On a variational approach for Stokes conjecture in water waves: existence of regular waves

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Abstract

Using the variational approach of Alt and Caffarelli in this paper we give an alternative proof of a theorem of Keady and Norbury on the existence of a family of regular water waves.

Dedicated to Paolo Marcellini on the occasion of his 60th birthday

1 Introduction

Consider an ideal liquid of either finite or infinite depth and consider a wave of constant shape that moves with speed c on the free surface of the liquid. Assume that the motion is two-dimensional (i.e. the motion is independent of the coordinate in the horizontal direction perpendicular to the velocity of the wave), irrotational and in a vertical plane. By taking axes fixed relative to a crest of the wave, the problem becomes one of steady motion. In the case of infinite depth the fluid domain is

$$D = \left\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R},\, y < f(x)\right\},$$

where $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$ is the free surface. For simplicity, assume that f has period ℓ, has a single crest per wavelength, and is symmetrical about that crest. If $\mathbf{v}$ is the velocity of the fluid and ρ its density, the conservation of mass and the fact that the fluid is irrotational, which are usually given in the form

$$\frac{d\rho}{dt} + \operatorname{div}(\rho\mathbf{v}) = 0 \quad \text{in } D,$$
$$\operatorname{curl}\mathbf{v} = 0 \quad \text{in } D,$$

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under the present assumptions simplify to $\operatorname{div}\mathbf{v} = 0$ and $\operatorname{curl}\mathbf{v} = 0$ in D. Hence, we may write $\mathbf{v} = (u_y, -u_x)$, where the potential u satisfies the following problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D, \\ \mathbf{v} = (u_y, -u_x) &\to (c, 0) \quad \text{as } y \to -\infty. \end{aligned}$$

Moreover, from Bernoulli's Theorem, in the steady motion of an inviscid fluid at every point of the same streamline one has $\frac{p}{\rho} + K$ =const, where p is the pressure and K is the energy per unit mass of the fluid. In the present setting this becomes

$$u = 0, \quad \frac{1}{2}|\nabla u|^2 + gy = \text{const} \quad \text{on } \Gamma,$$

where g is the gravitational acceleration and K has been taken as the sum of the kinetic and potential energy. Thus we are led to the following free boundary problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D, \\ u(x+\ell, y) &= u(x,y) \quad \text{in } D, \\ u = 0, &\quad \frac{1}{2}|\nabla u|^2 + gy = \text{const} \quad \text{on } \Gamma, \\ (u_y, -u_x) &\to (c, 0) \quad \text{as } y \to -\infty. \end{aligned} \tag{1.1}$$

In 1847 Stokes (see [Sto], [Sto2]) assumed that, for fixed c and ℓ, there exists a family of solutions for this problem which are parametrized by the height H of the wave, $H := \max f - \min f$, and he conjectured that there exists a wave of greatest height, which is characterized by the fact that its shape is not regular but has sharp crests of included angle $\frac{2}{3}\pi$. He also conjectured that for this wave $f'' > 0$ in $\left(0, \frac{\ell}{2}\right)$.

Stokes conjectures have been proved in a series of papers. The starting point was a paper of Nekrasov in 1922 (see [MT]), where he used a hodograph transformation to map the region under one period onto the unit circle, and the point at infinite depth onto the origin of the complex plane. More precisely, setting

$$\varphi(s) = \arctan f'(x), \tag{1.2}$$

one obtains the nonlinear integral equation

$$\varphi(s) = \frac{1}{3}\int_0^\pi K(s,t)\,\frac{\sin\varphi(t)}{\mu + \int_0^t \sin\varphi(\tau)\,d\tau}\,dt, \quad 0 < s \le \pi, \tag{1.3}$$

where

$$K(s,t) := \frac{1}{\pi}\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|,$$

$(s,t) \in [0,\pi] \times [0,\pi]$, $s \neq t$, and $\mu = \frac{2\pi q_0^3}{3g\ell c}$, with q_0 is the velocity of the fluid at the point $(0, f(0))$. Hence the value $\mu = 0$ corresponds to $q_0 = 0$ namely to a stagnation point of the fluid. Thus solutions of (1.3) with $\mu > 0$ correspond to *regular waves*, while in the case $\mu = 0$ one has a *Stokes wave*.

In the case of finite depth the kernel $K(s,t)$ should be replaced by

$$K(s,t) := \frac{1}{\pi} \log \left| \frac{\operatorname{sn} \frac{1}{\pi} T (s+t)}{\operatorname{sn} \frac{1}{\pi} T (s-t)} \right|,$$

where sn denotes the Jacobian elliptic function whose quarter periods T and iT' satisfy $\frac{T'}{T} = \frac{4h}{\ell}$ and h is the mean depth of the fluid.

The first existence result for solutions of Nekrasov's integral equation (1.3) is due to Krasovski [K] in 1961, who, using a degree theory argument, proved that for every angle $0 < \beta < \frac{\pi}{6}$ there exist $0 < \mu < \frac{1}{3}$ and a continuous solution φ_μ of (1.3) with

$$\sup_{s \in [0,\pi]} \varphi_\mu(s) = \beta.$$

The problem of existence of regular waves was completely settled by Keady and Norbury [KN] in 1978, who, again using degree theory arguments, proved that for every $0 < \mu < \frac{1}{3}$ there exists a continuous solution φ_μ of (1.3), while there are no solutions for $\mu \geq \frac{1}{3}$.

Toland [To] in 1978 and McLeod [ML2] in 1979 showed that as $\mu \to 0^+$ the regular waves φ_μ converge to a solution φ_0 of the limiting problem $\mu = 0$ and proved that if the limit $\lim_{s \to 0^+} \varphi_0(s)$ exists, then it must be $\frac{\pi}{6}$. The existence of the limit was proved by Amick, Fraenkel, and Toland [AF2] in 1982, and independently by Plotnikov [Pl] in 1982.

Finally, in 2004 Plotnikov and Toland [PlT] proved that there exist Stokes waves with $f'' > 0$ in $\left(0, \frac{\ell}{2}\right)$.

Although the hodograph transform (1.2) has proved to be quite successful in tackling Stokes conjectures, from an intuitive point of view it is difficult to visualize the qualitative properties of the solutions of the nonlinear integral equation (1.3).

In this paper we address the free boundary problem (1.1) using a variational approach. More precisely following the work of Alt and Caffarelli [AC], we minimize the functional

$$J_\lambda(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}} (\lambda - y)_+ \right) d\mathbf{x}, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2, \tag{1.4}$$

in the set

$$K := \Big\{ u \in L^1_{\mathrm{loc}}(\Omega),\ \nabla u \in \left(L^2_{\mathrm{loc}}(\Omega)\right)^2,\ u(x,0) = u_0(x) \text{ for } x \in (-1,1), \tag{1.5}$$
$$u(-1,y) = u(1,y) = 0 \text{ for } y \in (0,\infty) \},$$

where $u|_{\partial\Omega}$ denotes the trace of u, $u_0 \in C^1_c(-1,1)$, $u_0 \geqslant 0$ and u_0 is not identically 0. Here $\Omega = (-1,1) \times (0,\infty)$ and the parameter $\lambda > 0$ plays the role of the parameter μ in (1.3).

Existence of minimizers of (1.4) follows from [AC], where the authors study existence and regularity of minimizers of the functional

$$J_{\lambda}(u) = \int_{\Omega} \left(|\nabla u|^2 + \chi_{\{u>0\}} Q(\mathbf{x}) \right) d\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1.6}$$

where

$$0 < Q_{\min} \leq Q(\mathbf{x}) \leq Q_{\max}. \tag{1.7}$$

Note that in our case the function Q reduces to

$$Q(x, y) = (\lambda - y)_+$$

so that near $y = \lambda$ we lose the lower bound $Q_{\min} > 0$.

This is a crucial point in the analysis since it leads to a loss of regularity. Indeed, (1.7) was used in [AC] to prove that minimizers of (1.6) decay linearly near the free boundary, and thus they are regular. On the contrary, our results imply that minimizers u whose support touch the line $y = \lambda$ decay like $(y - \lambda)^{3/2}$. This is the expected decay of a Stokes wave u. Indeed, as shown, e.g., in the monograph of Grisvard [Gris], the solution of the Dirichlet boundary problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned}$$

where D is the domain given in polar coordinates by,

$$D = \{(r, \theta) : r > 0,\ 0 < \theta < \omega\}$$

is the function

$$u = r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi\theta}{\omega}\right). \tag{1.8}$$

Hence if u decays like $r^{\frac{3}{2}}$, than necessarily $\omega = \frac{2\pi}{3}$ as conjectured by Stokes. Note that in [AC] the linear decay of minimizers leads to an angle $\omega = \pi$, and thus one has a regular free boundary.

In this paper we prove the existence of a critical value $\lambda_c \in (0, \infty)$ with the property that:

- If $u_\lambda \in K$ is a minimizer of the functional J_λ for $\lambda > \lambda_c$, then the support of u_λ remains below the line $y = \lambda$, and thus u_λ and its free-boundary $\partial\{u_\lambda > 0\}$ are smooth by the regularity results of Alt and Caffarelli. Note that this gives a completely different proof of a theorem of Keady and Norbury [KN] on the existence of a family of regular water waves.

