HOMOGENIZATION OF FIBER REINFORCED MICROSTRUCTURES: THE EXTREMAL CASES

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Abstract. We analyze the asymptotic behavior of the antiplane deformations of a *fragile* material reinforced by a reticulated *elastic* structure. The microscopic geometry of this material is described by means of two "small" parameters: the size ε of the periodic grid and the ratio δ between the thickness of each of the fibers and their period of distribution. We show that this behavior depends dramatically on the relative size of the parameters. Indeed, in the two considered cases, *i.e.*, $\varepsilon \ll \delta$ and $\varepsilon \gg \delta$, we obtain two different limit models: a perfectly elastic model and an elastic model with macroscopic cracks, respectively.

Keywords: homogenization, free discontinuity, Γ-convergence

2000 Mathematics Subject Classification: 28A20, 35B27, 35B40, 49J45, 73B27, 73M25

Contents

1. Introduction

The aim of this paper is to study the asymptotic behavior of the antiplane deformations of a fragile material reinforced by a reticulated elastic structure (the dark and the white part in Figure 1, respectively). This structure is reinforced by thin unbreakable fibers disposed periodically along two orthogonal directions of the plane. Two parameters are involved: the size ε of the periodic grid and the ratio δ between the thickness of each of the fibers and their period of distribution.

We show how the overall behavior of the structure depends dramatically on the relative size of the parameters. If $\delta/\varepsilon \to +\infty$ the asymptotic behavior is perfectly elastic without cracks. Instead, if $\delta/\varepsilon \to 0$ we obtain in the limit an elastic material with brittle macroscopic cracks.

Let $\Omega \subseteq \mathbb{R}^2$ be the reference configuration of the material. Since we are taking into account the presence of cracks, the natural mathematical setting for our problem is the space

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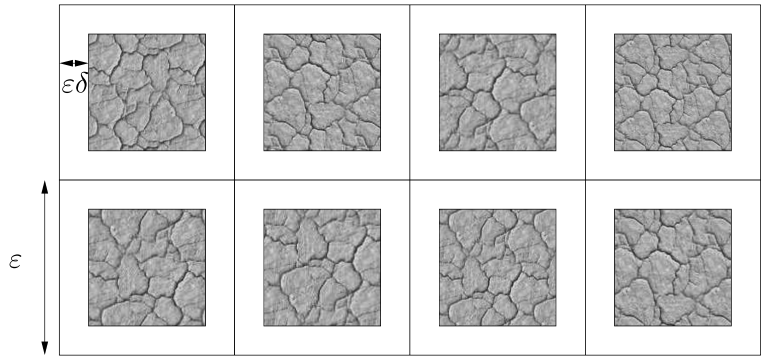

FIGURE 1. A representation of the composite material.

$GSBV(\Omega)$, introduced by De Giorgi and Ambrosio [10], where the asymptotic behavior of the composite can be described by the following family of functionals

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^1(S_u) & \text{if} \quad S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1}$$

where $\Omega_\varepsilon := \Omega \cap \varepsilon\left((\delta, 1-\delta)^2 + \mathbb{Z}^2\right)$ and $\delta = \delta(\varepsilon) \in (0, 1/2)$.

The function u denotes the displacement, while its discontinuity set S_u represents the crack. In addition to the volume term, which is the standard elastic energy in the antiplane case, the expression of the functional F_ε presents a surface term, which accounts for the energy needed to open the crack. The set Ω_ε represents the soft zone where the crack could lie while the set $\Omega \setminus \Omega_\varepsilon$ represents the unbreakable fibers inside the material.

Our purpose is to determine the Γ-limit F_{hom} of F_ε and distinguish the different asymptotic models according to the limit ϑ of $\delta(\varepsilon)/\varepsilon$. More precisely, we analyze the extremal cases

$$\vartheta = +\infty \quad \text{and} \quad \vartheta = 0.$$

In the first case, we show that

$$F_{\text{hom}}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx & \text{if} \quad u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

This functional describes a material without cracks. Indeed, even if at scale ε the material has microscopic cracks, these cannot glue together into a macroscopic one and they have not effect on the limit, since the elastic fibers well separate the brittle regions.

In the second case we prove that

$$F_{\text{hom}}(u) = \int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^1(S_u).$$

This means that, despite the presence of the unbreakable fibers, the collective behavior of microscopic cracks is equivalent in the limit to a macroscopic crack.

In the intermediate case $\vartheta \in (0, +\infty)$, under the assumption that the Γ-limit has an integral representation, we obtain that the surface term depends also on the size $[u]$ of the jump.

Remark 1.1. Homogenization in SBV setting has been developed in previous works [5, 3]. Anyway, these classical results do not apply to our particular case, because the surface energy in (1.1) does not satisfy the required hypotheses.

Remark 1.2. Another interesting situation occurs when the parameter δ is fixed and independent of ε and the family of functionals has the form

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx + a(\varepsilon)\mathcal{H}^1(S_u) & \text{if} \quad S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

where $a(\varepsilon) \searrow 0^+$ as $\varepsilon \searrow 0^+$. Recently, in [11, 12] a model of damage has been deduced from this case.

2. Problem setting

In the following we assume that Ω is a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary. We recall here some notation about the space of special functions of bounded variation on Ω, briefly $SBV(\Omega)$, and we refer to [1] for the definitions and the standard theory.

If $u \in SBV(\Omega)$,

- ∇u is the approximate gradient of u;
- S_u is the approximate discontinuity set of u;
- ν_u is the generalized normal to S_u;
- $u^\pm$ is the traces of u on both sides of S_u;
- $[u] := |u^+ - u^-|$ is the size of the jump.

The same notations are used when $u \in GSBV(\Omega)$, the space of generalized special functions of bounded variation on Ω. We recall that $GSBV(\Omega)$ is made of all measurable functions $u : \Omega \to \mathbb{R}$ such that $(u \wedge m) \vee (-m) \in SBV(U)$ for every $m \in \mathbb{N}$ and every open subset $U \subset\subset \Omega$.

Other spaces that we will frequently use are

$$SBV^2(\Omega) := \left\{ u \in SBV(\Omega) \ : \ \nabla u \in L^2(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^1(S_u) < +\infty \right\};$$
$$\mathcal{U}^p(\Omega) := \left\{ u \in GSBV(\Omega) \cap L^p(\Omega) \ : \ \nabla u \in L^2(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^1(S_u) < +\infty \right\},$$

where $p \in [1, +\infty)$. It is easy to see that $\mathcal{U}^p(\Omega) \cap L^\infty(\Omega) \subseteq SBV^2(\Omega)$.

