COARSENING IN NONLOCAL INTERFACIAL SYSTEMS

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Abstract. We consider coarsening in interfacial systems driven by nonlocal energies. Of particular interest are the nonlocal Cahn–Hilliard equation and models of biological aggregation. The energies considered cause the system to separate into phases. The pattern of interfaces evolves under nonlocal surface-tension-type effects. The typical length scales grow and the pattern coarsens. We prove a rigorous upper bound on the coarsening rate.

The proof uses the energy-based approach to estimates on rate of coarsening introduced by Kohn and Otto [10]. To show the required estimates on the flatness of the energy landscape we develop a geometric approach which is applicable to a wider class of problems, which includes ones based on local, gradient type energies.

1. Introduction

Our main focus are systems driven by nonlocal energies. Of particular interest are nonlocal Cahn–Hilliard type equations and equations modeling biological aggregation. Nonlocal Cahn–Hilliard equations that we consider were derived by Giacomin and Lebowitz [8] as limits of the lattice-gas dynamics modeling phase segregation in binary alloys. In this setting, they represent a refinement in modeling over the fourth order Cahn–Hilliard equations. The equations modeling biological aggregation were derived by Topaz, Bertozzi, and Lewis [18].

The mathematical descriptions of these systems are rather similar. Both equations are gradient flows of the same general energy:

$$E(u) := \iint (u(x) - u(y))^2 K(x-y)\,dx\,dy + \int W(u(x))\,dx. \tag{1}$$

Here $K \geq 0$ is the interaction kernel, and W is a double-well potential whose minima are at 0 and 1. We leave the domain of integration vague at the moment. Heuristically it is convenient to consider the domains to be $\mathbb{R}^N \times \mathbb{R}^N$ and $\mathbb{R}^N$, respectively. However for technical reasons, when stating and proving rigorous results we will consider the problem on a finite domain.

The equations we study are gradient flows of the energy, in the appropriate metrics:

$$u_t - \nabla \cdot \left(\mu(u) \nabla \left(\frac{\delta E}{\delta u} \right) \right) = 0. \tag{2}$$

That is

$$u_t - \nabla \cdot \left(\mu(u) \nabla \left(4 \int K(y) dy \, u - 4K * u + W'(u) \right) \right) = 0. \tag{3}$$

For both equations the mobility μ is a nonnegative function. More precisely for the aggregation equation $\mu(u) = u$ while for the nonlocal Cahn–Hilliard equation $\mu > 0$ on $[0, 1]$.

The second term of the energy causes the system to separate into phases, while the first term penalizes the existence of interfaces. The energy (1) is a nonlocal counterpart of the energy

$$E_{loc}(u) := \int \frac{1}{2} |\nabla u(x)|^2 + W(u(x)) dx. \tag{4}$$

Roughly speaking, both of the energies measure interfacial area. The longer the length scale in the system the better the approximation to interfacial area is. This fact is characterized by the fact that the Γ-limits of appropriately rescaled energies is the functional measuring perimeter of the set occupied by one of the phases

$$E \xrightarrow{\Gamma} const. \, E_{per}.$$

where E_{per} is defined for BV functions with the range $\{0, 1\}$. For E_{loc} this is the result of Modica and Mortola [15] (see also [14]), while for E it was proven by Alberti, Bellettini, Cassandro, and Presutti [1] (see also [2]). Moreover matched asymptotics arguments (by Giacomin and Lebowitz [9] and by Bertozzi and the author [4]) show that the sharp-interface limits of the dynamics described by (2) is the Mullins–Sekerka (MS) equation for the nonlocal Cahn–Hilliard equation and the Hele–Shaw (HS) equation for the aggregation model.

After the interfaces have formed the system slowly evolves reducing the interfacial area. During this process the length scales that characterize the coarseness of a configuration grow. We are interested in the rate at which these length scales grow — the rate of coarsening. The fact that the sharp interface limits, (MS) and (HS), are both invariant under the scaling $x \to \lambda x$, $t \to \lambda^3 t$ suggests that the typical length scale grow as $t^{1/3}$. We prove a weak formulation of this statement, following the technique of Kohn and Otto [10], who proved the result for gradient flows of local energies (4). We use the energy as the measure of the coarseness of the system. In particular let $\overline{E}$ be the energy density, that is the energy per unit volume. Note that $\overline{E}$ has units of $1/length$. We show a weak version of the statement

$$\overline{E} \gtrsim t^{-1/3}.$$

This provides an upper bound on rate of coarsening as it shows that the interfacial area cannot decay faster than the given rate.

Outline. In the remainder of the introduction we discuss the gradient-flow structure of the equations, and the framework for obtaining rigorous result on coarsening rates introduced by Kohn and Otto. We also introduce the two applications we have in mind in more detail. In Section 2 we list the assumptions needed and give the precise formulation of the result. In Section 3 we present the proof of the main result. The main technical ingredient, the interpolation inequalities, are proved in Section 4. The approach we take in proving the interpolation inequality is general; essentially the same proof covers both types of mobilities and both nonlocal and local energies. We illustrate the application to local energies in Subsection 4.1.

1.1. **Gradient flow structure.** We now introduce the geometric structure of the equation (2). It is based on formal Riemannian viewpoint developed by Otto [16]. The equation (2) can be understood as a gradient flow of the energy (1) on the manifold of configurations. Since the equation is in divergence form it preserves the integral of u over the space. Thus the solution of the equation is a path on the manifold of functions with the same integral.

At each point the tangent space is the set of possible perturbations, all of which have mean zero. The local metric is defined as follows: Let s_1, s_2 be two tangent vectors at u. Then

$$g_u(s_1, s_2) = \int \mu(u) \nabla p_1 \cdot \nabla p_2 \tag{5}$$

where

$$-\nabla \cdot (\mu(u) \nabla p_i) = s_i \quad \text{for } i = 1, 2.$$

The equation (2) is the gradient flow of energy (1) with respect to the metric (5), that is for every tangent vector s

$$g(u_t, s) = -\frac{\delta E}{\delta u}[s].$$

Considering the configuration space as a manifold enables us to, in a way, measure the steepness of the energy landscape, which in turn provides bounds on the speed of the dynamics.

In particular the local metric gives a rise to a global metric on the manifold. Given a regular enough path, $v(s)$ for say $s \in [0,1]$ on the manifold we can measure its length

$$\text{length}(v) = \int_0^1 \sqrt{g_{v(s)}(v', v')}\, ds.$$

We can then define the global metric on the manifold: Let the distance of u_1 and u_2 be

$$d(u_1, u_2) = \inf\{\text{length}(v) \;:\; v \text{ is a path connecting } u_1 \text{ and } u_2\}.$$

It turns out that when $\mu = const.$ then $d(u_1, u_2)$ is a multiple of the H^{-1} norm, while for $\mu(u) = u$ the distance becomes the Wasserstein metric.

1.2. **Kohn–Otto framework.** Kohn and Otto [10] introduced an approach to obtaining information on the flatness of the energy landscape, and consequently on the rate of coarsening. The approach is robust and has been applied to studies of coarsening in epitaxial growth, mean-field models, thin-liquid films and other systems [6, 7, 5, 11, 12, 17]. See also [13] for a related result.

We first present it in abstract setting used in [17], which applies to gradient flows. Consider energy E on a Riemannian manifold $(\mathcal{M}, g)$. The metric g introduces a global distance on $\mathcal{M}$, we denote it by d.