- If $u_\lambda \in K$ is a minimizer of the functional J_λ for $\lambda < \lambda_c$, then the support of u_λ crosses the line $y = \lambda$. We refer to this kind of waves as *non-physical waves.*

We also prove that if $\lambda_n \searrow \lambda_c$ and $\mu_n \nearrow \lambda_c$ then the corresponding sequences of minimizers $\{u_{\lambda_n}\}$ and $\{u_{\mu_n}\}$ converge strongly in $H^1_{\text{loc}}(\Omega)$ to two minimizers u^+ and $u^- \in K$ of J_{λ_c}, respectively. Moreover $\operatorname{supp} u^+ \subset \{y \leq \lambda_c\}$, while $\operatorname{supp} u^-$ intersects the line $y = \lambda_c$. We conjecture that $u^+ = u^-$. Note that if the conjecture were true, then the support of u^+ would touch the line $y = \lambda_c$ and be contained in the set $\{y \leq \lambda_c\}$. This would give an alternative proof of the theorems of Tolland and [To] in 1978 and McLeod [ML2] on existence of a Stokes wave. We have been unable to prove the conjecture.

In a forthcoming paper we will study the regularity of waves whose support touch the line $y = \lambda$.

2 Regular waves

We begin with some preliminary results that are due to Alt and Caffarelli [AC]. We present their proof for the convenience of the reader.

Theorem 2.1 (Theorem 1.3 in [AC]) *There exists an absolute minimizer $u \in K$ of the functional J_λ.*

Proof. Let $\alpha := \inf_{v \in K} J_\lambda(v)$ and let $\{u_n\} \subset K$ be a minimizing sequence for J_λ, that is, $J_\lambda(u_n) \to \alpha$ as $n \to \infty$. Since

$$J_\lambda((u_n)_+) \leq J_\lambda(u_n)$$

and $(u_n)_+$ is still admissible, without loss of generality we may assume that $u_n \geq 0$ and that

$$J_\lambda(u_n) \leq \alpha + 1 \text{ for all } n \in \mathbb{N}.$$

Then $\{\nabla u_n\}$ is bounded in $\left(L^2(\Omega)\right)^2$.

Let $\Omega_R := (-1,1) \times (0,R)$. By a standard slicing argument we have that $u_n(x,\cdot)$ is absolutely continuous in $(0,R)$ for $\mathcal{L}^1$ a.e. $x \in (-1,1)$ and

$$u_n(x,y) = u_0(x) + \int_0^y \frac{\partial u_n}{\partial y}(x,s)\,ds$$

for $\mathcal{L}^1$ a.e. $x \in (-1,1)$ and all $y \in (-1,1)$. If we square this identity and use Cauchy's inequality we get:

$$u_n^2(x,y) \leq 2\left(u_0^2(x) + \int_0^R |\nabla u_n(x,s)|^2\,ds\right).$$

Integrating in x gives

$$\begin{aligned}\int_{-1}^1 u_n^2(x,y)\,dx &\leq 2\int_{-1}^1 u_0^2(x)\,dx + 2\int_{\Omega_R} |\nabla u_n(x,s)|^2\,dsdx \\ &\leq C\end{aligned}$$

for all $y \in (0, R)$ and for all $n \in \mathbb{N}$. By integrating in y we conclude that:

$$\int_{-1}^{1} \int_{0}^{R} u_n^2(x, y)\, dxdy \leq CR$$

for all $n \in \mathbb{N}$. Therefore $\{u_n\}$ is bounded in $H^1(\Omega_R)$. Since $H^1(\Omega_R)$ is compactly embedded in $L^p(\Omega_R)$, $1 \leq p < \infty$, $\{u_n\}$ admits a subsequence (not relabeled) that converges weakly in $H^1(\Omega_R)$ and strongly in $L^p(\Omega_R)$ to a function $u^R \in H^1(\Omega_R)$.

If we now let $S > R$ and extract a further subsequence we may assume that $u_n \rightharpoonup u^R$ in $H^1(\Omega_R)$ and $u_n \rightharpoonup u^S$ in $H^1(\Omega_S)$. By the uniqueness of the weak limit we have that

$$u^R(\mathbf{x}) = u^S(\mathbf{x}) \text{ for } \mathcal{L}^2 \text{ a.e. } \mathbf{x} \in \Omega_R.$$

Taking a sequence $R_k\ (:= k) \nearrow \infty$ and using a diagonalization argument we may find a subsequence of $\{u_n\}$ (again not relabeled) weakly convergent in $H^1_{\mathrm{loc}}(\Omega)$ to the non negative function

$$u(x, y) := u^{R_k}(x, y) \text{ if } R_{k-1} \leq y < R_k,\ k \in \mathbb{N}.$$

Moreover, since $\left\{\chi_{\{u_n>0\}}\right\}$ is bounded in $L^\infty(\Omega)$, we may find a function $\gamma \in L^\infty(\Omega)$, $0 \leq \gamma \leq 1$ and yet another subsequence such that $\left\{\chi_{\{u_n>0\}}\right\}$ converges weakly star to γ in $L^\infty(\Omega)$.

Next we will prove that $\gamma \geq \chi_{\{u>0\}}$. Since $u_n \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\left\{\chi_{\{u_n>0\}}\right\}$ converges weakly star to γ in $L^\infty(\Omega)$, letting $n \to \infty$ in the identity

$$\int_{\Omega_R} u_n(1 - \chi_{\{u_n>0\}})\, d\mathbf{x} = 0$$

yields

$$\int_{\Omega_R} u(1 - \gamma)\, d\mathbf{x} = 0 \quad \text{for all } R > 0.$$

Since $u \geq 0$ and $\gamma \leq 1$, we conclude that $u(1-\gamma) = 0$ $\mathcal{L}^2$ a.e. in Ω. Therefore $\gamma = 1$ $\mathcal{L}^2$ a.e. in the set $\{u > 0\}$, and so

$$\gamma \geq \chi_{\{u>0\}}.$$

This is sufficient to show that u is a minimizer. Indeed, for any $R > 0$

$$\begin{aligned}
&\int_{\Omega_R} \left(|\nabla u|^2 + \gamma\,(\lambda - y)_+\right) d\mathbf{x} \\
&\quad \leq \liminf_{n\to\infty} \int_{\Omega_R} |\nabla u_n|^2\, d\mathbf{x} + \lim_{n\to\infty} \int_{\Omega_R} \chi_{\{u_n>0\}}(\lambda - y)_+\, d\mathbf{x} \\
&\quad \leq \lim_{n\to\infty} J_\lambda(u_n) = \alpha.
\end{aligned}$$

Letting $R \to \infty$ and using the fact that $\gamma \geq \chi_{\{u>0\}}$ yields

$$\begin{aligned}\alpha \leq J_\lambda(u) &= \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+)\, d\mathbf{x} \\ &\leq \int_\Omega (|\nabla u|^2 + \gamma(\lambda - y)_+)\, d\mathbf{x} \leq \alpha.\end{aligned}$$

Thus $J_\lambda(u) = \alpha$. ∎

Definition 2.2 *Let $D \subset \mathbb{R}^2$ be an open set. A function u in $L^1_{\mathrm{loc}}(D)$ is* subharmonic *if*

$$\int_D u\Delta\varphi\, d\mathbf{x} \geq 0 \text{ for all nonnegative functions } \varphi \in C_c^\infty(D).$$

Lemma 2.3 (Lemma 2.2 in [AC]) *Let $u \in K$ be a minimizer of J_λ. Then u is subharmonic and for any $\mathbf{x}_0 \in \Omega$ and $0 < S < R < \operatorname{dist}(\mathbf{x}_0, \partial\Omega)$ we have*

$$\frac{1}{|B_R(\mathbf{x}_0)|} \int_{B_R(\mathbf{x}_0)} u(\mathbf{y})\, d\mathbf{y} \geq \frac{1}{|B_S(\mathbf{x}_0)|} \int_{B_S(\mathbf{x}_0)} u(\mathbf{y})\, d\mathbf{y}.$$

Proof. For nonnegative functions $\varphi \in C_0^\infty(\Omega)$ and for any $\varepsilon > 0$ the function $u - \varepsilon\varphi$ belongs to K, and since $u - \varepsilon\varphi \leq u$, we have

$$\{\mathbf{x} \in \Omega : (u - \varepsilon\varphi)(\mathbf{x}) > 0\} \subset \{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\}.$$

Therefore

$$0 \leq J_\lambda(u - \varepsilon\zeta) - J_\lambda(u) \leq \varepsilon^2 \int_\Omega |\nabla\varphi|^2\, d\mathbf{x} - 2\varepsilon \int_\Omega \nabla u \cdot \nabla\varphi\, d\mathbf{x}.$$

Dividing by ε and letting $\varepsilon \searrow 0$ yields

$$0 \geq \int_\Omega \nabla u \cdot \nabla\varphi\, d\mathbf{x} = -\int_\Omega u\Delta\varphi\, d\mathbf{x}.$$

Hence u is subharmonic.

Note that a simple density argument yields

$$\int_\Omega u\Delta\varphi\, d\mathbf{x} \geq 0$$

for all nonnegative functions $\varphi \in H^2(\Omega)$ with compact support.