We say that a sequence $u_k \subset SBV^2(\Omega)$ converges weakly to some $u \in SBV^2(\Omega)$, and we write $u_k \rightharpoonup u$ in $SBV^2(\Omega)$, if

$$\begin{cases} u_k \text{ is bounded in } L^\infty(\Omega); \\ u_k \to u \text{ strongly in } L^1(\Omega); \\ \nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega, \mathbb{R}^2); \\ \sup\limits_{k \in \mathbb{N}} \mathcal{H}^1(S_{u_k}) < +\infty. \end{cases}$$

We define the Mumford-Shah functional $MS : L^p(\Omega) \to [0, +\infty]$ as

$$MS(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^1(S_u) & \text{if} \quad u \in \mathcal{U}^p(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We make explicit the dependence on the domain Ω with the notation $MS(\cdot, \Omega)$.

Finally, we consider the family of functionals $F_\varepsilon : L^p(\Omega) \to [0,+\infty]$ defined by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^1(S_u) & \text{if} \quad u \in \mathcal{U}^p(\Omega) \text{ and } S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\Omega_\varepsilon := \Omega \cap \varepsilon\left((\delta, 1-\delta)^2 + \mathbb{Z}^2\right)$ and $\delta = \delta(\varepsilon) \in (0, 1/2)$.

The goal of this paper is to analyze the Γ-limit of the family F_ε in the space $L^p(\Omega)$ endowed with the strong topology. We refer to [4] for the definition and properties of Γ-convergence. In Theorem 4.1 and Theorem 5.1 we show that the Γ-limit varies depending on $\vartheta := \lim \delta(\varepsilon)/\varepsilon$.

Throughout the paper, given a subset A of $\mathbb{R}^2$, we employ the following notations

- χ_A is the characteristic function of A;
- $B_\eta(A) := \{x \in \mathbb{R}^2 \ : \ \mathrm{dist}(x, A) < \eta\}$.

3. A lower bound estimate

The aim of this section is to provide a lower bound for the Γ-limit of the family F_ε.

Theorem 3.1. *Assume $\Omega := (-1/2, 1/2)^2$ and denote by $\mathbb{S}^1$ the unitary circle in $\mathbb{R}^2$. Given $t \geq 0$ and $\nu \in \mathbb{S}^1$, we define the function $u_{t,\nu} : \Omega \to \mathbb{R}$ by*

$$u_{t,\nu} := t\chi_{\{x \in \Omega \,:\, x\cdot\nu \leq 0\}}.$$

Then, for any sequence $\varepsilon_k \searrow 0^+$ and any sequence u_k in $L^p(\Omega)$ such that $u_k \to u_{t,\nu}$ strongly in $L^p(\Omega)$, we have the estimate

$$\liminf_{k \to +\infty} F_{\varepsilon_k}(u_k) \geq 10^{-6}\sqrt{\vartheta}\, t. \tag{3.1}$$

In the proof of Theorem 3.1 we will use the following two lemmas. The first is a well known trick that allows us to modify a sequence u_k keeping the limit. The second is an ad hoc adaptation of the argument used in [9, Subsection 4.2]. For the readers convenience, we prefer to present here a simplified proof with the suitable modifications.

Lemma 3.2. *Let B be a Borel subset of $(0,1)^2$ with $c := \mathcal{L}^2(B) > 0$ and let $\varepsilon_k \searrow 0^+$. Given two sequences u_k and v_k in $L^1(\Omega)$ such that $u_k \to u$ and $v_k \to v$ strongly in $L^1(\Omega)$, suppose that*

$$u_k = v_k \quad \text{in} \quad B_k := \Omega \cap \varepsilon_k(B + \mathbb{Z}^2).$$

Then $u = v$.

Proof. By the Riemann-Lebesgue lemma, χ_{B_h} tends to c weakly* in $L^\infty(\Omega)$. Since the sequence $(u_k - v_k)\chi_{B_k}$ converges in distribution to $c(u - v)$, for any $\varphi \in C_c^\infty(\Omega)$ we have

$$c\int_\Omega \varphi(u - v)dx = \int_\Omega \varphi(u_k - v_k)\chi_{B_k} dx = 0.$$

Since φ is arbitrary and c is strictly positive, we can conclude that $u = v$. □

Patching Lemma. Let $U := (0,1) \times (-1/2, 1/2)$. Then for any $u \in SBV^2(U) \cap L^\infty(U)$ with $\mathcal{H}^1(S_u) \leq 1/28$, there exists $v \in SBV^2(U) \cap L^\infty(U)$ such that

(i) v is constant in $(0,1) \times (-2/7, 2/7)$;
(ii) $v = u$ in $(0,1) \times [(-1/2, -3/7) \cup (3/7, 1/2)]$;

(iii) $\|v\|_{L^\infty(U)} \leq \|u\|_{L^\infty(U)}$;
(iv) $\|\nabla v\|_{L^2(U,\mathbb{R}^2)} \leq 10^3 \|\nabla u\|_{L^2(U,\mathbb{R}^2)}$;
(v) $S_v \subseteq S_u$;
(vi) if u is $(0,1)$-periodic in the first variable, then v has the same property.

Proof. The plan of the proof is the following. In the first step we carry out a suitable truncation of u so as to control $\|u\|_{L^\infty}$ with $\|\nabla u\|_{L^1}$. In the second step we obtain (i) thank to a simple cut-off argument. We remark that we need the first step because in the SBV setting Poincaré type inequalities in general do not work.

Step 1: Truncation. Since we are going to use the slicing procedure, we need to fix the precise representative u^* of u as defined in [1, Definition 3.63 and Corollary 3.80]. We make use of the following notations:

- $L_r := (0,1) \times \{r\}$ and $L^r := \{r\} \times (-1/2, 1/2)$, $r \in \mathbb{R}$;
- $I := \big\{r \in (5/14, 3/7) \,:\, L_r \cap S_{u^*} = \emptyset\big\}$;
- $J := \big\{r \in (-3/7, -5/4) \,:\, L_r \cap S_{u^*} = \emptyset\big\}$;
- $H := \big\{r \in (0,1) \,:\, L^r \cap S_{u^*} = \emptyset\big\}$;
- $\mathrm{osc}_r(u^*) := \sup_{L_r} u^* - \inf_{L_r} u^*$ the oscillation of u^* along L_r, $r \in (-1/2, 1/2)$;
- $\mathrm{osc}^r(u^*) := \sup_{L^r} u^* - \inf_{L^r} u^*$ the oscillation of u^* along L^r, $r \in (0,1)$.