Proposition 1. *Let $h^* \in \mathcal{M}$. Let $h : \mathbb{R}_+ \to \mathcal{M}$ be a solution of*

$$h_t \;=\; -\text{grad}E(h). \tag{6}$$

and $h(0) = h_0$.

Assume that for some $\alpha \geq 0$ the interpolation inequality

$$E(h)\,\text{dist}(h, h^*)^\alpha \geq 1 \qquad \text{for all } h \in \mathcal{M} \text{ with } E(h) \leq \varepsilon \tag{7}$$

holds. Then for $\sigma \in (1, 1 + \frac{2}{\alpha})$

$$\int_0^T E(h(t))^\sigma\, dt \gtrsim \int_0^T (t^{-\frac{\alpha}{\alpha+2}})^\sigma\, dt \tag{8}$$

provided $T \gg \text{dist}(h_0, h^*)^{\alpha+2}$ and $E(h(0)) \leq \varepsilon$.

Remark 1. *The precise meaning of $\gtrsim$ and $\gg$ is the following: For all $\sigma \in (1, 1 + \frac{2}{\alpha})$ there exists a constant $C = C(\alpha, \sigma)$ such that $\forall \delta > 0\ \exists C_\delta = C(\alpha, \sigma, \delta)$:*

$$\int_0^T E(h(t))^\sigma\, dt \geq (1 - \delta) C \int_0^T (t^{-\frac{\alpha}{\alpha+2}})^\sigma\, dt \tag{9}$$

provided $T \geq C_\delta\, \text{dist}(h_0, h^*)^{\alpha+2}$.

Proof of the Proposition is based on ODE arguments of [10] and can be found in [17].

1.3. **Nonlocal Cahn–Hilliard Equation.** The equation (2) is a rescaled version of the model by Giacomin and Lebowitz [8, 9]. We introduce it in original variables below and, for completeness, present the rescaling needed.

The free energy is given by

$$\mathcal{E} = \frac{1}{4}\int_{\Omega}\int_{\mathbb{R}^N} K(x-y)(\rho(x)-\rho(y))^2 dx dy + \int_{\Omega} f_c(\rho(x))dx.$$

Here $K \geq 0$ is a smooth kernel with symmetry $K(x) = K(-x)$. Giacomin and Lebowitz assume that K is compactly supported, with support contained in Ω. This assumption is physically quite reasonable, but it is not necessary from mathematical point of view. The function f_c is a double well potential, symmetric about $\frac{1}{2}$ with minima at $\frac{1}{2} \pm m$. The mobility function σ is assumed to be smooth, symmetric about $\frac{1}{2}$, positive on $(0,1)$,

$$\text{(GL1)} \qquad \sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 0.$$

Let

$$f(\rho) = f_c(\rho) + \frac{\int_{\mathbb{R}^N} K(x)dx}{2}\left(\rho - \frac{1}{2}\right)^2.$$

Giacomin and Lebowitz assume that for some $c > 0$ and for all $\rho \in (0,1)$

$$\text{(GL2)} \qquad \frac{1}{c} \leq \sigma(\rho) f''(\rho) \leq c.$$

This assumption is needed for existence/uniqueness theory they use (it makes the equation uniformly parabolic), but is not directly required for the coarsening estimates.

Under the assumptions above Giacomin and Lebowitz show that for initial datum $0 < \rho_0 < 1$, there exists a unique weak solution $\rho \in L^2([0,T], H^1(\Omega))$ with $\rho_t \in L^2([0,T], H^{-1}(\Omega))$ for any $T > 0$. Furthermore $0 < \rho < 1$.

Rescaling. We rescale the dependent variable, the potential, and the mobility so that the wells of the new potential are 0 and 1. Let

$$u = \frac{1}{2m}\left(\rho - \frac{1}{2}\right) + \frac{1}{2}$$
$$W_m(u) = \frac{1}{4m^2} f_c(\rho) = \frac{1}{4m^2} f_c\left(2m(u - \frac{1}{2}) + \frac{1}{2}\right)$$
$$\mu_m(u) = \sigma(\rho) = \sigma\left(2m(u - \frac{1}{2}) + \frac{1}{2}\right).$$

Under this rescaling u solves (2), and is hence a gradient flow of (1).

1.4. **Biological Aggregation.** Topaz, Bertozzi, and Lewis [18] introduced a model of biological aggregation that emerges due to "social forces" between individuals. That is the individuals are attracted to other individuals of their species, but avoid overcrowding. The population is modeled by its density u. The velocity of individuals is modeled as

$$v = v_a + v_r = \nabla(K * u) - \nabla g(u)$$

where $v_a = \nabla(K * u)$ is the term attraction to other individuals which are being sensed through the kernel K. The term modeling repulsion, v_r, is given by a local operator $v_r = -\nabla g(u)$, where g is an increasing function. The continuity equation then reads:

$$u_t + \nabla \cdot (u\, v) = u_t + \nabla \cdot (u \nabla(K * u - g(u))) = 0.$$

From biological perspective it is reasonable to assume that $g'(0) = 0$ and g is strictly convex. However, it is sufficient to assume that

$$\text{(BA)} \quad \begin{aligned} &\text{the function } g'(z) - \int_{R^N} K(x)dx \text{ has exactly one zero on } \mathbb{R}^+, \\ &g'(0) < \int_{R^N} K(x)dx \quad \text{and} \quad \liminf_{z\to\infty} g'(z) - \int_{\mathbb{R}^N} K(x)dx > 0. \end{aligned}$$

Under this assumption u solves (3) for some double well potential W. More precisely: Let $G(z) := \int_0^z g(s)ds$ and $\tilde{W}(z) := G(z) - \frac{1}{2}\int_{\mathbb{R}^N} K(x)dx\, z^2$. The condition (BA) implies that $\tilde{W}$ is concave at 0, and has exactly one inflection point. Thus we can define

$$W(z) := 4\left(\tilde{W}(z) - \left(\min_{s>0} \frac{\tilde{W}(s)}{s}\right) z\right).$$

Then W is a double well potential on $[0,\infty)$ with one well at 0. It follows that

$$u_t - \nabla \cdot \left(u \nabla \left(\int_{\mathbb{R}^N} K(y) dy u - K * u + \frac{1}{4} W'(u) \right) \right) = 0.$$

which after scaling the time by factor 4 is the equation (3).

On the level of the model, the equation provides information on why herds (or other animal groups) form, why they have an almost constant density, why they have sharp boundaries and how they evolve. Numerical simulations conducted in 1D in [18] also observe the coarsening phenomenon. The primary driving force for coarsening in 1D is the nonlocal interaction via kernel K, as there there are no surface-tension like effect. The rate of coarsening depends on the decay of K at ∞. Nevertheless the rigorous bounds we prove still apply and are in fact optimal for certain kernels.

2. Statement of the result

When thinking about coarsening we have in mind an infinite domain on which coarsening persists for all time. However building the theory for such solutions poses major challenges. We instead consider domains of finite size and prove results that are independent of the domain size. In particular we consider the domain $\Omega = [0,\Lambda]^N$. We investigate the dynamics of periodic configurations on $\mathbb{R}^N$ with period cell Ω. Thus we consider Ω with the topology of the torus $\mathbb{R}^N/(\Lambda\mathbb{Z})^N$. In particular the distances on Ω are measured on the torus, and are thus may be different from the ones measured in $\mathbb{R}^N$.