To prove the second part of the lemma, fix $\mathbf{x}_0 \in \Omega$ and

$$0 < S < R < \operatorname{dist}(\mathbf{x}_0, \partial\Omega)$$

as in the statement. We construct a suitable test function φ that relates both averages. We start with the fundamental solution $V(\mathbf{x}) := -\log|\mathbf{x}|$, and fit under its graph a paraboloid P_R tangent to V at $|\mathbf{x}| = R$. We compute this paraboloid to be

$$P_R(\mathbf{x}) = \log R + \frac{1}{2} - \frac{|\mathbf{x}|^2}{2R^2}.$$

We define

$$V_R(\mathbf{x}) := \begin{cases} V(\mathbf{x}-\mathbf{x}_0) - P_R(\mathbf{x}-\mathbf{x}_0) & \text{for } |\mathbf{x}-\mathbf{x}_0| < R, \\ 0 & \text{otherwise.} \end{cases}$$

Note that V_R is $C^{1,1}$ except near $\mathbf{x}_0$ and

$$\Delta V_R = \frac{2}{R^2}\chi_{B_R(\mathbf{x}_0)} \quad \text{in } \mathbb{R}^2.$$

Since $V_R \geq V_S$, we define the nonnegative function

$$\varphi := V_R - V_S.$$

Moreover, since

$$\varphi(\mathbf{x}) = P_S(\mathbf{x}-\mathbf{x}_0) - P_R(\mathbf{x}-\mathbf{x}_0) \text{ for } |\mathbf{x}-\mathbf{x}_0| < S,$$

it follows that $\varphi \in C_c^{1,1}(\Omega) \cap H^2(\Omega)$ with

$$\Delta\varphi = \frac{2}{R^2}\chi_{B_R} - \frac{2}{S^2}\chi_{B_S} \quad \text{in } \mathbb{R}^2.$$

Since u is subharmonic, we have

$$0 \leq \int_\Omega u\Delta\varphi \, d\mathbf{x} = 2\pi\left(\frac{1}{|B_R(\mathbf{x}_0)|}\int_{B_R(\mathbf{x}_0)} u \, d\mathbf{x} - \frac{1}{|B_S(\mathbf{x}_0)|}\int_{B_S(\mathbf{x}_0)} u \, d\mathbf{x}\right),$$

which proves the lemma. ■

Remark 2.4 *In view of the previous lemma we can work with the precise representative*

$$u(\mathbf{x}) := \lim_{R\to 0^+} \frac{1}{|B_R(\mathbf{x})|}\int_{B_R(\mathbf{x})} u(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in \Omega.$$

Note that the minimizer constructed in Theorem 2.1 was nonnegative. We now prove all minimizers are actually nonnegative and bounded.

Lemma 2.5 (Lemma 2.3 in [AC]) *Let $u \in K$ be a minimizer of J_λ. Then*

$$0 \leq u(\mathbf{x}) \leq \sup_{[-1,1]} u_0$$

for $\mathcal{L}^2$ a.e. $\mathbf{x}$ in Ω.

Proof. Since $u_0 \geq 0$, we have that $\min\{u,0\} = 0$ on $(-1,1)\times\{0\}$. Therefore $u_\varepsilon = u - \varepsilon \min\{u,0\}$ belongs to K and $\{u_\varepsilon > 0\} \subset \{u > 0\}$. Hence, since $J_\lambda(u) \leq J_\lambda(u_\varepsilon)$, it follows that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, d\mathbf{x} \geq \int_\Omega |\nabla u|^2 \, d\mathbf{x},$$

or, equivalently,

$$\int_\Omega |\nabla u|^2 \, d\mathbf{x} - \varepsilon \int_\Omega \nabla u \cdot \nabla (\min\{u,0\}) \, d\mathbf{x} + \varepsilon^2 \int_\Omega |\nabla (\min\{u,0\})|^2 \, d\mathbf{x} \geq \int_\Omega |\nabla u|^2 \, d\mathbf{x}.$$

Dividing by ε and letting ε go to 0, we obtain

$$\int_\Omega \nabla u \cdot \nabla (\min\{u,0\}) \, d\mathbf{x} \leq 0.$$

This implies that

$$0 \geq \int_\Omega \nabla u \cdot \nabla (\min\{u,0\}) \, d\mathbf{x} = \int_{\Omega \cap \{u<0\}} |\nabla (\min\{u,0\})|^2 \, d\mathbf{x}.$$

Hence $\min\{u,0\}$ is constant, which implies that $u \geq 0$ $\mathcal{L}^2$ a.e. in Ω.

To prove the other inequality, let $m := \sup_{[-1,1]} u_0$, $v := \max\{u - m, 0\}$, and $u_\varepsilon := u + \varepsilon v$. Again $v = 0$ on $(-1,1)\times\{0\}$ and $\{u_\varepsilon > 0\} \subset \{u > 0\}$, therefore $J_\lambda(u) \leq J_\lambda(u_\varepsilon)$ implies as before that

$$\int_\Omega |\nabla u|^2 \, d\mathbf{x} - \varepsilon \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} + \varepsilon^2 \int_\Omega |\nabla v|^2 \, d\mathbf{x} \geq \int_\Omega |\nabla u|^{\ 2} d\mathbf{x}$$

and as before we conclude that

$$\int_\Omega |\nabla v|^2 \, d\mathbf{x} = \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} \leq 0,$$

which means that v is a constant, thus $u \leq m$. ∎

Next we adapt the proof of Lemma 3.2 of [AC] to our setting.

Theorem 2.6 *There is a constant $C_{\max} > 0$ such that for every (small) ball $B_r(\mathbf{x}_0) \subset \Omega$, $\mathbf{x}_0 = (x_0, y_0)$, and for any minimizer $u \in K$ of J_λ, if*

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 \geq C_{\max} \sqrt{(\lambda - y_0 + r)_+},$$

then $u > 0$ in $B_r(\mathbf{x}_0)$.

Proof. **Step 1:** For simplicity we denote $B_r(\mathbf{x}_0)$ simply by B_r. Consider the harmonic function v in B_r with boundary values u. Since $u \geq 0$, from the maximum principle we have that $v > 0$ in B_r. Outside B_r define $v := u$. Since $J_\lambda(u) \le J_\lambda(v)$, we have

$$\int_{B_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+ \right) \, d\mathbf{x} \le \int_{B_r} \left(|\nabla v|^2 + (\lambda - y)_+ \right) \, d\mathbf{x}. \tag{2.1}$$

Since $\Delta v = 0$ in B_r and $u - v = 0$ on ∂B_r, by the first Green's formula we have

$$\int_{B_r} \nabla v \cdot (\nabla u - \nabla v) \, d\mathbf{x} = 0,$$

or, equivalently,

$$\int_{B_r} |\nabla v|^2 \, d\mathbf{x} = \int_{B_r} \nabla u \cdot \nabla v \, d\mathbf{x}.$$

In turn,

$$\begin{aligned}
\int_{B_r} (|\nabla u|^2 - |\nabla v|^2) \, d\mathbf{x} &= \int_{B_r} (|\nabla u|^2 + |\nabla v|^2 - 2\,|\nabla v|^2) \, d\mathbf{x} \\
&= \int_{B_r} (|\nabla u|^2 + |\nabla v|^2 - 2\nabla u \cdot \nabla v) \, d\mathbf{x} \\
&= \int_{B_r} |\nabla u - \nabla v|^2 \, d\mathbf{x}.
\end{aligned}$$

If we use this in (2.1), we find

$$\int_{B_r(\mathbf{x}_0)} |\nabla u - \nabla v|^2 \, d\mathbf{x} \le \int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}}(\lambda - y)_+ \, d\mathbf{x}. \tag{2.2}$$

We want to control the right-hand side of the previous inequality by the left-hand side.

Step 2: In this step, for simplicity in the notation, we take $\mathbf{x}_0$ to be the origin. For $|\mathbf{z}| \le \frac{1}{2}r$ consider the transformation $T : B_r \to B_r$ defined by

$$T(\mathbf{x}) :\, = \left(1 - \frac{|\mathbf{x}|}{r}\right) \mathbf{z} + \mathbf{x}, \quad \mathbf{x} \in B_r. \tag{2.3}$$

Note that $T(\mathbf{0}) = \mathbf{z}$. For $\mathbf{x} \in B_r$ define

$$u_{\mathbf{z}}(\mathbf{x}) := u(T(\mathbf{x})), \quad v_{\mathbf{z}}(\mathbf{x}) := v(T(\mathbf{x})), \tag{2.4}$$

and for $\mathbf{q} \in \partial B_1$ set

$$\rho_{\mathbf{q}} := \inf \left\{ \rho \in \left[\frac{1}{8}r, r\right] : \, u_{\mathbf{z}}(\rho \mathbf{q}) = 0 \right\}$$

if this set is nonempty, and $\rho_{\mathbf{q}} := r$ if the set is empty. By a slicing argument for $\mathcal{H}^1$ a.e. $\mathbf{q} \in \partial B_1$ the function

$$g(\rho) := u_{\mathbf{z}}(\rho \mathbf{q}) - v_{\mathbf{z}}(\rho \mathbf{q}), \quad 0 < \rho \le r,$$

is absolutely continuous in $(0,r)$, and so, using also the facts that $g(r)=0$ (since $u=v$ on ∂B_r) and that $u_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q})=0$, we have

$$v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q}) = g(r)-g(\rho_{\mathbf{q}}) = \int_{\rho_{\mathbf{q}}}^{r} g'(\rho)d\rho = \int_{\rho_{\mathbf{q}}}^{r} \nabla(u_{\mathbf{z}}-v_{\mathbf{z}})(\rho\mathbf{q})\cdot\mathbf{q}\,d\rho.$$