We have $\mathcal{L}^1(I)$, $\mathcal{L}^1(J)$ and $\mathcal{L}^1(H) \geq 1/28$. Moreover, by [1, Theorems 3.28, 3.107 and 3.108], for $\mathcal{L}^1$ *a.e.* $r \in I$ $u^*(\cdot, r)$ is absolutely continuous with derivative given $\mathcal{L}^1$ *a.e.* by $\partial_{x_1} u^*(\cdot, r)$. Then

$$\mathrm{osc}_r(u^*) \leq \int_0^1 |\partial_{x_1} u^*(x_1, r)|\, dx_1$$

and so, integrating over I, we obtain $\int_I \mathrm{osc}_r(u^*) dr \leq \|\nabla u\|_{L^1(U,\mathbb{R}^2)}$. By the Mean Value Theorem, there exists $r_1 \in (5/14, 3/7)$ such that

$$\mathrm{osc}_{r_1}(u^*) \leq 28 \|\nabla u\|_{L^1(U,\mathbb{R}^2)}\,.$$

Similarly, it is possible to prove that there exist $r_2 \in (-3/7, -5/4)$ and $r_3 \in (0,1)$ such that $\mathrm{osc}_{r_1}(u^*)$ and $\mathrm{osc}^{r_3}(u^*)$ are smaller than $28 \|\nabla u\|_{L^1(U,\mathbb{R}^2)}$.

Since $L := L_{r_1} \cup L_{r_2} \cup L^{r_3}$ is a connected set, if $m_{u^*} := (\sup_L u^* + \inf_L u^*)/2$, we have that

$$|u^*(x) - m_{u^*}| \leq c_{u^*} \quad \text{for } \mathcal{H}^1 \, a.e. \ x \in L,$$

where $c_{u^*} := 42 \|\nabla u\|_{L^1(U,\mathbb{R}^2)}$.

In this way we can truncate the function u^* on $(0,1) \times [r_2, r_1]$ (the light grey part of Figure 2) without generate new fracture along L_{r_1} and L_{r_2}:

$$w := \begin{cases} [u^* \wedge (m_{u^*} + c_{u^*})] \vee (m_{u^*} - c_{u^*}) & \text{in } (0,1) \times [r_2, r_1] \\ u^* & \text{in } (0,1) \times [(-1/2, r_2) \cup (r_1, 1/2)]. \end{cases}$$

Obviously w satisfies (ii)-(vi).

Step 2: Cut-Off. Let $\phi \in C^1((1/2, 1/2), [0,1])$ be a cut-off function such that

$$\phi(t) := \begin{cases} 0 & \text{if } t \in (-2/7, 2/7) \\ 1 & \text{if } t \in (-1/2, -5/14) \cup (5/14, 1/2), \end{cases}$$

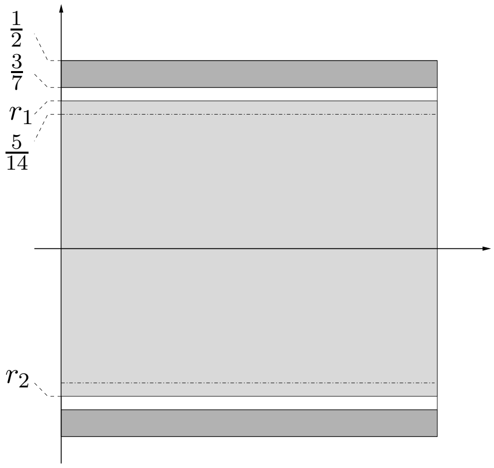

FIGURE 2

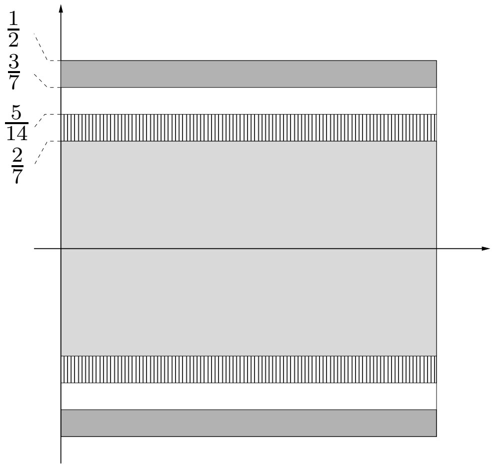

FIGURE 3

and $|\phi'(t)| \leq 15$. If we define the function v on U as $v(x) := \phi(x_2)[w(x) - m_{u^*}] + m_{u^*}$, by construction v satisfies (i)-(iii), (v) and (vi). In particular v is constant on $(0,1) \times (-2/7, 2/7)$ (the light grey part of Figure 3).

Let $\widetilde{U} := (0,1) \times [(-5/14, -2/7) \cup (2/7, 5/14)]$. Since $\nabla v = \phi \nabla w + \phi'(w - m_{u^*})$ and

$$\begin{aligned}\left\|\phi'(w - m_{u^*})\right\|_{L^\infty(U)} &\leq 15 \sup_{\widetilde{U}} |w - m_{u^*}| \\ &\leq 630 \left\|\nabla v\right\|_{L^1(U,\mathbb{R}^2)} \leq 630 \left\|\nabla v\right\|_{L^2(U,\mathbb{R}^2)},\end{aligned}$$

we have that v satisfies also (iv):

$$\|\nabla v\|_{L^2(U,\mathbb{R}^2)} \leq 631 \|\nabla u\|_{L^2(U,\mathbb{R}^2)}.$$

□

Proof of Theorem 3.1. Let $\Theta := \liminf_k F_{\varepsilon_k}(u_k)$. If Θ is finite, we can assume $u_k \in \mathcal{U}^p(\Omega)$ and $MS(u_k)$ bounded. By truncating u_k between 0 and t, we can also assume $u_k \in SBV^2(\Omega) \cap L^\infty(\Omega)$ and $\|u_k\|_{L^\infty(\Omega)}$ bounded. Thanks to the compactness result in $SBV(\Omega)$ stated in [1, Theorem 4.8], we have that u_k converges weakly to $u_{t,\nu}$ in $SBV^2(\Omega)$ and $\lim_k MS(u_k) = \Theta$.

The plan of the proof is to handily modify both u and u_k keeping a control on the energies. We split the argument in two steps. We assume initially that

$$\varepsilon_k^{-1} \text{ is an even integer.} \tag{3.2}$$

Step 1: Symmetrization. With suitable ninety degree rotations, we can suppose that $\nu = (\cos\gamma, \sin\gamma)$ with $\gamma \in [\pi/4, 3\pi/4]$. Starting from u_k we will construct a new sequence u_k^{sym} having more symmetry and converging to a function u^{sym} with an horizontal crack.

We fix a small $\eta > 0$ and we choose a sequence m_k in $\mathbb{N}$ such that $m_k \nearrow +\infty$ and

$$m_k \int_\Omega |u - u_k|^p \, dx \leq \eta. \tag{3.3}$$

For any integer $j \in \{-\varepsilon_k^{-1}, \dots, \varepsilon_k^{-1} - 1\}$ we consider in $SBV^2(\Omega, \mathbb{R}^2)$ the functional

$$F_j^k(v^{(1)}, v^{(2)}) := MS(v^{(2)}, X_j^k) + m_k \int_{X_j^k} \left|v^{(1)} - v^{(2)}\right|^p dx,$$

where X_j^k denotes the vertical strip $(j\,\varepsilon_k/2, (j+1)\varepsilon_k/2) \times (-1/2, 1/2)$.