Throughout the paper we use the following, somewhat nonstandard, notation. For $U \subseteq \Omega$, and a function u

$$⨍_U f(x) dx := \frac{1}{|\Omega|} \int_U f(x) dx \quad \text{and} \quad \|U\| := \frac{|U|}{|\Omega|}.$$

Let P be the maximal interval containing 1 on which $\mu > 0$:

$$P = \{z \,:\, z \leq 1, \;\; \mu|_{[z,1]} > 0\} \cup \{z \,:\, z \geq 1, \;\; \mu|_{[1,z]} > 0\}.$$

The configuration space is

$$\mathcal{M} := L^1(\Omega, \overline{P}).$$

The configurations u are functions defined on Ω, but when convenient we will also consider them as being periodic functions of $\mathbb{R}^N$.

To a configuration, u, we associate energy density

$$\overline{E}(u) := ⨍_\Omega \int_{\mathbb{R}^N} (u(x) - u(y))^2 K(x-y) dx dy + ⨍_\Omega W(u(x)) dx. \tag{10}$$

If the expression is not defined we say that the energy density is infinite. Conditions on the interaction kernel, K, and the double-well potential, W are described below.

We assume the following of the interaction kernel K:

(K1) K is nonnegative and $K \in L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$.

(K2) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N$. (This condition insures the symmetry of the interaction term in (10) with respect to x and y.)

(K3) $K(0) > 0$.

(K4) $K \in W^{2,1}(\mathbb{R}^N)$ and $\|K\|_{C^2(\mathbb{R}^N)} < \infty$.

The last condition is only needed for the existence theory, [4].

The condition (K3) is not essential either, but significantly simplifies parts of the presentation. In particular it enables us to associate a length scale to a kernel in the following way: For $r > 0$ let

$$\kappa(x) := \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x) \quad \text{and} \quad \kappa_r(x) := \frac{1}{r^N}\kappa\left(\frac{x}{r}\right) \tag{11}$$

where χ_U is the characteristic function of the set U. Given $r > 0$ let $h_K(r) := \sup\{c \,:\, K \geq c\kappa_r\}$. Note that by assumption (K3), $h_K(r) > 0$ for r small. It is not hard to prove that $h_K(r) \to 0$ as $r \to 0$ and also as $r \to \infty$. Consider the location of the maximum of $h_K(r)$. If there is more than one maximum we pick the first one. More precisely let

$$r_K := \min\{r_{max} \mid h_K(r_{max}) = \max_{r>0} h_K(r)\}. \tag{12}$$

From this point on, we write H_K for $h_K(r_K)$.

We now turn to conditions on the potential W. Condition on growth of W are only needed if P is infinite. To be able to precisely state when is a value of u close to one of the equilibria, we need to fix the average:

$$a := \fint_\Omega u(x,0)dx.$$

This is relevant when a is itself close to one of the equilibria. To simplify the presentation, from here onwards, we restrict our attention only to

$$0 < a \leq \frac{1}{2}. \tag{13}$$

We assume:

(W1) W is a nonnegative continuous function.

(W2) $W(0) = W(1) = 0$ and $W > 0$ on $\overline{P}\backslash\{0,1\}$.

(W3) At least linear growth at $\pm\infty$: There exists a constant h_W such that $W(z) \geq h_W(z-1)$ for all $z \in P \cap (\frac{9}{8}, \infty)$ and $W(z) \geq h_W|z|$ for all $z \in P \cap (-\infty, -\frac{a}{4})$. We can furthermore require

$$h_W \leq W(z) \text{ for all } z \in \left[\frac{a}{4}, \frac{7}{8}\right].$$

We now state the main result; first in a general form. The classes of solutions of (3) studied in [9] and [4] satisfy the conditions of the theorem and thus the bounds on coarsening hold. These results are presented in Corollary 3.

Theorem 2. *Let $\Omega = [0,\Lambda]^N$. Assume that conditions (K1)-(K3) and (W1)-W3) hold and that mobility $\mu(z) \equiv z$ or that $\mu \in C(P,(0,c_\mu])$. Suppose that $u \in C^{weak}([0,\infty, L^1(\Omega,\overline{P}))$ is a weak solution of*

$$u_t + \nabla \cdot J = 0$$

for some flux $J \in L^1(\Omega,\mathbb{R}^N)$. Assume further that $\limsup_{t\to 0+} \overline{E}(u(\,\cdot\,,t)) \ll 1$ and (13) holds. Assume that energy dissipation inequality holds: For almost all $0 \leq t_1 < t_2$

$$\int_{t_1}^{t_2}\int_\Omega \frac{1}{\mu(u)}|J|^2 dxdt \leq -(E(u(t_2)) - E(u(t_1))). \tag{14}$$

In the case $\mu(z) \not\equiv z$ we also need the regularity assumption that $u(t) \in L^2(\Omega)$ for all $t \geq 0$ and that the range of $u(\,\cdot\,,0)$ is contained in P. Then for all $\sigma \in (1,2)$ there exists a constant $C = C(\sigma, a, H_K, r_K, c_\mu)$ such that for all $T \gg 1$

$$\int_0^T \overline{E}(u(t))^\sigma dt \geq C\int_0^T \left(t^{-\frac{1}{3}}\right)^\sigma dt. \tag{15}$$

By weak solution we mean that for all $\phi \in C_c^\infty(\Omega\times[0,\infty))$

$$\iint_{[0,\infty)\times\Omega} u\phi_t + J\cdot\nabla\phi\, dxdt + \int_\Omega u(x,0)\phi(x,0)dx = 0.$$

Recall that we consider Ω with the topology of a torus, which means that the test functions used in the definition of a weak solution are also defined on the torus; in other words they are periodic.

By C^{weak} we mean continuous with respect to weak topology of the target space. Together with the form of the equation this implies that

$$\int_\Omega u(x,t)dx = const.$$

Note that the dissipation inequality (14) is an equality if u is a classical solution of the gradient flow (2), when $J = -\mu(u)\nabla \frac{\delta E}{\delta u}$.

Corollary 3. *Let $\Omega = [0,\Lambda]^N$. Assume that conditions (K1)-(K4) and (W1)-W3) hold. Assume either of the following holds*

(i) *$\mu(z) \equiv z$, $W \in C^2([0,\infty))$ with $W'' > -4\int_{\mathbb{R}^N} K(y)dy$ on $\mathbb{R}^+$, and u is the solution of* (3) *in the sense of* [4] *with $u(\,\cdot\,,0) \in L^\infty(\Omega)$.*

(ii) *$\mu(z) \le c_\mu$ for all z, conditions (GL1) and (GL2) hold, and u is a solution of* (3) *in the sense of* [9]. *Furthermore assume that the range of $u(\,\cdot\,,0)$ is contained in the interior of P.*

(iii) *$0 < \mu(z) \le c_\mu$ for all z, (GL2) holds on $\mathbb{R}$ and u is a solution of* (3) *in the sense of* [9]. *Furthermore assume that $u(\,\cdot\,,0) \in L^\infty(\Omega)$.*

Assume that $\limsup_{t\to 0+} \overline{E}(u(\,\cdot\,,t)) \ll 1$ and (13) *holds. Then for all $\sigma \in (1,2)$ there exists a constant $C = C(\sigma, a, H_K, r_K, c_\mu)$ such that for all $T \gg 1$*

$$\int_0^T \overline{E}(u(t))^\sigma dt \ge C \int_0^T \left(t^{-\frac{1}{3}}\right)^\sigma dt. \tag{16}$$

Proof. (of corollary) In the first case the conditions of W imply that associated $g(z) := \int_{\mathbb{R}^N} K(x)dx + \frac{1}{4}W'(z)$ is an increasing function. This in turn implies that the conditions under which Bertozzi and the author [4] proved existence of solutions of (3) hold. Properties of solutions ensuring that assumptions of Theorem 2 were also established. This implies the claim of the corollary.

