .$$

Taking the limit as $k \to \infty$, by the Lebesgue dominated convergence theorem we obtain (5.2) when B is an open set. Since both sides of (5.2) are Radon measures with respect to B, the equality on open sets implies (5.2) for every Borel set. ☐

The following proposition will be used to prove Theorem 5.5. For a proof see, e.g., [16].

Proposition 5.4. *Let $A \subset \mathbb{R}^n$, $n \geq 2$, be an open subset with compact boundary and satisfying the segment property and let* Id *be the identity map on $\mathbb{R}^n$. Then for every $\eta > 0$ there exists a $\Phi_\eta \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $\Phi_\eta\left(\overline{A}\right) \subset A$, $\Phi_\eta - \mathrm{Id}$ has compact support in $\mathbb{R}^n$, and $\Phi_\eta - \mathrm{Id} \to 0$ in $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ as $\eta \to 0$.*

The following theorems were used in the proof of Theorem 3.3. Given a set $A \subset \mathbb{R}^2$ and $\eta > 0$, we use the notation

$$A^\eta := \{y \in \omega : \operatorname{dist}(y, A) < \eta\} .$$

Theorem 5.5. *Let $\Omega := \omega \times I$, where $\omega \subset \mathbb{R}^2$ is an open bounded domain with Lipschitz boundary, and let $A \subset \omega$ be an open set satisfying the segment property. Let $u \in W^{1,q}\left(\Omega;\mathbb{R}^3\right)$ be such that $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $A \times I$. Then there exists an extension, still denoted u, which belongs to $W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ and satisfies $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $A \times \mathbb{R}$. Moreover, for every $\eta > 0$ there exists a function $u_\eta \in W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ such $D_3 u_\eta = 0$ $\mathcal{L}^3$ a.e. in $A^\eta \times \mathbb{R}$ and $u_\eta \to u$ strongly in $W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ as $\eta \to 0^+$.*

Proof. Since ω is Lipschitz, we can extend u to a function in $W^{1,q}\left(\mathbb{R}^2 \times I;\mathbb{R}^3\right)$. By a reflection argument, we can further extend it to a function $u \in W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ such that $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $A \times \mathbb{R}$. Let $\Phi_\eta \in C^\infty\left(\mathbb{R}^2;\mathbb{R}\right)$ be given by Proposition 5.4. Since the compact set $\Phi_\eta\left(\overline{A}\right)$ is contained in A, we may find $\delta_\eta > 0$ so small that $\Phi_\eta\left(A^{\delta_\eta}\right) \subset A$. Hence, up to a change in the parameter η, it is not restrictive to assume that $\Phi_\eta(A^\eta) \subset A$. Define $\Psi_\eta(x_1, x_2, x_3) := (\Phi_\eta(x_1, x_2), x_3)$. It is easy to see that $u_\eta := u \circ \Psi_\eta$ has the required properties. □

Theorem 5.6. *Let $\Omega := \omega \times I$, where $\omega \subset \mathbb{R}^2$ is an open bounded domain and let $A \subset \omega$ be an open set with $\mathcal{L}^2(\partial A) = 0$. Let $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ be such that $D_3 b \in L^2\left(A \times I;\mathbb{R}^3\right)$. Then for every $\eta > 0$ there exists an extension, still denoted b, which belongs to $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ and satisfies $D_3 b \in L^2\left(A^\eta \times \mathbb{R};\mathbb{R}^3\right)$. Moreover, there exists a function $b_\eta \in L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ with $D_3 b_\eta \in L^2\left(A^\eta \times \mathbb{R};\mathbb{R}^3\right)$ such $b_\eta \to b$ strongly in $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ and $D_3 b_\eta \to D_3 b$ strongly in $L^2\left(A \times \mathbb{R};\mathbb{R}^3\right)$ as $\eta \to 0^+$.*

Proof. By a reflection argument, we may extend b to a function in $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ such that $D_3 b \in L^2\left(A \times \mathbb{R};\mathbb{R}^3\right)$. Define

$$b_\eta(x) := \begin{cases} 0 & \text{if } x \in (A^\eta \setminus A) \times \mathbb{R}, \\ b(x) & \text{otherwise.} \end{cases}$$

Then $b_\eta \to b$ in $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$. Moreover, by Proposition 5.3, we have

$$D_3 b_\eta(x) = \begin{cases} 0 & \text{if } x \in (A^\eta \setminus A) \times \mathbb{R}, \\ D_3 b(x) & \text{if } x \in A \times \mathbb{R} \end{cases}$$

for $\mathcal{L}^3$ a.e. $x \in A_\eta \times \mathbb{R}$. □

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