For any $k \in \mathbb{N}$ let j_k be an integer realizing

$$F_{j_k}^k(u, u_k) = \min\left\{F_j^k(u, u_k) \;:\; j \in \{-\varepsilon_k^{-1}, \dots, \varepsilon_k^{-1} - 1\}\right\}$$

and let u_k^{sym} (resp. w_k) be the extension of $u_k|_{X_{j_k}^k}$ (resp. $u_{t,\nu}|_{X_{j_k}^k}$) to Ω obtained with horizontal reflections. Notice that $S_{u_k^{sym}} \subseteq \varepsilon_k \left((\delta_k, 1 - \delta_k)^2 + \mathbb{Z}^2\right)$ and that

$$u_k(x) = u_k(x_1 + \varepsilon_k, x_2) \quad \forall x \in \Omega \text{ and } \forall k \in \mathbb{N}. \tag{3.4}$$

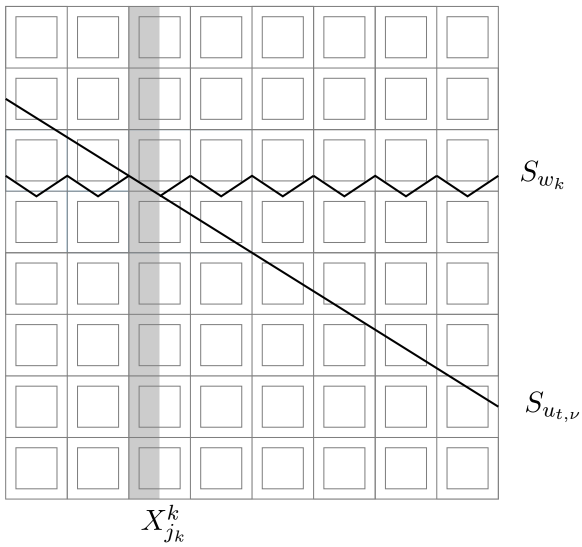

FIGURE 4

Obviously we have $MS(w_k, \Omega) = MS(u_{t,\nu}, \Omega)$. Moreover, from the minimality on $X_{j_k}^k$ we deduce that

$$MS(u_k^{sym}, \Omega) \leq MS(u_k, \Omega) + \eta.$$

Since w_k and u_k^{sym} are also bounded in $L^\infty(\Omega)$, thanks again to the compactness result in $SBV(\Omega)$, there exist w and u^{sym} in $SBV^2(\Omega)$ such that (up to subsequences not relabeled) $w_k \rightharpoonup w$ and $u_k^{sym} \rightharpoonup u^{sym}$ weakly in $SVB^2(\Omega)$. Again from the minimality on $X_{h_k}^k$ and (3.3), we deduce that

$$\sup_{k \in \mathbb{N}} \left\{ m_k \left\|w_k - u_k^{sym}\right\|_{L^p(\Omega)}^p \right\} < +\infty$$

and therefore $w = u^{sym}$. Let now $g_k : (-1/2, 1/2) \to (-1/2, 1/2)$ be a Lipschitz map such that $\mathrm{Graf} g_k = S_{w_k}$. The oscillation of g_k is bounded by $\varepsilon_k / \left|\cos\gamma\right|$ and therefore (up to subsequences not relabeled) g_k converges uniformly to a certain constant β. If $\gamma \in (\pi/4, 3\pi/4)$, then $\beta \in (-1/2, 1/2)$. Instead, if $\gamma = \pi/4$ or $3\pi/4$, with the previous construction we could have $\beta = \pm 1/2$. To avoid this possibility, when $\gamma = \pi/4$ or $3\pi/4$ we change a little the construction of u_k^{sym}. Recalling that ε_k^{-1} is an even integer, we choose the vertical strips X_j^k only in $(-1/4,1/4)\times(-1/2,1/2)$ so that $\beta\in(-1/4,1/4)$. In this case we get

$$MS(u_k^{sym},\Omega)\le 2MS(u_k,\Omega)+2\eta.$$

Given $\xi>0$, for k sufficiently large

$$w_k(x)=\begin{cases}0 & \text{if } x\in(-1/2,1/2)\times[\beta+\xi,1/2)\\ t & \text{if } x\in(-1/2,1/2)\times(-1/2,\beta-\xi],\end{cases}$$

therefore $u^{sym}=t\chi_{(-1/2,1/2)\times(-1/2,\beta)}$.

Step 2: Regularization. Here the idea is to smooth u_k^{sym} on those cubes where $S_{u_k^{sym}}$ is "small" by using the Patching Lemma.

For any $i,j\in I_k:=\{-\varepsilon_k^{-1}/2,\dots,\varepsilon_k^{-1}/2-1\}$ we define the following sets:

- $Q_{i,j}:=\varepsilon_k(0,1)^2+(i\varepsilon_k,j\varepsilon_k)$;
- $Q'_{i,j}:=\varepsilon_k\left((0,1)\times(1/14,13/14)\right)+(i\varepsilon_k,j\varepsilon_k)$;
- $Q''_{i,j}:=\varepsilon_k\left((0,1)\times(3/14,11/14)\right)+(i\varepsilon_k,j\varepsilon_k)$;
- $T_j:=(0,1)\times(j\varepsilon_k,(j+1)\varepsilon_k)$.

Notice that by the periodicity property (3.4), if

$$\mathcal{H}^1(S_{u_k^{sym}}\cap Q_{i,j})<\varepsilon_k/28 \tag{3.5}$$

for a certain (i,j), then $\mathcal{H}^1(S_{u_k^{sym}}\cap Q_{h,j})<\varepsilon_k/28$ for all $h\in I_k$, i.e., condition (3.5) is satisfied by all the cubes in the strip T_j. Let J_k be the set of the indices $j\in\{1-\varepsilon_k^{-1},\dots,\varepsilon_k^{-1}-2\}$ such that all the cubes in the strips T_{j-1}, T_j and T_{j+1} satisfy (3.5).