Using Holder's inequality we have

$$v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q}) \le \sqrt{r-\rho_{\mathbf{q}}}\left(\int_{\rho_{\mathbf{q}}}^{r} \left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\,d\rho\right)^{\frac{1}{2}},$$

while, by Poisson's formula for the harmonic function v and the fact that $\frac{1}{8}r\le\rho_{\mathbf{q}}$,

$$\begin{aligned}
v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q}) &= \frac{\left(r^2-\rho_{\mathbf{q}}^2\right)}{2\pi r}\int_{\partial B_r}\frac{u(\mathbf{y})}{\left|\rho_{\mathbf{q}}\mathbf{q}-\mathbf{y}\right|^2}\,d\mathcal{H}^1(\mathbf{y}) \\
&\ge \frac{32}{49\pi}\frac{\left(r^2-\rho_{\mathbf{q}}^2\right)}{r^3}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y}) \\
&= \frac{64}{49}\frac{(r-\rho_{\mathbf{q}})(r+\rho_{\mathbf{q}})}{r^2}\frac{1}{|\partial B_r|}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y}) \\
&\ge \frac{72}{49}\frac{(r-\rho_{\mathbf{q}})}{r}\frac{1}{|\partial B_r|}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y}).
\end{aligned}$$

Squaring and combining the two inequalities we obtain

$$\begin{aligned}
(r-\rho_{\mathbf{q}})\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2 &\le C\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\,d\rho \qquad (2.5)\\
&\le \frac{C}{r}\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\rho\,d\rho.
\end{aligned}$$

Integrating the previous inequality in $\mathbf{q}$ yields

$$\begin{aligned}
\int_{\partial B_1}(r-\rho_{\mathbf{q}})\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2 d\mathcal{H}^1(\mathbf{q}) \qquad (2.6)\\
\le \frac{C}{r}\int_{\partial B_1}\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\rho\,d\rho d\mathcal{H}^1(\mathbf{q}) \\
= \frac{C}{r}\int_{B_r\setminus B_{r/8}}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|(\mathbf{x})^2\,d\mathbf{x},
\end{aligned}$$

where we have used the fact that $\rho_{\mathbf{q}}$ is bounded from below by $\frac{1}{8}r$. Since the Jacobian of T can be bounded independently of r (see (2.3) and (2.4)), changing variable on the right-hand side gives

$$\begin{aligned}
\int_{\partial B_1}(r-\rho_{\mathbf{q}})\,d\mathcal{H}^1(\mathbf{q})\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2 \\
\le C\frac{1}{r}\int_{B_r}\left|\nabla v-\nabla u\right|^2(\mathbf{x})\,d\mathbf{x}.
\end{aligned}$$

Now

$$\int_{B_r(\mathbf{0})\setminus B_{\frac{r}{8}}(\mathbf{z})} \chi_{\{u_{\mathbf{z}}=0\}}(\mathbf{x})\, d\mathbf{x} = \int_{\partial B_1} \int_{r/8}^{r} \chi_{\{u_{\mathbf{z}}=0\}}(\rho \mathbf{q})\rho \, d\rho d\mathcal{H}^1(\mathbf{q})$$
$$\leq r \int_{\partial B_1} (r - \rho_{\mathbf{q}})\, d\mathcal{H}^1(\mathbf{q}).$$

Therefore

$$\left(\int_{B_r(\mathbf{0})\setminus B_{\frac{r}{8}}(\mathbf{z})} \chi_{\{u_{\mathbf{z}}=0\}} d\mathbf{x}\right) \left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2$$
$$\leq Cr \int_{\partial B_1} (r-\rho_{\mathbf{q}})\, d\mathcal{H}^1 \left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2$$
$$\leq C \int_{B_r} |\nabla v - \nabla u|^2 \, d\mathbf{x}.$$

We now need to replace $u_{\mathbf{z}}$ by u on the left hand side of the previous inequality. We begin by showing that

$$B_r(\mathbf{0}) \setminus B_{\frac{3}{16}r}(\mathbf{z}) \subseteq T\left(B_r(\mathbf{0}) \setminus B_{\frac{1}{8}r}(\mathbf{z})\right)$$

or, equivalently,

$$T\left(B_{\frac{1}{8}r}(\mathbf{z})\right) \subseteq B_{\frac{3}{16}r}(\mathbf{z}).$$

Indeed,

$$|T(\mathbf{x}) - \mathbf{z}| = \left|\left(1 - \frac{|\mathbf{x}|}{r}\right)\mathbf{z} + \mathbf{x} - \mathbf{z}\right| \leq |\mathbf{x}|\left(\frac{|\mathbf{z}|}{r} + 1\right) \leq \frac{r}{8}\left(\frac{1}{2}+1\right) = \frac{3}{16} r.$$

Hence by the change of variables $\mathbf{x} = T^{-1}(\mathbf{y})$ we get

$$\begin{aligned}\int_{B_r\setminus B_{\frac{r}{8}}} \chi_{\{u_{\mathbf{z}}=0\}}(\mathbf{x})\, d\mathbf{x} &= \int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u_z=0\}}(T^{-1}(\mathbf{y}))J\left(T^{-1}(\mathbf{y})\right)\, d\mathbf{y} \\ &= \int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u=0\}}(\mathbf{y})J\left(T^{-1}(\mathbf{y})\right)\, d\mathbf{y} \qquad (2.7) \\ &\geq c\int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u=0\}}(\mathbf{y})\, d\mathbf{y},\end{aligned}$$

where we used the fact that $J\left(T^{-1}(\mathbf{y})\right)$ bounded from below by $\frac{1}{2}$ since the Jacobian of T is bounded by 2 (see (2.3)). Hence

$$\int_{B_r\setminus B_{\frac{3}{16}r}(\mathbf{z})} \chi_{\{u=0\}}\, d\mathbf{y} \left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2 \leq C\int_{B_r} |\nabla v - \nabla u|^2 \, d\mathbf{x}.$$

If we write this inequality for two values of $\mathbf{z}$ such that $B_{\frac{3}{16}r}(\mathbf{z}_1) \cap B_{\frac{3}{16}r}(\mathbf{z}_2) = \emptyset$, say $\mathbf{z}_1 = (\frac{r}{2}, 0)$, $\mathbf{z}_2 = (-\frac{r}{2}, 0)$ and add the two relations we have

$$\int_{B_r} \chi_{\{u=0\}} \, d\mathbf{y} \left(\frac{1}{r\,|\partial B_r|} \int_{\partial B_r} u \, d\mathcal{H}^1 \right)^2 \leq C \int_{B_r} |\nabla v - \nabla u|^2 \, d\mathbf{x}.$$

Using this and the inequality (2.2) we get

$$\begin{aligned} &\int_{B_r} \chi_{\{u=0\}} \, d\mathbf{y} \left(\frac{1}{r\,|\partial B_r|} \int_{\partial B_r} u \, d\mathcal{H}^1 \right)^2 \\ &\leq C \int_{B_r} |\nabla v - \nabla u|^2 \, d\mathbf{x} \leq \int_{B_r} \chi_{\{u=0\}} (\lambda - y)_+ \, d\mathbf{x}. \end{aligned} \tag{2.8}$$

Step 3: If $B_r(\mathbf{x}_0) \subseteq \{y \geq \lambda\}$, then by (2.2)

$$\int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} = 0,$$

and so $u = v > 0$ $\mathcal{L}^2$ a.e. in $B_r(\mathbf{x}_0)$. If $B_r(\mathbf{x}_0) \cap \{y < \lambda\} \neq \emptyset$, then by (2.8),

$$\begin{aligned} &\int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}} \, d\mathbf{y} \left(\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 \right)^2 d\mathbf{y} \\ &\quad \leq C \int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} \\ &\quad \leq C(\lambda - y_0 + r)_+ \int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}} \, d\mathbf{x}. \end{aligned}$$

Hence, if

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 > \sqrt{C(\lambda - y_0 + r)_+},$$

then necessarily

$$\int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}} \, d\mathbf{x} = 0,$$

and so again

$$\int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} = 0$$

and we proceed as before to conclude $u = v > 0$ in $B_r(\mathbf{x}_0)$.

We denote $C_{\max} := \sqrt{C}$. ■

Remark 2.7 *(i) The previous theorem implies that if $B_r(\mathbf{x}_0)$ intersects the free boundary $\partial\{u > 0\}$, then*

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 \leq C_{\max} \sqrt{(\lambda - y_0 + r)_+}.$$

In particular, if $\lambda - y_0 \leq r$, then

$$\frac{1}{|\partial B_r(\mathbf{x}_0)|}\int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 \leq C_{\max} r^{\frac{3}{2}}.$$

(ii) *It follows from the previous theorem that if $u(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 = (x_0, y_0)$ with $y_0 > \lambda$, then u is positive and harmonic in the whole set $(-1,1) \times (\lambda, \infty)$.*

Theorem 2.8 *Let $u \in K$ be a minimizer of J_λ. Then the set $\{u > 0\}$ is open and u is harmonic in $\{u > 0\}$. Moreover, if the set $\{x \in (-1,1) : u_0(x) > 0\}$ is connected, then so is the set $\{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\}$.*

Proof. Fix any $\mathbf{x}_0 \in \Omega$ such that $u(\mathbf{x}_0) > 0$. If $\mathbf{x}_0 = (x_0, y_0)$, where $y_0 > \lambda$, then we can find a small ball $B_r(\mathbf{x}_0) \subseteq \{y > \lambda\}$, and in view of the last remark, u is positive and harmonic in this ball. Thus in what follows it suffices to assume that $y_0 \leq \lambda$.