For case *(ii)* the existence theory needed for Theorem 2 to apply was established by Giacomin and Lebowitz [9].

The case *(iii)* is in principle simpler than the case *(ii)*, and includes the constant mobility case. The only technical issue is that the L^∞ bounds used in [9] follow from condition (GL1). In our case appropriate bounds can be established, for example as in [4]. □

3. Proof of Theorem 2

We seek to apply the framework of the Proposition 1. However the configuration space, $\mathcal{M}$, is not a true Riemannian manifold and the only remnant of the gradient flow structure is the energy dissipation inequality (14). Nevertheless arguments of the proof of the proposition, can be adapted to include this setting. We define the "geodesic distance" on $\mathcal{M}$ as follows: Given $u_0, u_1 \in \mathcal{M}$ let us first define a representation of admissible paths between u_0 and u_1:

$$\mathcal{A}(u_0,u_1) := \Big\{(u,J)\ : u:[0,1]\to\mathcal{M}, J\in L^1(\Omega\times[0,1],\mathbb{R}^N) \text{ such that } \\ u_t + \nabla\cdot J = 0 \quad \text{on } \Omega\times[0,1] \text{ weakly}, \\ u\in C^{weak}([0,1],L^1(\Omega)) \text{ and } u(0)=u_0, u(1)=u_1, \text{ and} \\ \int_0^1\int_\Omega \frac{1}{\mu(u(x,t))}\,|J(x,t)|^2 dxdt < \infty\Big\}.$$

We define

$$d^2(u_0,u_1) := \inf_{(u,J)\in\mathcal{A}} \int_0^1\int_\Omega \frac{1}{\mu(u(x,t))}\,|J(x,t)|^2 dxdt. \tag{17}$$

Here $\frac{0}{0}=0$. We note that d may, in general, be infinite. It follows from the definition that d satisfies the triangle inequality. This can be shown by concatenating the appropriate test flows (with optimally rescaled times).

Let u be as in the statement of the theorem. We define

$$L(t) := d(u(t),a) \qquad \overline{L}(t) := \frac{1}{\sqrt{|\Omega|}}\, d(u(t),a). \tag{18}$$

In the case $\mu(u)=u$ it follows from the characterization of d given below in (19), that $L(t)$ is finite for all t. In the other case, from the assumption on range of u_0 follows that $L(0)$ is finite. To see this it is enough to consider the test pair $(\tilde{u},\tilde{J})$ with $\tilde{u}(s)=u_0+s(a-u_0)$ for $s\in[0,1]$ and $\tilde{J}=\nabla p$ where p solves $-\Delta p = a-u_0$. The fact that $L(t)$ is finite for all t then follows from the argument for continuity of L given in Lemma 4.

Let

$$\overline{E}(t) := \overline{E}(u(t)).$$

From (14) we have that $\overline{E}$ is nonincreasing almost everywhere. We now modify $\overline{E}$ on set of measure 0 to insure that it is nonincreasing

$$\overline{E}_{new}(t) = \min\{\liminf_{s\to t-} \overline{E}(s), \overline{E}(t)\}.$$

Inspecting the proof of Proposition 1 from [17] shows that in addition to the interpolation inequality, one only needs the inequality

$$\left(\frac{d\overline{L}}{d\overline{E}}\right)^2 \leq -\frac{dt}{d\overline{E}}$$

(since $\overline{E}$ is nonincreasing, $\overline{L}$ can be considered as a function of $\overline{E}$) which follows from the more familiar form of the dissipation inequality

$$\left(\frac{d\overline{L}}{dt}\right)^2 \leq -\frac{d\overline{E}}{dt}$$

where both inequalities are to be understood as a comparison of measures (with given densities). The latter inequality in turn follows form the assumption (14). We prove these claims in Lemmas 4 and 5.

To be able to prove the interpolation inequalities we need a more workable form of d. In the case $\mu(u) = u$ the distance d is nothing else than the Wasserstein distance. This was shown by Benamou and Brenier [3] (see also Section 8.1 in Villani's book [19]):

(19)

$$d_W(u_0, u_1)^2 \;=\; \inf\left\{ \iint_{\Omega\times\Omega} |x-y|^2 d\pi(x,y) \;\Big|\; \int_\Omega d\pi(\cdot, y) = u_0,\; \int_\Omega d\pi(x, \cdot) = u_1 \right\}.$$

The distance above, $|x-y|$, is taken on torus Ω.

In the case $\mu(u) \leq c_\mu$ first note that from the definition of the distance (17) follows that the distance corresponding to $\mu(u)$ is greater than the distance corresponding to constant mobility c_μ. Thus

$$\overline{L}_\mu \geq \overline{L}_{c_\mu} = \frac{1}{c_\mu}\overline{L}_1.$$

Thus it is enough to establish the interpolation inequality for $\overline{L} = \overline{L}_1$. But for mobility equal to one, the distance is the H^{-1} norm. More precisely, for $u_0, u_1 \in \mathcal{M} \cap L^2(\Omega)$, $\int_\Omega u_1 - u_0 dx = 0$ and hence we can consider the following representation of the H^{-1} norm

$$d(u_0, u_1)^2 = \|u_0 - u_1\|^2_{H^{-1}} = \int_\Omega |\nabla p|^2 dx$$

where $p \in H^2(\Omega)$ (p is periodic by topology of Ω) is a solution of

$$-\Delta p = u_1 - u_0.$$

Proof of this claim is straightforward, it relies on convexity in J of the action functional on the right hand side of (17), and the observation that $J = \nabla p$, along with $u(t) = u_0 + t(u_1 - u_0)$, minimizes the action functional. One can also show that for $u \in \mathcal{M} \cap L^2(\Omega)$

$$\overline{L}(u) = \max_{\xi \in H^1(\Omega),\, \xi \not\equiv const.} \frac{⨍_\Omega (u-a)\xi dx}{\sqrt{⨍_\Omega |\nabla \xi|^2 dx}}. \tag{20}$$

by using Cauchy-Schwarz inequality to show that $\xi = p$ (with $u_1 = u$ and $u_2 = a$) is the maximizing function. Given that $u(t) \in \mathcal{M} \cap L^2(\Omega)$ for all t we can use this characterization.

To complete the proof of the theorem now only need the interpolation inequalities established in Section 4.