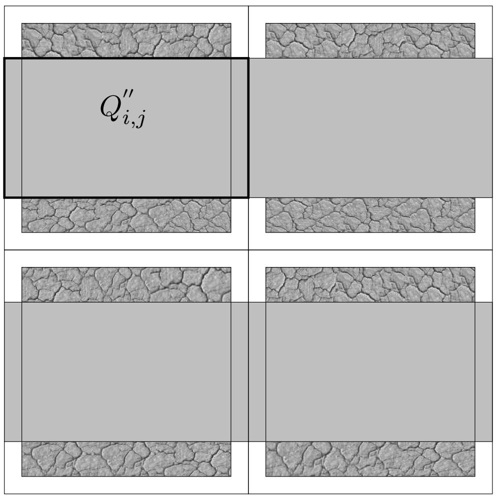

FIGURE 5

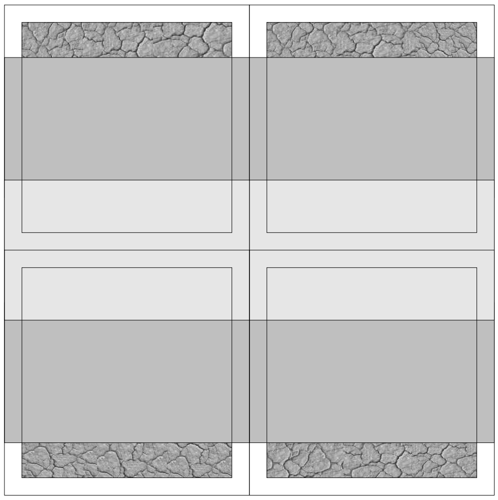

FIGURE 6

By a rescaling argument, thanks to the Patching Lemma, inside at every cube $Q_{i,j}$ satisfying (3.5), we can change u_k^{sym} on $Q'_{i,j}$ so that it becomes smooth (because constant) in $Q''_{i,j}$ (see Figure 5). We remark in particular that by keeping the periodic condition (3.4), we did not created new cracks along the vertical boundary of the cubes $Q_{i,j}$. Moreover, since the sequence is unchanged on $Q_{i,j}\setminus Q'_{i,j}$, by Lemma 3.2 the limit remains u^{sym}.

To erase the crack also in $Q_{i,j}\setminus Q''_{i,j}$ for $i\in J_k$, we proceed in a similar way, by using the Patching Lemma to modify u_k so that it becomes constant on the light grey part of

Figures 6 and 7. We denote u_k^{reg} the sequence obtained regularizing u_k^{sym} through the above two modifications. By construction u_k^{reg} converges weakly to u^{sym} in $SBV^2(\Omega)$, $S_{u_k^{reg}} \subseteq \varepsilon_k \left((\delta_k, 1-\delta_k)^2 + \mathbb{Z}^2\right)$ and

$$MS(u_k^{reg}, \Omega) \leq 10^6 MS(u_k^{sym}, \Omega).$$

Moreover, for any $j \in J_k$ the function u_k^{reg} belongs to $H^1(T_j)$ and it is constant along the horizontal boundary of the strip T_j.

Let now $v_k \in H^1(\Omega)$ be a function with zero average such that $\nabla v_k = \nabla u_k^{reg}$ on the strips T_j when $j \in J_k$ and $\nabla v_k = 0$ otherwise. Up to a subsequence, v_k converges weakly to a certain v in $H^1(\Omega)$. We denote by $\tilde{u}_k := u_k^{reg} - v_k$ the sequence obtained by flattening u_k^{reg} on the strips T_j, $j \in J_k$. We have that $\tilde{u}_k$ converges weakly to $\tilde{u} := u^{sym} - v$ in $SBV^2(\Omega)$. Moreover $S_{\tilde{u}} = S_{u^{sym}}$ and $[\tilde{u}] = t$ on $S_{\tilde{u}}$.

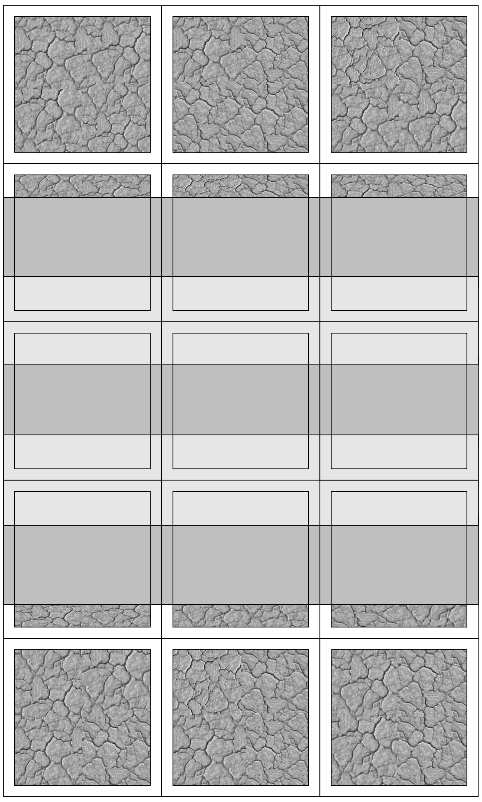

FIGURE 7

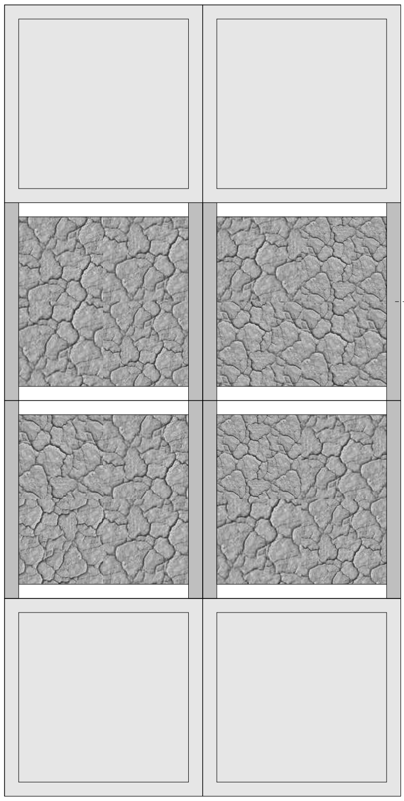

FIGURE 8

Let $m_k^1 := \min J_k$ (resp. $m_k^2 := \max J_k$) and c_k^1 (resp. c_k^2) be the constant value of $\tilde{u}_k$ on $x_2 = \varepsilon_k m_k^1$ (resp. $x_2 = \varepsilon_k m_k^2$). Since $T_{m_k^1} \subseteq (-1/2, 1/2) \times (-1/2, \beta)$ (resp. $T_{m_k^2} \subseteq (-1/2, 1/2) \times (\beta, 1/2)$) for k sufficiently large, we have $\left|c_k^2 - c_k^1\right| > t/2$. Consider now the set

$$U_k := \bigcup_{i \in I_k} \bigcup_{j \in I_k^c} \varepsilon_k \left[((0, \delta_k) \cup (1-\delta_k, 1)) \times (0,1)\right] + (i\varepsilon_k, j\varepsilon_k),$$

where $J_k^c := \{j \in I_k : j \notin J_k \text{ and } m_k^1 < j < m_k^2\}$. The set U_k is constituted by the vertical frame of the cubes belonging at the strips T_j, $j \in J_k^c$ (the dark grey vertical zones of Figure 8).