By Remark 2.4, for all r sufficiently small

$$\frac{1}{|B_r(\mathbf{x}_0)|}\int_{B_r(\mathbf{x}_0)} u\, d\mathbf{y} > \frac{1}{2}u(\mathbf{x}_0) > 0. \tag{2.9}$$

Fix $r > 0$ so small that (2.9) holds and such that

$$\frac{u(\mathbf{x}_0)}{r} > C_{\max}\sqrt{(\lambda - y_0 + r)}, \tag{2.10}$$

and define

$$g(\rho) := \int_{\partial B_\rho(\mathbf{x}_0)} u\, d\mathcal{H}^1, \quad 0 < \rho \leq r.$$

Using a slicing argument we have that

$$\frac{1}{\pi r^2}\int_{B_r(\mathbf{x}_0)} u\, d\mathbf{y} = \frac{1}{r}\int_0^r \frac{1}{\pi r} g(\rho)\, d\rho > \frac{1}{2}u(\mathbf{x}_0) > 0,$$

and thus we may find some $0 < \rho < r$ such that

$$\frac{1}{\pi\rho}\int_{\partial B_\rho(\mathbf{x}_0)} u\, d\mathcal{H}^1 > \frac{1}{\pi r}\int_{\partial B_\rho(\mathbf{x}_0)} u\, d\mathcal{H}^1 > \frac{1}{2}u(\mathbf{x}_0) > 0.$$

Hence, also by (2.10) and the fact that $\rho < r$, we have

$$\frac{1}{\rho\,|\partial B_\rho(\mathbf{x}_0)|}\int_{\partial B_\rho(\mathbf{x}_0)} u\, d\mathcal{H}^1 \geq \frac{u(\mathbf{x}_0)}{\rho} > C_{\max}\sqrt{(\lambda - y_0 + r)},$$

It follows by the previous theorem that $u > 0$ in $B_\rho(\mathbf{x}_0)$ and u is harmonic in $B_\rho(\mathbf{x}_0)$.

To prove the last part of the theorem, it is enough to observe that if we consider a function v that is equal to u in the connected part of $\{u > 0\}$ that contains the set $\{u_0 > 0\}$ and zero otherwise, then $v \in K$ and $J_\lambda(v) < J_\lambda(u)$. ■

From now on we assume that the set $\{u_0 > 0\}$ is connected. The next result is based on Corollary 3.3 in [AC].

Corollary 2.9 *Let $u \in K$ be a minimizer of J_λ. Then u is locally Lipschitz.*

Proof. Fix $\varepsilon > 0$ and let $\Omega_\varepsilon := (-1+\varepsilon, 1-\varepsilon) \times \left(\varepsilon, \frac{1}{\varepsilon}\right)$. We claim that u is Lipschitz in Ω_ε. To see this, fix $\mathbf{x}_0 \in \Omega_\varepsilon$ such that $u(\mathbf{x}_0) > 0$. Since the set $\{u > 0\} \cap \Omega_\varepsilon$ is open, we may find $B_r(\mathbf{x}_0) \subset \{u > 0\} \cap \Omega_\varepsilon$.

Let $\rho_\varepsilon(\mathbf{x}_0) := \sup\{r > 0 : B_r(\mathbf{x}_0) \subset \{u > 0\} \cap \Omega_\varepsilon\}$. By the previous theorem for any $r < \rho_\varepsilon$, u is harmonic in $B_r(\mathbf{x}_0)$. In turn $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also harmonic in $B_r(\mathbf{x}_0)$, and so, by the mean value and divergence theorems,

$$\frac{\partial u}{\partial x}(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{x}_0)} \frac{\partial u}{\partial x}(\mathbf{y})\, d\mathbf{y} = \frac{1}{\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u v_1 \, d\mathcal{H}^1, \tag{2.11}$$

where

$$v(\mathbf{x}) = (v_1, v_2) = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}.$$

Similarly

$$\frac{\partial u}{\partial y}(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u v_2 \, d\mathcal{H}^1. \tag{2.12}$$

There are now two cases. If $B_{\rho_\varepsilon}(\mathbf{x}_0)$ touches $\partial\Omega_\varepsilon$, then by Lemma 2.5 we get

$$\left|\frac{\partial u}{\partial x}(\mathbf{x}_0)\right|, \left|\frac{\partial u}{\partial y}(\mathbf{x}_0)\right| \leq \frac{\sup u_0}{\pi d_\varepsilon(\mathbf{x}_0)}, \tag{2.13}$$

where $d_\varepsilon(\mathbf{x}_0) := \mathrm{dist}(\mathbf{x}_0, \partial\Omega_\varepsilon)$.

A similar estimate holds if $\rho_\varepsilon(\mathbf{x}_0) \geq \frac{1}{8} d_\varepsilon(\mathbf{x}_0)$ (replacing $\sup u_0$ with $8 \sup u_0$ in (2.13)). If $\rho_\varepsilon(\mathbf{x}_0) < \frac{1}{8} d_\varepsilon(\mathbf{x}_0)$, then $B_{\rho_\varepsilon}(\mathbf{x}_0)$ touches $\partial\{u > 0\}$ at some point $\mathbf{x}_1$ in which $u(\mathbf{x}_1) = 0$. Let $r = \frac{1}{4}\rho_\varepsilon(\mathbf{x}_0)$. If $\mathbf{y} \in \partial B_r(\mathbf{x}_0)$, then, since u is subharmonic in $B_s(\mathbf{y})$, where $\frac{5}{4}\rho_\varepsilon(\mathbf{x}_0) < s \leq \frac{3}{2}\rho_\varepsilon(\mathbf{x}_0)$, by Poisson's formula we have

$$u(\mathbf{y}) \leq \frac{1}{|\partial B_s(\mathbf{y})|} \int_{\partial B_s(\mathbf{y})} u \, d\mathcal{H}^1.$$

Since $|\mathbf{x}_1 - \mathbf{y}| \leq |\mathbf{x}_1 - \mathbf{x}_0| - |\mathbf{x}_0 - \mathbf{y}|$, then $\mathbf{x}_1 \in B_s(\mathbf{y})$ and

$$\frac{1}{|\partial B_s(\mathbf{y})|} \int_{B_r(\mathbf{y})} u \, d\mathcal{H}^1 \leq C_{\max} \sqrt{(\lambda - y + s)_+},$$

where $\mathbf{y} = (x, y)$. Hence

$$u(\mathbf{y}) \leq C_{\max} s \sqrt{(\lambda - y + s)}.$$

Thus from formulas (2.11) and (2.12) we get

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(\mathbf{x}_0)\right| &\le \frac{C_{\max}s}{\pi r^2}\int_{\partial B_r(\mathbf{x}_0)}\sqrt{(\lambda-y+s)_+}\,d\mathcal{H}^1(x,y)\\ &\le \frac{C_{\max}\frac{3}{2}\rho_\varepsilon(\mathbf{x}_0)}{\pi\frac{1}{4}\rho_\varepsilon(\mathbf{x}_0)r}\int_{\partial B_r(\mathbf{x}_0)}\sqrt{(\lambda-y+s)_+}\,d\mathcal{H}^1(x,y) \qquad (2.14)\\ &\le 12C_{\max}\sqrt{(\lambda-y_0+r+s)_+}\\ &\le 12C_{\max}\sqrt{(\lambda-y_0+2\rho_\varepsilon(\mathbf{x}_0))_+}.\end{aligned}$$

Similarly

$$\left|\frac{\partial u}{\partial y}(\mathbf{x}_0)\right| \le 12C_{\max}\sqrt{(\lambda-y_0+2\rho_\varepsilon(\mathbf{x}_0))_+}. \qquad (2.15)$$

Since $\rho_\varepsilon(\mathbf{x}_0)<\frac{1}{8}d_\varepsilon(\mathbf{x}_0)$, also by (2.13) we obtain

$$|\nabla u(\mathbf{x}_0)|\le\max\left\{\frac{\sup u_0}{\pi d_\varepsilon(\mathbf{x}_0)},C\sqrt{\lambda+d_\varepsilon(\mathbf{x}_0)}\right\}.$$

Since $\nabla u(\mathbf{x})=0$ for $\mathcal{L}^2$ a.e. $\mathbf{x}\in\Omega_\varepsilon$ such that $u(\mathbf{x})=0$, we have proved that u is Lipschitz in Ω_ε.

If now $\Omega'\subset\subset\Omega$ we can choose ε so small that $\Omega'\subset\Omega_\varepsilon$. Then $|\nabla u|\le C(\Omega_\varepsilon)$, and so u is locally Lipschitz continuous. ■

Next we adapt the proof of Lemma 3.4 of [AC] to our setting.

Theorem 2.10 *For any $k\in(0,1)$ there exists a positive constant $C(k)$ such that for any minimizer u of J_λ and for every (small) ball $B_r(\mathbf{x}_0)\subset\Omega$, if*

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|}\int_{\partial B_r(\mathbf{x}_0)}u\,d\mathcal{H}^1\le C(k)\sqrt{(\lambda-y_0-kr)_+}, \qquad (2.16)$$

then $u=0$ in $B_{kr}(\mathbf{x}_0)$.

Proof. The idea of the proof is that if the average of u on $\partial B_r(\mathbf{x}_0)$ is small, then replacing u by a function w that vanishes in $B_{kr}(\mathbf{x}_0)$ will decrease J_λ.

All the balls used in this proof are centered at $\mathbf{x}_0$, therefore, for simplicity, we write B_r for $B_r(\mathbf{x}_0)$.