Lemma 4. *Assume that u satisfies the conditions of Theorem 2 and $\overline{L}$ is defined in* (18). *Then $\overline{L}$ is a continuous function and for almost all $t \geq 0$ and $h > 0$*

$$\left(\frac{\overline{L}(u(t+h)) - \overline{L}(u(t))}{h} \right)^2 \leq -\frac{\overline{E}(u(t+h)) - \overline{E}(u(t))}{h}.$$

Proof. By the assumptions of the theorem u is a distributional solution of

$$u_t + \nabla \cdot J = 0 \qquad \text{on } \Omega \times [t, t+h].$$

Moreover it follows from assumption (14) for all $t \geq 0$ and all $h > 0$

$$\int_t^{t+h} \int_\Omega \frac{1}{\mu(u)} |J|^2 dx dt \leq \limsup_{s \to 0+} E(u(s)) < \infty.$$

Note that it was also assumed that $u \in C^{weak}([t, t+h], L^1(\Omega))$. Thus (u, J), after appropriate rescaling in time, belongs to $\mathcal{A}(u(t), u(t+h))$. By triangle inequality,

$$\begin{aligned} \left(\overline{L}(u(t+h)) - \overline{L}(u(t))\right)^2 &\leq \inf_{(\tilde{u}, \tilde{J}) \in \mathcal{A}(u(t), u(t+h))} \int_0^1 \int_\Omega \frac{1}{\mu(\tilde{u}(x,s))} \, |\tilde{J}(x,s)|^2 \, dx \, ds \\ &\leq h \int_0^h \int_\Omega \frac{1}{\mu(u(x, t+s))} \, |J|^2 dx ds. \end{aligned}$$

Thus $\overline{L}$ is a continuous function. Dividing the above by h^2 and using (14) gives that for almost all $t \geq 0$ and $h > 0$:

$$\left(\frac{\overline{L}(u(t+h)) - \overline{L}(u(t))}{h}\right)^2 \leq \frac{1}{h}\int_0^h \int_\Omega \frac{1}{\mu(u(x,t+s))}|J|^2 dxds$$
$$\leq -\frac{\overline{E}(u(t+h)) - \overline{E}(u(t))}{h}.$$

□

For a function e on $\mathbb{R}$ let us define $e(t+) := \lim_{s\to t+} e(s)$ and $e(t-) := \lim_{s\to t-} e(s)$.

Lemma 5. *Let e be a nonnegative, nonincreasing function on $[0,\infty)$. Let l be a continuous function on $[0,\infty)$, such that*

$$\left(\frac{l(t_2) - l(t_1)}{t_2 - t_1}\right)^2 \leq -\frac{e(t_2) - e(t_1)}{t_2 - t_1} \qquad \textit{for almost all } t_2 > t_1 \geq 0. \tag{21}$$

Then $l(t)$ is an absolutely continuous function on $[0,\infty)$ and for all $\tau_2 > \tau_1 \geq 0$

$$\int_{\tau_1}^{\tau_2} \left(\frac{dl}{dt}\right)^2 dt \leq e(\tau_1+) - e(\tau_2-).$$

Furthermore, consider $t(e) := \sup\{t \,:\, e(t) \geq e\}$, the "inverse" of the function e and $l(e) := l(t(e))$. Then

$$\left(\frac{l(e_2) - l(e_1)}{e_2 - e_1}\right)^2 \leq -\frac{t(e_2) - t(e_1)}{e_2 - e_1} \qquad \textit{for all } e(0) \geq e_1 > e_2 \geq 0.$$

Consequently l is an absolutely continuous function of e and for all $e(0) \geq e_1 > e_2 \geq 0$

$$\int_{e_2}^{e_1} \left(\frac{dl}{de}\right)^2 de \leq t(e_2+) - t(e_1-).$$

Proof. If $e(0) = 0$ the proof is trivial. So assume $e(0) > 0$. Continuity of l implies that

$$\left(\frac{l(t_2) - l(t_1)}{t_2 - t_1}\right)^2 \leq -\frac{e(t_2-) - e(t_1+)}{t_2 - t_1} \qquad \text{all } t_2 > t_1 \geq 0.$$

Let $\varepsilon > 0$. Let $\delta := \varepsilon^2/e(0)$. Let $[x_i, y_i]$ for $i = 1, \ldots, m$ be a family of disjoint intervals on $[0,\infty)$ of total length less than δ:

$$\sum_{i=1}^m y_i - x_i < \delta.$$

Then

$$\begin{aligned}\sum_{i=1}^m |l(y_i) - l(x_i)| &\le \sum_{i=1}^m \sqrt{e(x_i) - e(y_i)}\,\sqrt{y_i - x_i} \\ &\le \sqrt{\sum_{i=1}^m e(x_i) - e(y_i)}\,\sqrt{\sum_{i=1}^m y_i - x_i} \\ &\le \sqrt{e(0)}\,\sqrt{\delta} = \varepsilon.\end{aligned}$$

So l is absolutely continuous.

To prove the second claim note that for any $h > 0$

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \left(\frac{l(t+h) - l(t)}{h}\right)^2 dt &\le \int_{\tau_1}^{\tau_2} \frac{e(t+) - e(t+h-)}{h} dt \\ &\le \frac{1}{h}\left(\int_{\tau_1}^{\tau_1+h} e(t+)dt - \int_{\tau_2}^{\tau_2+h} e(t-)dt\right).\end{aligned}$$

By taking the $\liminf_{h\to 0}$ and using the Fatou's lemma we obtain

$$\int_{\tau_1}^{\tau_2} \left(\frac{dl}{dt}\right)^2 dt \le e(\tau_1+) - e(\tau_2+).$$

Now use this claim on interval $(\tau_1, \tau_2 - \varepsilon)$ and take the limit as $\varepsilon \to 0$.

To prove the remaining claims, note that

$$(l(t(e_2)) - l(t(e_1)))^2 \le (e(t(e_2)-) - e(t(e_1+))(t(e_2) - t(e_1)).$$

Observing that $e(t(e_2)-) \ge e_2$ and $e(t(e_1)+) \le e_1$ yields the desired inequality. The claims then follow from the arguments presented above. □

4. Interpolation inequalities

In this section we prove the interpolation inequalities needed. As shown in Section 3 we only need to consider the $\overline{L}$ corresponding Wasserstein distance (19) and to H^{-1} norm (20).

The proof we present is general and extends to local energies, which we discuss in Subsection 4.1. It also captures the improved constants established in [5], see

Remark 2. It is based on simple geometric heuristic. Consider function $\tilde{u}$ with range $\{0, 1\}$, and $\kappa_r * \tilde{u}$, its average over ball of radius r. Then for r small

$$\int |\tilde{u} - \kappa_r * \tilde{u}| dx$$

contains information about the interfacial area, while for r large it carries information on the distance $d(\tilde{u}, a)$. This allows us to interpolate between energy and the distance. We divided the proof into steps and present the motivation at their beginning.

Theorem 6 (Interpolation inequality)**.** *Let $0 < a \leq \frac{1}{2}$. Assume K satisfies conditions (K1)-(K3) and W satisfies (W1)-(W3). There exists a constant $C = C(a, h_W, r_K, H_K) > 0$ such that for all $\Lambda > 0$ and all configurations $u \in \mathcal{M}$ for which $\overline{E}(u) < \frac{a}{64}\left(\frac{h_W}{1+h_W}\right) H_K$, and in the H^{-1} case also $\overline{E}(u) < \frac{1}{2^{N+2}}\frac{a h_W}{20} H_K$, the following holds*

$$\overline{E}(u)\overline{L}(u) \geq C. \tag{22}$$

The constant $C = c(N) r_K \left(\frac{h_W}{1+h_W}\right) a^{3/2} H_K$.

Proof. **Step 1: Reduction.** Let κ_{r_K} be as defined by (11) and (12). We use the notation $\kappa_r := \kappa_{r_K}$. To make the distinction between energies, let $\overline{E}_K$ and $\overline{E}_{\kappa_r}$ be the energy densities corresponding to kernels K and κ_r respectively. Note that $\overline{E}_K \geq H_K \overline{E}_{\kappa_r}$. So it is enough to show the above claim for κ_r, with $r_K = r$ and $H_K = 1$. Therefore from here on we only consider $K = \kappa_r$.