Denoted by n_k the number of strips T_j where the cubes $Q_{i,j}$ do not satisfy (3.5), we trivially have the estimate $\mathcal{H}^1(S_{\tilde{u}_k}) \geq 28n_k$. Moreover, noted that J_k^c has at most $3n_k$ elements, since $\left|c_k^2 - c_k^1\right| > t/2$ and $\tilde{u}_k$ is constant on the strips T_j when $j \in J_k$, by comparing $\tilde{u}_k$ with the affine junction on U_k (the lower volume energy configuration on U_k), we get

$$\int_{U_k} |\tilde{u}_k|^2\, dx \geq \frac{\vartheta\, t^2}{6n_k}.$$

Gathering all the previous estimates, we obtain

$$\begin{aligned}\Theta + \eta &\geq \liminf_{k\to+\infty} \frac{1}{2} MS(u_k^{sym}, \Omega) \geq \frac{10^{-6}}{2} \liminf_{k\to+\infty} MS(u_k^{reg}, \Omega)\\ &\geq \frac{10^{-6}}{2} \liminf_{k\to+\infty} MS(\tilde{u}_k, \Omega) \geq \frac{10^{-6}}{2} \liminf_{k\to+\infty} \Big(\frac{\vartheta\, t^2}{6n_k} + 28n_k\Big) \geq 10^{-6}\sqrt{\vartheta}\, t,\end{aligned}$$

where the last inequality is obtained by minimizing with respect to n_k. Being $\eta > 0$ arbitrary, the previous inequality gives estimate (3.1).

In order to remove assumption (3.2) we consider, for $k \in \mathbb{N}$, the largest even integer m_k such that $m_k \leq \varepsilon_k^{-1}$. Let $s_k := m_k\varepsilon_k$. We define $\hat{u}_k : \Omega \to \mathbb{R}$ by

$$\hat{u}_k(x) := u_k(s_k\, x)/\sqrt{s_k}.$$

Since s_k converges to 1, we have that $\hat{u}_k$ tends to $u_{t,\nu}$ strongly in $L^p(\Omega)$. Moreover $S_{\hat{u}_k} \subseteq m_k^{-1}\left((\delta_k, 1-\delta_k)^2 + \mathbb{Z}^2\right)$ and $\lim_k \delta_k m_k = \vartheta$. By applying the previous result we get

$$\liminf_{k\to+\infty} F(u_k) \geq \liminf_{k\to+\infty} s_k MS(\hat{u}_k) \geq 10^{-6}\sqrt{\vartheta}\, t.$$

□

Remark 3.3. Given a sequence $\varepsilon_k \searrow 0^+$, by a well known compactness argument (see [4, Proposition 2.14]), there exists a subsequence ε_{k_j} such that $F_{\varepsilon_{k_j}}$ Γ-converges to some functional $F : L^p(\Omega) \to [0, +\infty]$. When $\vartheta \in (0, +\infty)$, it is proved in [8] that the functional F admits an integral representation: there exists a Borel function $g : [0, +\infty) \times \mathbb{S}^1 \to [0, +\infty)$ such that

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\, dx + \int_{S_u} g\big([u], \nu_u\big)\mathcal{H}^1 & \text{if} \quad u \in \mathcal{U}^p(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Estimate (3.1) shows that $g = g(t, \nu)$ effectively depends on the variable t. In fact, fixed $\nu \in \mathbb{S}^1$ and denoted by l the length of $S_{u_{1,\nu}}$, for t sufficiently large we have

$$g(t, \nu)l = F(u_{t,\nu}) \geq 10^{-6}\sqrt{\vartheta}\, t > g(1, \nu)l.$$

4. The perfectly elastic case

The main result of the present section is the following.

Theorem 4.1. *Let F_ε be defined as in (2.1). If $\vartheta = +\infty$, then the family F_ε Γ-converges to*

$$F_{\text{hom}}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx & \text{if} \quad u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let $\varepsilon_k \searrow 0^+$ and let u_k be a sequence in $\mathcal{U}^p(\Omega)$ such that u_k tends to u strongly in $L^p(\Omega)$ and $\limsup_k F_{\varepsilon_k}(u_k)$ is finite. We want to show that $u \in H^1(\Omega)$. We start by assuming in addition that

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(\Omega)} < +\infty \tag{4.1}$$

and, consequently, $u_k \in SBV^2(\Omega)\cap L^\infty(\Omega)$. Thanks to the compactness result in $SBV(\Omega)$ stated in [1, Theorem 4.8], we have that $u \in SBV^2(\Omega)$ and u_k converges weakly to u in $SBV^2(\Omega)$. To prove that $u \notin SBV^2(\Omega)\setminus H^1(\Omega)$ we proceed by contradiction, by assuming that

$$\mathcal{H}^1(S_u) > 0.$$

The idea is to carry out a blow up around a suitable jump point of u, so to be in position to use Theorem 3.1.

Since the sequence of measures $\mathcal{H}^1\lfloor_{S_{u_k}}$ is bounded, up to a subsequence (not relabeled) we have that

$$\mu_k := \mathcal{H}^1\lfloor_{S_{u_k}} \rightharpoonup \mu \quad \text{weakly* in the sense of measures}$$

for some finite non-negative Radon measure μ. For any $x \in \Omega$ we define

$$l(x) := \limsup_{r\to 0^+} \frac{\mu\big(B_r(x)\big)}{r}.$$

By applying [1, Theorem 2.56], if $l(x) = +\infty$ on a Borel set B with $\mathcal{H}^1(B) > 0$, then necessarily $\mu(B) = +\infty$, in contradiction with $\mu(\Omega) < +\infty$. Therefore $l(x) < +\infty$ for $\mathcal{H}^1$ *a.e.* $x \in \Omega$.

Let $\overline{x}$ be an approximate jump point for u such that $l(\overline{x}) < +\infty$ holds. Given $r > 0$ sufficiently small so that $B_r(\overline{x})$ is included in Ω, we define on $B_1(0)$ the following real functions

- $u_k^r(x) := u_k(\overline{x} + rx)$;
- $u^r(x) := u(\overline{x} + rx)$;
- $\tilde{u} := \chi_{\{x\in B_1(0)\,:\,x\cdot\nu_u(\overline{x})\le 0\}}$.