Step 1: In this step we find a lower bound for $\int_{\partial B_{kr}}u\,d\mathcal{H}^1$. Define

$$v(\mathbf{x}):=\frac{\ell_u\sqrt{k}}{\log\left(\frac{1}{\sqrt{k}}\right)}\max\left\{\log\frac{|\mathbf{x}-\mathbf{x}_0|}{kr},0\right\},\quad \mathbf{x}\in B_r,$$

where

$$\ell_u:=\frac{1}{\sqrt{k}}\sup_{B_{r\sqrt{k}}}u. \qquad (2.17)$$

Note that

$$\nabla v(\mathbf{x})=\begin{cases}\frac{\ell_u\sqrt{k}}{\log\left(\frac{1}{\sqrt{k}}\right)}\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^2} & \text{if } |\mathbf{x}-\mathbf{x}_0|>kr,\\ 0 & \text{if } |\mathbf{x}-\mathbf{x}_0|\le kr.\end{cases} \qquad (2.18)$$

We claim that

$$\ell_u \leq C(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} u \, d\mathcal{H}^1. \tag{2.19}$$

To see this, let V be the harmonic function that is equal with u on $\partial B_r(x_0)$. By Harnack 's inequality we have

$$\begin{aligned} \sup_{B_{r\sqrt{k}}} V &\leq C\,(k) \inf_{B_{r\sqrt{k}}} V \leq C\,(k)\, V\,(\mathbf{x}_0) = C(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} V \, d\mathcal{H}^1 \\ &= C(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} u \, d\mathcal{H}^1. \end{aligned}$$

Since u is subharmonic, we have that $u \leq V$, and so, possibly changing $C\,(k)$, we obtain

$$\frac{1}{\sqrt{k}} \sup_{B_{r\sqrt{k}}} u \leq \frac{1}{\sqrt{k}} \sup_{B_{r\sqrt{k}}} V \leq C(k) \frac{1}{|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1,$$

which proves (2.19).

Define now

$$w := \begin{cases} \min\{u, v\} & \text{in } B_{\sqrt{k}r}, \\ u & \text{outside } B_{\sqrt{k}r}. \end{cases}$$

Since $w \in K$, we have $J_\lambda(u) \leq J_\lambda\,(w)$, which implies that

$$\int_{B_{\sqrt{k}r}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+ \right) d\mathbf{x} \leq \int_{B_{\sqrt{k}r}} \left(|\nabla w|^2 + \chi_{\{w>0\}}(\lambda - y)_+ \right) d\mathbf{x}.$$

Notice that $w = 0$ in B_{kr} and outside this ball $w = 0$ whenever $u = 0$. Hence

$$\begin{aligned} \int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+ \right) d\mathbf{x} &\leq \int_{B_{\sqrt{k}r} \setminus B_{kr}} \left(|\nabla w|^2 - |\nabla u|^2 \right) d\mathbf{x} \\ &= \int_{B_{\sqrt{k}r} \setminus B_{kr}} \left(|\nabla w|^2 - |\nabla u|^2 \right) d\mathbf{x} \qquad (2.20) \\ &\leq 2 \int_{B_{\sqrt{k}r} \setminus B_{kr}} (\nabla w - \nabla u) \cdot \nabla w \, d\mathbf{x}, \end{aligned}$$

where the last inequality follows from the fact that if we move all terms to the right-hand side we obtain a perfect square. On the other hand, since v is harmonic,

$$\begin{aligned} 0 &= \int_{B_{\sqrt{k}r} \setminus B_{kr}} (w - u)\, \Delta v \, d\mathbf{x} \qquad (2.21) \\ &= -\int_{B_{\sqrt{k}r} \setminus B_{kr}} (\nabla w - \nabla u) \cdot \nabla v \, dx + \int_{\partial\left(B_{\sqrt{k}r} \setminus B_{kr}\right)} (w - u)\, \nabla v \cdot \nu \, d\mathcal{H}^1. \end{aligned}$$

Therefore

$$\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot\nabla v\,dx = \int_{\partial\left(B_{\sqrt{k}r}\setminus B_{kr}\right)} (w-u)\,\nabla v\cdot\nu\,d\mathcal{H}^1 =$$
$$= -\int_{\partial B_{kr}} u(\nabla v\cdot v)\,d\mathcal{H}^1$$
$$\leq \frac{C(k)\ell_u}{r}\int_{\partial B_{kr}} u\,d\mathcal{H}^1,$$

where we have used the fact that $w = u$ on ∂B_{kr} and (2.18). We also have

$$\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot\nabla w\,d\mathbf{x} = \int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot\nabla v\,dx. \tag{2.22}$$

To see this, we split $B_{\sqrt{k}r}\setminus B_{kr}$ in the three sets $\{u > v\}$, $\{u < v\}$ and $\{u = v\}$. When $u > v$, we have that $w = v$ therefore $\nabla w = \nabla v$ $\mathcal{L}^2$ a.e. in $\{u > v\}$. When $u < v$, we have that $w = u$, and so both integrals over the set $\{u < v\}$ are 0. Finally $\nabla w = \nabla u$ $\mathcal{L}^2$ a.e. in the set where $u = v$.

We conclude from (2.20), (2.21), and (2.22) that

$$\int_{B_{kr}} \left(|\nabla u|^2 + \chi(\{u>0\})(\lambda - y)_+\right)\,d\mathbf{x} \leq \frac{C(k)\ell_u}{r}\int_{\partial B_{kr}} u\,d\mathcal{H}^1. \tag{2.23}$$

Step 2: Define

$$Q_{\min} := \sqrt{(\lambda - y_0 - kr)_+}. \tag{2.24}$$

If $Q_{\min} = 0$, then the result follows immediately from (2.16) and (2.19). Thus assume that $Q_{\min} > 0$.

By the trace theorem, (2.17), Young's inequality, and the fact that $Q_{\min} \leq \sqrt{(\lambda - y)_+}$ in B_{kr}, we have

$$\int_{\partial B_{kr}} u\;d\mathcal{H}^1 \leq C(k)\left(\frac{1}{r}\int_{B_{kr}} u\,d\mathbf{x} + \int_{B_{kr}} |\nabla u|\;d\mathbf{x}\right)$$
$$= C(k)\left(\frac{1}{r}\int_{B_{kr}} u\chi_{\{u>0\}}\,d\mathbf{x} + \int_{B_{kr}} |\nabla u|\,\chi_{\{u>0\}}\,d\mathbf{x}\right)$$
$$\leq C(k)\left[\frac{\ell_u}{rQ_{\min}^2}\int_{B_{kr}} \chi_{\{u>0\}}(\lambda - y)_+\,d\mathbf{x}\right.$$
$$\left.+\frac{1}{Q_{\min}}\left(\int_{B_{kr}} |\nabla u|^2\;d\mathbf{x} + \int_{B_{kr}} \chi_{\{u>0\}}(\lambda - y)_+\,d\mathbf{x}\right)\right]$$
$$\leq \frac{C(k)}{Q_{\min}}\left(\frac{\ell_u}{rQ_{\min}} + 1\right)\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{(\{u>0\})}(\lambda - y)_+\right)\,d\mathbf{x}.$$

Combined with (2.23), the previous inequality yields

$$\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right)\,d\mathbf{x} \tag{2.25}$$
$$\leq \frac{C(k)\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}} + 1\right)\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{(\{u>0\})}(\lambda - y)_+\right)\,d\mathbf{x}.$$

It follows from (2.19) that if

$$\frac{1}{rQ_{\min}}\frac{1}{|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1$$

is small, then so is $\frac{\ell_u}{rQ_{\min}}$. Hence by taking the constant $C(k)$ in (2.16) sufficiently small, we can ensure that

$$\frac{C(k)\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}}+1\right)<1. \tag{2.26}$$

This, together with the previous inequality, implies that $u=0$ in B_{kr}. ■

Remark 2.11 *By extending u to zero in $\mathbb{R}\times[0,\infty)\setminus\overline{\Omega}$, it is easy to see that the previous theorem remains valid if $B_r(\mathbf{x}_0)$ intersects the lateral boundary of Ω.*

Using the previous theorem and Remark 2.11 we can prove the first main result of the paper, namely the existence of regular waves (cf. Keady and Norbury [KN]).

Theorem 2.12 (Existence of regular waves) *There exists $\lambda_0 >> 1$, depending on the initial datum u_0, such that for all $\lambda\geq\lambda_0$ and for any minimizer $u\in K$ of J_λ, the support of u is contained in the set $[-1,1]\times[0,\lambda)$.*

Proof. Fix any $y_0>0$, and let $k:=\frac{1}{2}$ and $r:=\frac{y_0}{2}$. Then for any $x_0\in[-1,1]$,

$$B_r(x_0,y_0)\subset\mathbb{R}\times\left[\frac{y_0}{2},\infty\right),$$

and thus we are in a position to apply Remark 2.11. By Theorem 2.10 for any minimizer $u\in K$ of J_λ we have, (see (2.17)),

$$\ell_u=\sqrt{2}\sup_{B_{\sqrt{2}r}(x_0,y_0)} u\leq\sqrt{2}\max_{[-1,1]} u_0,$$

while (see (2.24))

$$Q_{\min}=\sqrt{\left(\lambda-\frac{3}{2}y_0\right)_+}.$$

It now follows from (2.26) that

$$\frac{C(\frac{1}{2})\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}}+1\right)\leq\frac{C(\frac{1}{2})\sqrt{2}\,\|u_0\|_\infty}{\frac{y_0}{2}\sqrt{\left(\lambda-\frac{3}{2}y_0\right)_+}}\left(\frac{\sqrt{2}\,\|u_0\|_\infty}{\frac{y_0}{2}\sqrt{\left(\lambda-\frac{3}{2}y_0\right)_+}}+1\right)<1$$

for all $\lambda\geq\lambda_0=\lambda_0\left(\|u_0\|_\infty,y_0\right)$.