Step 2: u is separated into phases. We show that any low-energy-density configuration, u, has significant portion of the mass on the set where values of u are close to 1. More precisely:

Claim. Let $A := \{x \,:\, u(x) \geq \frac{7}{8}\}$ and $\tilde{u} := \chi_A$. For future reference let $\underline{A} := \{x \,:\, u(x) < \frac{a}{4}\}$ and let I be the interfacial region, $I := \Omega \backslash (A \cup \underline{A})$. Assume $\overline{E}(u) < \frac{3}{32} a h_W$. Then

$$\|A\| = \frac{|A|}{|\Omega|} = ⨍_\Omega \tilde{u}(x) dx > \frac{1}{2} ⨍_\Omega u(x) dx = \frac{a}{2} \quad \text{and} \quad \|A\| \leq 2a. \tag{23}$$

Proof. Due to assumption (W3)

$$\overline{E}(u) \geq h_W \left(\left\| \left\{ \frac{a}{4} < u \leq \frac{7}{8} \right\} \right\| + \left\| \left\{ \frac{9}{8} < u \right\} \right\| \right)$$

Consequently

$$
\begin{aligned}
a = \fint_\Omega u dx &= \fint_{\{u \le \frac{a}{4}\}} u dx + \fint_{\{\frac{a}{4} < u \le \frac{7}{8}\}} u dx + \fint_{\{\frac{7}{8} < u \le \frac{9}{8}\}} u dx + \fint_{\{\frac{9}{8} < u\}} u dx \\
&\le \frac{a}{4} + \frac{7}{8} \left\| \left\{ \frac{a}{4} < u \le \frac{7}{8} \right\} \right\| + \frac{9}{8} \fint_\Omega \tilde{u} dx + \left\| \left\{ \frac{9}{8} < u \right\} \right\| + \frac{1}{h_W} \fint_{\{\frac{9}{8} < u\}} W(u) dx \\
&\le \frac{a}{4} + \frac{\overline{E}}{h_W} + \frac{9}{8} \fint_\Omega \tilde{u} dx + \frac{\overline{E}}{h_W} \\
&< \frac{a}{4} + \frac{3a}{16} + \frac{9}{8} \fint_\Omega \tilde{u} dx.
\end{aligned}
$$

Therefore $\fint_\Omega \tilde{u} dx > \frac{a}{2}$.

To prove the second claim, note

$$
\begin{aligned}
a = \fint_\Omega u(x) dx &\ge \frac{7}{8} \|A\| + \fint_{\{-\frac{a}{4} \le a < 0\}} u(x) dx + \fint_{\{u < -\frac{a}{4}\}} u(x) dx \\
&\ge \frac{7}{8} \|A\| - \frac{a}{4} - \frac{1}{h_W} \fint_\Omega W(u(x)) dx \\
&\ge \frac{7}{8} \|A\| - \frac{a}{4} - \frac{\overline{E}}{h_W} \\
&\ge \frac{7}{8} \|A\| - \frac{a}{2}.
\end{aligned}
$$

□

Step 3: Energy bounds a measure of interfacial area.
Claim:

$$
\fint_\Omega |\tilde{u} - \kappa_r * \tilde{u}| dx \le \left(\frac{16}{9} + \frac{2}{h_W} \right) \overline{E}.
$$

Heuristically, the expression on the left-hand side measures r times the area of the boundary of $\{\tilde{u} = 1\}$, while neglecting features of size less than r.

Proof. We use the following notation for the sum of sets $X + Y := \{x + y \mid x \in X,\ y \in Y\}$. Using that $\tilde{u}$ takes only values 0 and 1 we obtain:

$$\begin{aligned}
&⨍_{\Omega} \left|\tilde{u}(x) - \int_{\mathbb{R}^N} \kappa_r(x-y)\tilde{u}(y)dy\right| dx \\
&\qquad = ⨍_{\Omega} \int_{\mathbb{R}^N} |\tilde{u}(x) - \tilde{u}(y)|\kappa_r(x-y)dydx \\
&\qquad \leq \frac{1}{|\Omega|}\Big[\int_A \int_{\underline{A}+\Lambda\mathbb{Z}^N} + \int_{\underline{A}} \int_{A+\Lambda\mathbb{Z}^N} |\tilde{u}(x) - \tilde{u}(y)|^2 \kappa_r(x-y)dydx + \\
&\qquad\qquad \int_I \int_{\mathbb{R}^N} + \int_{\Omega} \int_{I+\Lambda\mathbb{Z}^N} \kappa_r(x-y)dydx \Big] \\
&\qquad \leq \left(\frac{3}{4}\right)^{-2} ⨍_{\Omega} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 \kappa_r(x-y)dydx + 2 \left\| \left\{ \frac{a}{4} \leq u \leq \frac{7}{8} \right\} \right\| \\
&\qquad \leq \left(\frac{16}{9} + \frac{2}{h_W} \right) \overline{E}.
\end{aligned}$$

□

Step 4. *Claim.* $\phi : (0,\infty) \to [0,\infty)$ defined by

$$\phi(s) := ⨍_{\Omega} |\tilde{u} - \tilde{u} * \kappa_s| dx$$

is subadditive.

One should note that some other possible measures of surface area (for example the volume of appropriate tubular neighborhood) do not have this property in general and may have a superlinear growth (for appropriate range of r).

Proof. Let $s = p + q$ for some $p, q > 0$. As in Step 3 we have

$$\phi(s) = ⨍_{\Omega} \int_{\mathbb{R}^N} |\tilde{u}(x) - \tilde{u}(y)|\kappa_s(y)dydx.$$

Now let $z = x - \frac{p}{s}y$. Using periodicity and the scaling properties of kernel κ_s one finds

$$\begin{aligned}
\phi(s) &\leq ⨍_{\Omega} \int_{\mathbb{R}^N} (|\tilde{u}(x) - \tilde{u}(z)| + |\tilde{u}(z) - \tilde{u}(y)|)\kappa_s(y)dydx \\
&= ⨍_{\Omega} \int_{\mathbb{R}^N} |\tilde{u}(x) - \tilde{u}(x - \frac{p}{s}y)|\kappa_s(y)dydx + ⨍_{\Omega} \int_{\mathbb{R}^N} |\tilde{u}(z) - \tilde{u}(z - \frac{q}{s}y)|\kappa_s(y)dydz
\end{aligned}$$

substitute $\tilde{y} = \frac{p}{s}\, y$ in the first integral and $\tilde{y} = \frac{q}{s}\, y$ and $x = z$ in the second

$$= \fint_\Omega \int_{\mathbb{R}^N} |\tilde{u}(x) - \tilde{u}(\tilde{y})| \kappa_p(\tilde{y}) d\tilde{y} dx + \fint_\Omega \int_{\mathbb{R}^N} |\tilde{u}(x) - \tilde{u}(\tilde{y})| \kappa_q(\tilde{y}) d\tilde{y} dx$$
$$= \phi(p) + \phi(q)$$

□

Step 5. $\kappa_l * \tilde{u} \sim \tilde{u}$ for some l of size $\frac{1}{E}$. More precisely