With respect the strong topology of $L^p(B_1(0))$, we have that u_k^r tends to u^r as $k \nearrow +\infty$ and (see [1, Remark 3.72]) u^r tends to $\tilde{u}$ as $r \searrow 0^+$. Moreover we have that

$$\lim_{k\to+\infty} MS\big(u_k^r, B_1(0)\big) = \int_{B_r(\overline{x})} |\nabla u(x)|^2\,dx + \frac{\mu(B_r(\overline{x}))}{r}.$$

Let $r_h \searrow 0^+$ be a sequence such that $l(\overline{x})$ is the limit of $\mu(B_{r_h}(\overline{x}))/r_h$ for $h \nearrow +\infty$. Setted $\delta_k := \delta(\varepsilon_k)$, by applying a diagonalization argument (see [2, Corollary 1.18]) to the double indexed sequence $(u_k^{r_h}, MS(u_k^{r_h}, B_1(0)), \frac{\varepsilon_k}{\delta_k r_h})$, we can find a sequence $h_k \nearrow +\infty$ such that

$$u_k^{r_{h_k}} \xrightarrow{L^p} \tilde{u}, \quad MS\big(u_k^{r_{h_k}}, B_1(0)\big) \to l(\overline{x}) \quad \text{and} \quad \frac{\varepsilon_k}{\delta_k r_{h_k}} \searrow 0^+.$$

Notice that the last condition implies also that $\tilde{\varepsilon}_k := \varepsilon_k / r_{h_k} \searrow 0^+$.

Let $\widetilde{\Omega} := (-1/2, 1/2)^2$ and let V be the set of vertices of $(0,1)^2$. Chosen for any $k \in \mathbb{N}$ a point $\overline{x}_k$ in $\Omega \cap \varepsilon_k(V + \mathbb{Z}^2)$ with minimal distance from $\overline{x}$, we define $\tilde{u}_k : \widetilde{\Omega} \to \mathbb{R}$ by

$$\tilde{u}_k(x) := u_k^{r_{h_k}}\Big(x - \frac{\overline{x} - \overline{x}_k}{r_{h_k}}\Big).$$

The definition of $\tilde{u}_k$ is well posed: $(\overline{x} - \overline{x}_k)/r_{h_k}$ tends to zero as $k \nearrow +\infty$ and so, for k sufficiently large, the cube with side r_{h_k} centered in $\overline{x}_k$ is included in $B_{r_{h_k}}(\overline{x})$. Moreover, we have that $\tilde{u}_k$ tends to $\tilde{u}$ strongly in $L^p(\widetilde{\Omega})$, $MS(\tilde{u}_k, \widetilde{\Omega})$ remains bounded and $S_{\tilde{u}_k} \subseteq \tilde{\varepsilon}_k \big((\delta_k, 1-\delta_k)^2 + \mathbb{Z}^2\big)$. Then, since $\lim_k \delta_k / \tilde{\varepsilon}_k = +\infty$, we have a contradiction with estimate (3.1).

In order to remove assumption (4.1) we consider, for $m \in \mathbb{N}$, the truncations of the functions u and u_k:

- $u_k^m := (u_k \wedge m) \vee (-m)$;
- $u^m := (u \wedge m) \vee (-m)$.

By applying the previous result, we have $u^m \in H^1(\Omega)$. Moreover

$$\begin{aligned}\|u^m\|_{L^p(\Omega)} + \|\nabla u^m\|_{L^2(\Omega,\mathbb{R}^2)} &\le \liminf_{k\to+\infty}\big(\|u_k^m\|_{L^p(\Omega)} + \|\nabla u_k^m\|_{L^2(\Omega,\mathbb{R}^2)}\big)\\ &\le \liminf_{k\to+\infty}\big(\|u_k\|_{L^p(\Omega)} + \|\nabla u_k\|_{L^2(\Omega,\mathbb{R}^2)}\big) < +\infty.\end{aligned}$$

As Ω has Lipschitz boundary, $\|\cdot\|_{L^p(\Omega)} + \|\nabla\cdot\|_{L^2(\Omega,\mathbb{R}^2)}$ is an equivalent norm in $H^1(\Omega)$ and therefore, up to a subsequence, u^m is weakly convergent in $H^1(\Omega)$ to some $\hat{u} \in H^1(\Omega)$. Since u^m converges pointwise *a.e.* to $\hat{u}$, necessarily $u = \hat{u}$. □

5. The brittle case

The main result of the present section is the following.

Theorem 5.1. *Let F_ε be defined as in* (2.1). *If $\vartheta = 0$, then the family F_ε Γ-converges to MS.*

We shall use the following two lemmas about SBV functions. The first is an extension result, while the second is an approximation argument.

Lemma 5.2. (See [6, Theorem 3.1]). *Let $U \subseteq \mathbb{R}^2$ be a bounded open set and assume that $\Omega \subset\subset U$. Then there exists a constant $b = b(\Omega, U) > 0$ and an extension operator $T : SBV^2(\Omega) \cap L^\infty(\Omega) \to SBV^2(U) \cap L^\infty(U)$ such that*

(i) *$Tu = u$ a.e. in Ω;*
(ii) *$\|Tu\|_{L^\infty(U)} \le b\,\|u\|_{L^\infty(\Omega)}$;*
(iii) *$MS(Tu, U) \le b\,MS(u, \Omega)$.*

Lemma 5.3. (See [7, Corollary 3.11]). *Let $U \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and let $u \in SBV^2(U) \cap L^\infty(U)$. For every $\eta > 0$, there exists a function $v \in SBV^2(U) \cap L^\infty(U)$ such that*

(i) *S_v is essentially closed, i.e., $\mathcal{H}^1(\overline{S}_v \setminus S_v) = 0$;*
(ii) *$\overline{S}_v$ coincides with the intersection of U with the union of a finite number of disjoint segments;*

(iii) $v \in W^{1,\infty}(U \setminus \overline{S}_v)$;
(iv) $\|u - v\|_{L^p(U)} < \eta$;
(v) $\|\nabla u - \nabla v\|_{L^2(U,\mathbb{R}^2)} < \eta$;
(vi) $\left|\mathcal{H}^1(S_u) - \mathcal{H}^1(S_v)\right| < \eta$.

Within this section, we shall use the compact notation $\mathcal{W}(U)$ to denote the space of all measurable functions for which conditions (i), (ii), (iii) hold.

Proof of Theorem 4.1. The liminf inequality is a straight consequence of [1, Theorem 4.36], so we only need to construct a recovery sequence u_k for any $u \in \mathcal{U}^p(\Omega)$. Thanks to a truncation argument we can suppose that $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Moreover, fixed a small $\eta > 0$, thanks to Lemmas 5.2 and 5.3, we can suppose that u is the restriction to Ω of a function $v \in \mathcal{W}(B_{2\eta}(\Omega))$. Accordingly to the definition of $\mathcal{W}(B_{2\eta}(\Omega))$, there exist $S^1, \dots, S^l$ disjoint segments such that $\bigcup_{i=1}^l S^i = \overline{S}_v$.