Thus by the previous theorem (see (2.25)), we have that $u\equiv 0$ in $[-1,1]\times\left[\frac{3y_0}{4},\frac{5y_0}{4}\right]$, which implies that $u\equiv 0$ in $[-1,1]\times\left[\frac{3y_0}{4},\infty\right)$, (recall Theorem 2.8). ■

3 Stokes waves

In the previous section we have shown that for all λ sufficiently large the support of any minimizer $u \in K$ of J_λ remains well-below the line $y = \lambda$. Next we prove that for λ very small the support of u crosses the line $y = \lambda$. To highlight the dependence on the parameter λ in what follows we denote by $u_\lambda \in K$ a minimizer of the functional J_λ.

Following Theorem 10.2 in [Fr] we have the following result.

Theorem 3.1 (Monotonicity) *Consider* $0 < \mu < \lambda$ *and let* u_λ, $u_\mu \in K$ *be minimizers of* J_λ *and* J_μ, *respectively. Then*

$$u_\lambda < u_\mu. \tag{3.1}$$

Proof. Define $v_1 := \min\{u_\lambda, u_\mu\}$ and $v_2 := \max\{u_\lambda, u_\mu\}$. Since v_1 and v_2 belong to K,

$$J_\lambda(u_\lambda) + J_\mu(u_\mu) \le J_\lambda(v_1) + J_\mu(v_2).$$

Let $A_1 := \{u_\mu < u_\lambda\}$ and $A_2 := \{u_\mu \ge u_\lambda\}$. Then the previous inequality becomes

$$\begin{aligned}
&\int_{A_1\cup A_2} \Big(|\nabla u_\lambda|^2 + \chi(\{u_\lambda > 0\})(\lambda - y)_+\Big)\, d\mathbf{x} \\
&\qquad + \int_{A_1\cup A_2} \Big(|\nabla u_\mu|^2 + \chi(\{u_\mu > 0\})(\mu - y)_+\Big)\, d\mathbf{x} \\
&\qquad \le \int_{A_1\cup A_2} \Big(|\nabla v_1|^2 + \chi(\{v_1 > 0\})(\lambda - y)_+\Big)\, d\mathbf{x} \\
&\qquad + \int_{A_1\cup A_2} \Big(|\nabla v_2|^2 + \chi(\{v_2 > 0\})(\mu - y)_+\Big)\, d\mathbf{x}.
\end{aligned}$$

Since $v_1 = u_\mu$, $v_2 = u_\lambda$ in A_1 and $v_1 = u_\lambda$, $v_2 = u_\mu$ in A_2, the integrals containing gradients cancel out. Therefore

$$\begin{aligned}
&\int_{A_1\cup A_2} \chi(\{u_\lambda > 0\})(\lambda - y)_+\, d\mathbf{x} + \int_{A_1\cup A_2} \chi(\{u_\mu > 0\})(\mu - y)_+\, d\mathbf{x} \\
&\qquad \le \int_{A_1} \chi(\{u_\mu > 0\})(\lambda - y)_+\, d\mathbf{x} + \int_{A_2} \chi(\{u_\lambda > 0\})(\lambda - y)_+\, d\mathbf{x} \\
&+ \int_{A_1} \chi(\{u_\lambda > 0\})(\mu - y)_+\, d\mathbf{x} + \int_{A_2} \chi(\{u_\mu > 0\})(\mu - y)_+\, d\mathbf{x}.
\end{aligned}$$

The integrals over A_2 cancel out, therefore

$$\begin{aligned}
&\int_{A_1} \chi(\{u_\lambda > 0\})(\lambda - y)_+\, d\mathbf{x} + \int_{A_1} \chi(\{u_\mu > 0\})(\mu - y)_+\, d\mathbf{x} \\
&\qquad \le \int_{A_1} \chi(\{u_\mu > 0\})(\lambda - y)_+\, d\mathbf{x} + \int_{A_1} \chi(\{u_\lambda > 0\})(\mu - y)_+\, d\mathbf{x},
\end{aligned}$$

which implies

$$\int_{A_1} (\chi(\{u_\lambda > 0\}) - \chi(\{u_\mu > 0\}))\,(\lambda - y)_+ \,d\mathbf{x}$$
$$\leq \int_{A_1} (\chi(\{u_\lambda > 0\}) - \chi(\{u_\mu > 0\}))\,(\mu - y)_+ \,d\mathbf{x},$$

or, equivalently,

$$\int_{A_1} (\chi(\{u_\lambda > 0\}) - \chi(\{u_\mu > 0\}))\,((\lambda - y)_+ - (\mu - y)_+)\ d\mathbf{x} \leq 0. \tag{3.2}$$

Since $A_1 = \{u_\mu < u_\lambda\}$ and $\mu < \lambda$, we have that the integrand is non negative, which implies that it is actually zero $\mathcal{L}^2$ a.e. in A_1. By the continuity of u_μ and u_λ we have that

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \cap \{u_\mu < u_\lambda\} \subset \{u_\mu > 0\} \cap \{y < \lambda\} \cap \{u_\mu < u_\lambda\}.$$

On the other hand, since $A_2 = \{u_\mu \geq u_\lambda\}$, we have that

$$\{u_\lambda > 0\} \cap \{u_\mu \geq u_\lambda\} \subset \{u_\mu > 0\} \cap \{u_\mu \geq u_\lambda\}$$

and so

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subset \{u_\mu > 0\} \cap \{y < \lambda\} \tag{3.3}$$

Since equality holds in (3.2), we have actually proved that

$$J_\lambda(u_\lambda) + J_\mu(u_\mu) = J_\lambda(v_1) + J_\mu(v_2),$$

which implies that $J_\lambda(u_\lambda) = J_\lambda(v_1)$ and $J_\mu(u_\mu) = J_\mu(v_2)$.

Hence v_1 and v_2 are minimizers for J_λ and J_μ, respectively. In particular, by Theorem 2.8, they are harmonic in the set where they are positive.

If $0 < u_\lambda(\mathbf{x}_0) = u_\mu(\mathbf{x}_0)$ for some $\mathbf{x}_0 = (x_0, y_0) \in (-1,1) \times (0,\infty)$, then in a neighborhood of $\mathbf{x}_0$ the functions $u_\lambda - v_2 \leq 0$ and $u_\mu - v_2 \leq 0$ are harmonic and attain a maximum in an interior point. It follows by the maximum principle that $u_\lambda - v_2 = u_\mu - v_2 \equiv 0$ in the connected component of $\{v_2 > 0\}$ that contains $\mathbf{x}_0$. By Theorem 2.8 we have that $u_\mu = u_\lambda$ in Ω. This is a contradiction. Indeed, at a positive distance below the line $y = \mu$, we can apply the regularity results from [AC] to obtain that $\partial\{u_\lambda > 0\}$ is an analytic curve. By classical regularity results we can write the Euler-Lagrange equations of the functional J_λ to deduce in particular that $\frac{\partial u_\lambda}{\partial \nu}(x,y) = \lambda - y$ and $\frac{\partial u_\mu}{\partial \nu}(x,y) = \mu - y$ on $\partial\{u_\lambda > 0\}$, which contradicts the fact that $u_\mu = u_\lambda$ in Ω.

If $u_\lambda(x,y) \neq u_\mu(x_0,y_0)$ for all $(x,y) \in (-1,1) \times (0,\infty)$ with $y > 0$, then, since $\{u_\lambda > 0\}$ is open and connected by Theorem 2.8, we must have that either $u_\mu > u_\lambda$ or $u_\mu < u_\lambda$. In view of (3.3), necessarily $u_\lambda < u_\mu$. This concludes the proof. ∎

We now prove the existence of a critical level λ_c, which should correspond to a *Stokes wave*.

Theorem 3.2 *For every λ let $u_\lambda \in K$ be a minimizer of the functional J_λ. Let*

$$\lambda_c := \inf\{\lambda \geq 0 : \operatorname{supp} u_\lambda \subset \{y \leq \lambda\}\}.$$

Then $0 < \lambda_c < \infty$, u_λ is a regular wave for any $\lambda > \lambda_c$, while u_λ is a non-physical wave *for $\lambda < \lambda_c$, in the sense that the support of u_λ crosses the line $y = \lambda$.*

Proof. By Theorem 2.12 we have that $\lambda_c < \infty$. If $\lambda > \lambda_c$, by the definition of λ_c there exists $\lambda_c < \mu < \lambda$ such that $\operatorname{supp} u_\mu \subset \{y \leq \mu\}$.

By Theorem 3.1

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subset \{u_\mu > 0\} \cap \{y < \lambda\} = \{u_\mu > 0\} \cap \{y \leq \mu\},$$

and so

$$\{u_\lambda > 0\} \subset \{y \leq \mu\} \subset \{y < \lambda\}.$$

Thus u_λ is a regular wave.