Claim. Let $\mu > 2$ be a constant, which we will specify later. If $\overline{E} < \frac{a}{\mu}\left(\frac{h_W}{1+h_W}\right)$ then for

$$l = \left\lfloor \frac{a}{\mu} \left(\frac{h_W}{1+h_W} \right) \frac{1}{\overline{\overline{E}}} \right\rfloor r \;\; =: i\, r \tag{24}$$

the following holds

$$\phi(l) = \fint_\Omega |\tilde{u} - \kappa_l * \tilde{u}|\, dx < \frac{2}{\mu}\, a. \tag{25}$$

Proof. Assumption on $\overline{E}$ implies that $l > \frac{1}{2}\frac{a}{\mu}\left(\frac{h_W}{1+h_W}\right)\frac{1}{\overline{E}}\, r > 0$. Subadditivity of ϕ established in Step 4. and the bound of Step 3. imply

$$\fint_\Omega |\tilde{u} - \kappa_l * \tilde{u}|\, dx \le i \fint_\Omega |\tilde{u} - \kappa_r * \tilde{u}|\, dx \le \left\lfloor \frac{a}{\mu} \left(\frac{h_W}{1+h_W} \right) \frac{1}{\overline{\overline{E}}} \right\rfloor 2 \left(1 + \frac{1}{h_w}\right) \overline{E} \le \frac{2}{\mu}\, a.$$

□

Step 6. If for some $l > 1$,

$$\kappa_l * \tilde{u} \sim \tilde{u} \quad \text{then} \quad \overline{L} \gtrsim l.$$

More precisely:

Claim. Set $\mu = 64$. There exists a constant c, depending only on dimension, N, and on a, such that for any $l > 1$:

$$\text{If} \quad \fint_\Omega |\tilde{u} - \kappa_l * \tilde{u}|\, dx < \frac{2}{\mu}\, a \quad \text{then} \quad \overline{L} > cl. \tag{26}$$

We split the proof of this claim in four parts. First we establish two auxiliary claims. Then we prove the claim (26) for the Wasserstein metric case and for the H^{-1} metric case separately.

Step 6a. Let $A_l := \{x \in \Omega \; : \; \tilde{u} * \kappa_l > \frac{7}{8}\}$. If

$$\fint_\Omega |\tilde{u} - \kappa_l * \tilde{u}|\, dx < \frac{2}{\mu}\, a$$

Then

$$\|A_l\| > \frac{\mu - 32}{2\mu}\, a. \tag{27}$$

By the assumption

$$\frac{2}{\mu}\, a > \fint_\Omega |\tilde u - \kappa_l * \tilde u|\, dx \geq \fint_{A\setminus A_l} \frac{1}{8}\, dx = \frac{1}{8}\|A\setminus A_l\|.$$

From (23) we have $\|A\| > \frac{a}{2}$. Combining the two inequalities gives

$$\|A_l\| \geq \|A\| - \|A\setminus A_l\| > \left(\frac{1}{2} - \frac{16}{\mu}\right) a.$$

Remark. From this point on the proof does not require the closeness of $\tilde u$ and $\kappa_l * \tilde u$ explicitly, rather it only uses the fact that A_l is large, as described by (27). That is we only need that after $\tilde u$ is averaged over radius l it still has well developed interfaces.

Step 6b. Significant subset of A_l can be well approximated by balls of radius l. More precisely:

Claim. Set $\mu = 64$. There exists a finite subset, J, of A_l such that for $A_{ball} = \cup_{x\in J} B(x,l)$:

$$\frac{8}{7}\|A_{ball} \cap A\| \geq \|A_{ball}\| > \frac{1}{2^{N+2}}\, a. \tag{28}$$

This claim is has its roots in [5], see also [17]. Let J be a maximal family of points in A_l such that balls in $\{B(x,l)\}_{x\in J}$ are disjoint. Then $A_l \subset \cup_{x\in J} B(x,2l)$, by definition. Therefore, using (27)

$$\|A_{ball}\| \geq \frac{1}{2^N}\|A_l\| > \frac{1}{2^{N+2}} a.$$

Since $J \subset A_l$, for all $x \in J$ we have that

$$\frac{7}{8} \leq k_l * \tilde u(x) \leq \frac{|B(x,l) \cap A|}{|B(x,l)|}$$

Summing over $x \in J$ gives the first inequality.

Step 6c (Wasserstein). Let $\gamma = \left(\frac{9}{8}\right)^{1/N} - 1$. For set U, and $\lambda \geq 0$ let

$$U^\lambda := \{x \in \Omega \,:\, \operatorname{dist}(x,U) \leq \lambda\}.$$

Let $\lambda = \gamma l$. Using that $u \geq 0$ and Lemma 8

$$\begin{aligned}
\overline{L}^2 = \frac{d_{Wass}(u,a)^2}{|\Omega|} &\geq \lambda^2 \left(\fint_{A_{ball}} u(x)dx - a\|A^\lambda_{ball}\| \right) \\
&\geq \lambda^2 \left(\frac{7}{8}\|A \cap A_{ball}\| - a\left(1 + \frac{\lambda}{l}\right)^N \|A_{ball}\| \right) \\
&\geq \gamma^2 l^2 \left(\frac{49}{64} - \frac{1}{2}(1+\gamma)^N \right) \|A_{ball}\| \\
&\geq \gamma^2 \frac{1}{5}\frac{1}{2^{N+2}} a\, l^2.
\end{aligned}$$

Combining the conclusions of Steps 5 and 6 now proves the interpolation inequality (22). In particular

$$\overline{L} \geq c(N) r \left(\frac{h_W}{1+h_W} \right) a^{3/2} \frac{1}{\overline{\overline{E}}}.$$

Step 6d (H^{-1}) Assume $\overline{E} < \frac{1}{20}\frac{1}{2^{N+2}} a h_w$. To obtain a lower bound on $\overline{L}$, given by (20), we first build a local test function. For $\gamma > 1$ to be determined, let $\eta : [0,\infty) \to [0,1]$ be defined by

$$\eta(z) := \begin{cases} 1 & \text{if } z \in [0,l] \\ l - \frac{z-l}{\gamma l - l} & \text{if } l < z < \gamma l \\ 0 & \text{if } \gamma l \leq z. \end{cases}$$

Let $\bar{\xi}(x) := \eta(|x|)$. This is the local test function. Assume, for the moment, that $0 \in A_l$. Let $\hat{u}(x) := \max\left\{u(x), -\frac{a}{4}\right\}$. Then

$$\int_{B(0,\gamma l)} |\nabla \bar{\xi}|^2 dx = (\gamma^N - 1)|B(0,l)| \frac{1}{(\gamma-1)^2 l^2} \tag{29}$$

Also

$$\begin{aligned}
&\int_{B(0,\gamma l)} (\hat{u}-a)\bar{\xi}dx \\
&\qquad \geq \frac{7}{8}|A\cap B(0,l)| - \frac{a}{4}\left(|B(0,l)\backslash A| + (\gamma^N-1)|B(0,l)|\right) - a\gamma^N|B(0,l)| \\
&\qquad \geq \left(\left(\frac{7}{8}\right)^2 - \frac{a}{32} - \frac{a(\gamma^N-1)}{4} - a\gamma^N\right)|B(0,l)|
\end{aligned}$$

Let us now set $\gamma = \left(\frac{8}{7}\right)^{1/N}$. Then

$$\int_{B(0,\gamma l)} (\hat{u}-a)\bar{\xi}dx \geq \frac{3}{20}|B(0,l)|. \tag{30}$$