Let $\varepsilon_k \searrow 0^+$ and let $\delta_k := \delta(\varepsilon_k)$. To construct a recovery sequence u_k the idea is to smooth the function u in the unbreakable zone $\Omega \setminus \Omega_{\varepsilon_k}$. To obtain this easily, we would need that the extreme points of $\overline{S}_u$ do not fall in $\Omega \setminus \Omega_{\varepsilon_k}$ and that large vertical (resp. horizontal) portions of S_u do not fall in $B_{\varepsilon_k\delta_k}(\varepsilon_k\mathbb{Z} \times \mathbb{R})$ (resp. $B_{\varepsilon_k\delta_k}(\mathbb{R} \times \varepsilon_k\mathbb{Z})$). We can obviate with a shift of u. Let $w := v|_{B_\eta(\Omega)}$. Since $\delta_k \searrow 0^+$, we can find a sequence $t_k \searrow 0^+$ in $[0, \eta)$ such that

- the extreme points of $\overline{S}_w + (t_k, t_k)$ are in $\varepsilon_k((\delta_k, 1-\delta_k)^2 + \mathbb{Z}^2)$;
- if S^i is a vertical segment, then $B_{\varepsilon_k\eta}\left(S^i + (t_k, t_k)\right) \cap B_{\varepsilon_k\delta_k}(\varepsilon_k\mathbb{Z} \times \mathbb{R}) = \emptyset$;
- if S^i is an horizontal segment, then $B_{\varepsilon_k\eta}\left(S^i + (t_k, t_k)\right) \cap B_{\varepsilon_k\delta_k}(\mathbb{R} \times \varepsilon_k\mathbb{Z}) = \emptyset$.

We define $w_k(x) := v(x + (t_k, t_k))$ for $x \in B_\eta(\Omega)$.

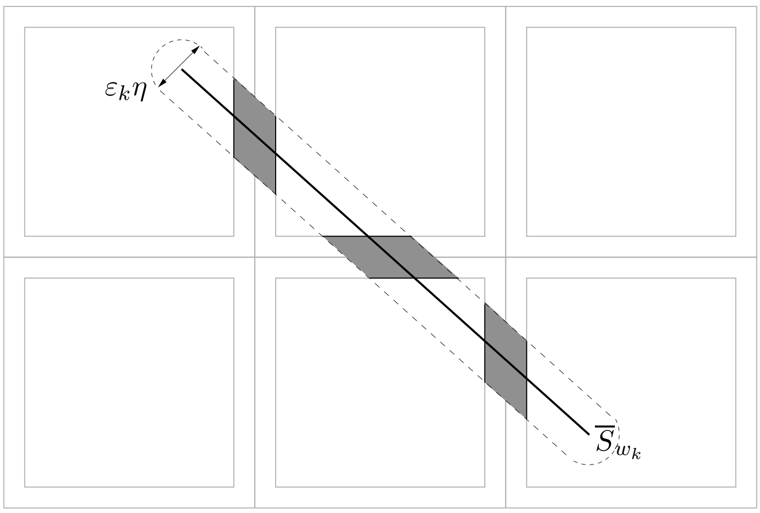

FIGURE 9

Since the function $v|_{B_{2\eta}(\Omega)\setminus B_{\varepsilon_k\eta}(\overline{S}_v)}$ has Lipschitz constant bounded from above by $c\,\varepsilon_k^{-1}\,\|\nabla v\|_{L^\infty(B_{2\eta}(\Omega),\mathbb{R}^2)}$, with c depending only on η, we can find a Lipschitz extension v_k to $B_{2\eta}(\Omega)$ satisfying

$$\|\nabla v_k\|_{L^\infty(B_{2\eta}(\Omega),\mathbb{R}^2)} \le \frac{c}{\varepsilon_k}\,\|\nabla v\|_{L^\infty(B_{2\eta}(\Omega),\mathbb{R}^2)}\,.$$

Now we can obtain a recovery sequence u_k for u by modifying the functions w_k on $B_{\varepsilon_k\eta}(\overline{S}_{w_k})\cap(\Omega\setminus\Omega_{\varepsilon_k})$ (the grey zones in Figure 9) in the following way:

$$u_k(x) := \begin{cases} v_k(x+(t_k,t_k)) & \text{if} \quad x\in B_{\varepsilon_k\eta}(\overline{S}_v)\cap(\Omega\setminus\Omega_{\varepsilon_k}) \\ w_k(x) & \text{otherwise.} \end{cases}$$

Note that u_k could present an additional fracture with respect to $w_k|_\Omega$ along $B_{\varepsilon_k\eta}(\overline{S}_{w_k})\cap\Omega\cap\varepsilon_k\left(\partial(\delta_k,1-\delta_k)^2+\mathbb{Z}^2\right)$ (the vertical and horizontal boundary of the grey zones in Figure 9).

By noting that $B_{\varepsilon_k\eta}(S^i)\cap B_{\varepsilon_k\eta}(S^h)=\emptyset$ if $i\neq h$ and k is large enough, in the rest of the proof we can assume for simplicity that $i=1$. Let $\gamma\in[0,\pi)$ be the angle formed by S^1 with a horizontal line and let l be the length of S^1.

Since S^1 intersects both $\mathbb{R}\times\varepsilon_k\mathbb{Z}$ and $\varepsilon_k\mathbb{Z}\times\mathbb{R}$ at most $[\varepsilon_k^{-1}]l$ times, we have the following estimates:

$$\mathcal{L}^1\left(B_{\varepsilon_k\eta}(\overline{S}_{w_k})\cap(\Omega\setminus\Omega_{\varepsilon_k})\right)\le c_1\eta\, l\,\delta_k\,\varepsilon_k$$
$$\mathcal{H}^1\left(B_{\varepsilon_k\eta}(\overline{S}_{w_k})\cap\Omega\cap\varepsilon_k\left(\partial(\delta_k,1-\delta_k)^2+\mathbb{Z}^2\right)\right)\le 2c_1\eta\, l,$$

where

$$c_1 := \begin{cases} 1 & \text{if} \quad \gamma=0 \text{ or } \pi/2 \\ \frac{1}{|\cos\gamma|}+\frac{1}{\sin\gamma} & \text{otherwise.} \end{cases}$$

Therefore u_k tends to u strongly in $L^p(\Omega)$ and

$$\limsup_{k\to+\infty} F_{\varepsilon_k}(u_k)\le MS(u)+\eta\, l(c^2\,\vartheta\,\|\nabla v\|^2_{L^\infty(B_{2\eta}(\Omega),\mathbb{R}^2)}+2c_1)=MS(u)+2c_1\eta\, l.$$

Being $\eta>0$ arbitrary, the previous inequality completes the proof. □

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