Now let us construct the (global) test function on Ω. Let

$$\xi(x) = \sup_{y\in J} \bar{\xi}(x-y).$$

Using (29), (23), and (28) one obtains

$$⨍_\Omega |\nabla\xi|^2 dx \leq \frac{\gamma^N-1}{(\gamma-1)^2 l^2}\|A_{ball}\| \leq \frac{a}{(\gamma-1)^2 l^2}. \tag{31}$$

Using that balls of radius l centered at points in J are disjoint we obtain

$$\begin{aligned}
⨍_\Omega (u-a)\xi dx &\geq ⨍_\Omega (\hat{u}-a)\xi dx + ⨍_{\{u<-\frac{a}{4}\}} u dx \\
&\geq \frac{3}{20}\|A_{ball}\| - \frac{1}{h_W}⨍_\Omega W(u)dx \\
&\geq \frac{3}{20}\,\frac{1}{2^{N+2}}a - \frac{\overline{E}}{h_w} \\
&\geq \frac{1}{10}\,\frac{1}{2^{N+2}}a.
\end{aligned}$$

Here we used the assumption $\overline{E} < \frac{1}{20}\frac{1}{2^{N+2}}ah_w$. Therefore

$$\overline{L} \geq \frac{⨍_\Omega (u-a)\xi dx}{\sqrt{⨍_\Omega |\nabla\xi|^2 dx}} \geq \tilde{c}(N)\sqrt{a}l.$$

The definition of l now implies

$$\overline{L} \geq c(N) r \left(\frac{h_W}{1+h_W} \right) a^{3/2} \frac{1}{\overline{E}}$$

which proves the interpolation inequality. □

Remark 2. *In the case $N = 2$, one can obtain a sharper result with respect to scaling in a (as $a \to 0^+$) by considering more carefully constructed test functions. This was done for the Mullins–Sekerka evolution by Conti, Niethammer, and Otto in* [5]*. In particular if γ is taken of size $a^{-1/2}$, and on $[l, cl]$, we replace the linear η by the optimal one, $\eta(z) = (\ln \gamma l - \ln z)/\ln \gamma$. By using such test function one obtains that*

$$\overline{L} \geq c(N) r \left(\frac{h_W}{1+h_W} \right) a^{3/2} |\ln a|^{1/2} \frac{1}{\overline{E}}.$$

4.1. **Interpolation inequalities for the local energy.** Let us now consider the case of the local energy density:

$$\overline{E}(u) := \fint_\Omega \frac{1}{2} |\nabla u(x)|^2 + W(u(x)) dx. \tag{32}$$

The method developed in the proof of Theorem 6 applies to the local energy with minor modifications.

Corollary 7 (Interpolation inequality)**.** *Let $0 < a \leq \frac{1}{2}$. Assume W satisfies (W1)-(W3). There exists a constant $C = C(a, h_W) > 0$ such that for all $\Lambda > 0$ and all configurations $u \in \mathcal{M}$ for which $\overline{E}(u) < \frac{a}{64}\left(\frac{h_W}{1+h_W}\right)$, and in the H^{-1} case also $\overline{E}(u) < \frac{1}{2^{N+2}} \frac{a h_W}{20}$, the following holds*

$$\overline{E}(u) \overline{L}(u) \geq C. \tag{33}$$

The constant $C = c(N) \left(\frac{h_W}{1+h_W}\right) a^{3/2}$.

Proof. Note that Step 1 is not needed, while the estimate of Step 2 only used the W term which is the same for both the local and the nonlocal energy. The main fact we need to check is statement of Step 3, which we prove below for $r^2 = \frac{1}{2}$. Steps 4, 5, and 6 do not require any modifications.

To prove that

$$\fint_\Omega |\tilde{u} - \kappa_r * \tilde{u}| dx \leq \left(\frac{16}{9} + \frac{2}{h_W} \right) \overline{E}$$

with $r^2 = \frac{1}{2}$ we begin as in Step 3 of Theorem 6:

$$\begin{aligned}
&\fint_\Omega \left|\tilde{u}(x) - \int_{\mathbb{R}^N} \kappa_r(x-y)\tilde{u}(y)dy\right| dx \\
&\qquad \le \left(\frac{3}{4}\right)^{-2} \fint_\Omega \int_{\mathbb{R}^N} |u(x)-u(y)|^2 \kappa_r(x-y) dydx + 2\left\|\left\{\frac{a}{4} \le u \le \frac{7}{8}\right\}\right\| \\
&\qquad \le \frac{16}{9} \fint_\Omega \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(x)-u(y)|^2 dydx + \frac{2}{h_W}\overline{E}.
\end{aligned}$$

It remains to further estimate the first term:

$$\begin{aligned}
&\fint_\Omega \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(x)-u(y)|^2 dydx \\
&\qquad \le \fint_\Omega \frac{1}{|B(0,r)|} \int_{B(x,r)} |x-y|^2 \left|\int_0^1 \nabla u(x - s(x-y))ds\right|^2 dydx \\
&\qquad \le \frac{1}{2} \fint_\Omega \frac{1}{|B(0,r)|} \int_{B(0,r)} \int_0^1 |\nabla u(x - sz)|^2 dsdzdx \\
&\qquad \le \frac{1}{2} \frac{1}{|B(0,r)|} \int_{B(0,r)} \int_0^1 2\overline{E} dsdz = \overline{E}.
\end{aligned}$$

□

5. Appendix

5.1. **A property of Wasserstein distance.** The following lemma is analogous to Lemma 5 in [17]: We state it in large generality, the reason being that we want to consider Ω with metric from the torus $R^N/(\Lambda\mathbb{Z})^N$. In applications σ is the Labesgue measure.

Lemma 8. *Let (Ω, d, σ) be a metric space endowed with finite measure σ. Let $u \in L^1(\Omega)$ be a nonnegative function with average $a := \fint_\Omega u(x)d\sigma(x)$. Let $A \subset \Omega$ measurable, and let $A^l := \{x \in \Omega \,:\, d(x,A) \le l\}$. Then*

$$d^2_{Wass}(u,a) \ge l^2 \left(\int_A u(x)d\sigma(x) - a\sigma(A^l)\right).$$

Proof. We use the definition of Wasserstein distance. Let π be an admissible transportation plan, that is a measure on $\Omega \times \Omega$ with marginals $u(x)d\sigma(x)$ and

$a\sigma(y)$. Then

$$\begin{aligned}\int_{\Omega\times\Omega}|x-y|^2 d\pi(x,y) &= \int_{A\times(\Omega\setminus A^l)}|x-y|^2 d\pi(x,y)\\ &\geq l^2\,\pi(A\times(\Omega\setminus A^l))\\ &\geq l^2(\pi(A\times\Omega)-\pi(\Omega\times A^l))\\ &= l^2\left(\int_A u(x)d\sigma(x)-\int_{A^l} a d\sigma(y)\right)\\ &= l^2\left(\int_A u(x)d\sigma(x)-a\sigma(A^l)\right).\end{aligned}$$

□

Acknowledgements. The author would like to thank Andrea Bertozzi and Chad Topaz for helpful discussions. The research was partially supported by NSF grant DMS-0638481. The author would also like to thank the Center for Nonlinear Analysis (NSF grants DMS-0405343 and DMS-0635983) for its support during the preparation of this paper.

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