To prove that $\lambda_c > 0$, fix any $\lambda > \lambda_c$ and let $0 < \lambda_0 \leq \lambda_c$ be such that the line $y = \lambda_0$ intersects the set $\{u_\lambda > 0\}$. By Theorem 3.1 once more, for any $0 < \mu \leq \lambda_0$

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subset \{u_\mu > 0\} \cap \{y < \lambda\},$$

and, since the line $y = \lambda_0$ intersects the set $\{u_\lambda > 0\}$, it also intersects the set $\{u_\mu > 0\}$. Thus $\lambda_c \geq \lambda_0 > 0$.

It follows from the definition of λ_c that for any $0 < \lambda < \lambda_c$, the support of u_λ is not contained in $\{y \leq \lambda\}$, and so u_λ is a *nonphysical wave.* ■

Next we prove that as $\lambda \searrow \lambda_c$ and $\lambda \nearrow \lambda_c$, corresponding minimizers u_λ approach two minimizers at level $y = \lambda_c$.

Theorem 3.3 *Let $\{\lambda_n\} \subset (0, \infty)$ be any sequence such that $\lambda_n \to \lambda_c$ and let $\{u_{\lambda_n}\} \subseteq K$ be minimizers of the functionals J_{λ_n}. Then (up to a subsequence) $\{u_{\lambda_n}\}$ converges strongly in $H^1_{\mathrm{loc}}(\Omega)$ to a minimizer $u \in K$ of J_{λ_c}.*

Proof. Extend u_0 to a function $u_0 \in C^1\left(\overline{\Omega}\right)$ such that $\operatorname{supp} u_0$ is contained in $[-1, 1] \times \left[0, \frac{\lambda_c}{2}\right]$. Let $n_1 \in \mathbb{N}$ so large that $\lambda_n > \frac{\lambda_c}{2}$ for all $n \geq n_1$. As in the proof of Theorem 2.1 we may extract a subsequence (not relabeled) $\{u_{\lambda_n}\}$ such that $\{u_{\lambda_n}\}$ converges weakly to some function $u_{\lambda_c} \in K$, while $\left\{\chi_{\{u_{\lambda_n} > 0\}}\right\}$ converges weakly star to a function γ in $L^\infty(\Omega)$ with

$$\gamma(\mathbf{x}) \geq \chi_{\{u_{\lambda_n} > 0\}}(\mathbf{x}) \quad \text{for } \mathcal{L}^2 \text{ a.e. } \mathbf{x} \in \Omega.$$

It remains to show that $u_{\lambda_c}$is a minimizer for J_{λ_c}. As in the last part of the proof of the Theorem 2.1, for any $R > 0$ and any $u \in K$, we have

$$\begin{aligned}
\int_{\Omega_R} \left(|\nabla u_{\lambda_c}|^2 + \chi_{\{u_{\lambda_n} > 0\}} (\lambda_c - y)_+\right) d\mathbf{x} &\leq \int_{\Omega_R} \left(|\nabla u_{\lambda_c}|^2 + \gamma (\lambda_c - y)_+\right) d\mathbf{x} \\
&\leq \liminf_{n \to \infty} \int_{\Omega_R} \left(|\nabla u_{\lambda_n}|^2 + \gamma (\lambda_n - y)_+\right) d\mathbf{x} \qquad (3.4) \\
&\leq \liminf_{n \to \infty} J_{\lambda_n}(u_{\lambda_n}) \leq \limsup_{n \to \infty} J_{\lambda_n}(u_{\lambda_n}) \\
&\leq \limsup_{n \to \infty} J_{\lambda_n}(u) = J_{\lambda_c}(u).
\end{aligned}$$

Letting $R \nearrow \infty$, we conclude that

$$J_{\lambda_c}(u_c) \leq J_{\lambda_c}(u)$$

for all $u \in K$. Since $u_{\lambda_c} \in K$, we have that u_{λ_c} is a minimizer for J_{λ_c}.

Note that taking $u = u_{\lambda_c}$ in (3.4) and letting $R \nearrow \infty$, gives

$$J_{\lambda_c}(u) = \lim_{n\to\infty} \int_\Omega \left(|\nabla u_{\lambda_n}|^2 + \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \right) d\mathbf{x}.$$

On the other hand, since

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{\lambda_n}|^2 \, d\mathbf{x} \geq \int_\Omega |\nabla u_{\lambda_c}|^2 \, d\mathbf{x}$$

and

$$\liminf_{n\to\infty} \int_\Omega \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \, d\mathbf{x} \geq \int_\Omega \chi_{\{u_{\lambda_c}>0\}} (\lambda_c - y)_+ \, d\mathbf{x},$$

it follows that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_{\lambda_n}|^2 \, d\mathbf{x} = \int_\Omega |\nabla u_{\lambda_c}|^2 \, d\mathbf{x}$$

and

$$\lim_{n\to\infty} \int_\Omega \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u_{\lambda_c}>0\}} (\lambda_c - y)_+ \, d\mathbf{x}.$$

It follows that $\{\nabla u_{\lambda_n}\}$ converges strongly to ∇u_{λ_c} in $\left(L^2(\Omega)\right)^2$, and hence $\{u_{\lambda_n}\}$ converges strongly to u_{λ_c} in $H^1_{\text{loc}}(\Omega)$. ∎

Corollary 3.4 *Let $\{\lambda_n\}$, $\{\mu_n\} \subset (0,\infty)$ be such that $\lambda_n \searrow \lambda_c$ and $\mu_n \nearrow \lambda_c$. Then $\{u_{\lambda_n}\}$ and $\{u_{\mu_n}\}$ converge strongly in $H^1_{\text{loc}}(\Omega)$ and uniformly to two minimizers u^+ and $u^- \in K$ of J_{λ_c}, respectively. Moreover* $\operatorname{supp} u^+ \subset \{y \leq \lambda_c\}$, *while* $\operatorname{supp} u^-$ *intersects the line $y = \lambda_c$.*

Proof. By Theorem 3.1 the sequence $\{u_{\lambda_n}\}$ is increasing, while the sequence $\{u_{\mu_n}\}$ is decreasing. Thus for all $\mathbf{x} \in \Omega$ there exist

$$\lim_{n\to\infty} u_{\lambda_n}(\mathbf{x}) = u^+(\mathbf{x}), \qquad \lim_{n\to\infty} u_{\mu_n}(\mathbf{x}) = u^-(\mathbf{x}).$$

It follows by the previous theorem, that u^+ and u^- are minimizers of J_{λ_c}. Since u^+ and u^- are continuous (see Theorem 2.9), by Dini's monotone convergence theorem, the convergence is uniform.

To prove the second part of the statement, assume by contradiction that there exists $\mathbf{x}_0 = (x_0, y_0) \in (-1,1) \times (\lambda_c, \infty)$ such that $u^+(\mathbf{x}_0) > 0$. Since $\{u_{\lambda_n}\}$ converges uniformly to u^+, we have that

$$u_{\lambda_n}(\mathbf{x}_0) > \frac{u^+(\mathbf{x}_0)}{2}$$

for all n sufficiently large. Since $\lambda_n \searrow \lambda_c$, taking so large that $\lambda_n < y_0$, we have contradicted the fact that $\operatorname{supp} u_{\lambda_n} \subset \{y < \lambda_n\}$. Thus $\operatorname{supp} u^+ \subset \{y \leq \lambda_c\}$.

Next, assume by contradiction that

$$\operatorname{supp} u^- \subset \{y < \lambda_c\}.$$

Fix $\varepsilon > 0$ such that $\operatorname{supp} u^- \subset \{y < \lambda_c - \varepsilon\}$. Let $C\left(\frac{1}{2}\right)$ be the constant given in Theorem 2.10 with $k = \frac{1}{2}$.

Since $\mu_n \nearrow \lambda_c$ and $\{u_{\mu_n}\}$ converges uniformly to zero in $[-1,1]\times[\lambda_c - \varepsilon, \lambda_c]$, we may find n_1 so large that

$$\mu_n > \lambda_c - \frac{\varepsilon}{4}$$

and

$$u_{\mu_n} < C\left(\tfrac{1}{2}\right)\frac{\varepsilon}{4}\sqrt{\frac{3}{8}\varepsilon} \text{ in } [-1,1]\times[\lambda_c - \varepsilon, \lambda_c] \tag{3.5}$$

for all $n \geq n_1$.

We now apply Theorem 2.10 and Remark 2.11 to u_{μ_n} taking $x_0 \in (-1,1)$, $y_0 = \lambda_c - \frac{3}{4}\varepsilon$, $r = \frac{\varepsilon}{4}$. By (3.5), for all $n \geq n_1$, we have

$$\begin{aligned}
&\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|}\frac{1}{\sqrt{(\lambda - y_0 - \frac{1}{2}r)_+}}\int_{\partial B_r(\mathbf{x}_0)} u_{\mu_n}\, d\mathcal{H}^1 \\
&= \frac{1}{\frac{\varepsilon}{4}\left|\partial B_{\frac{\varepsilon}{4}}(\mathbf{x}_0)\right|}\frac{1}{\sqrt{(\mu_n - \lambda_c + \frac{5}{8}\varepsilon)_+}}\int_{\partial B_{\frac{\varepsilon}{4}}(\mathbf{x}_0)} u_{\mu_n}\, d\mathcal{H}^1 \\
&< C\left(\tfrac{1}{2}\right)\frac{\sqrt{\frac{3}{8}\varepsilon}}{\sqrt{(\mu_n - \lambda_c + \frac{5}{8}\varepsilon)_+}} \leq C\left(\tfrac{1}{2}\right),
\end{aligned}$$

where in the last inequality we have used (3.5) and the fact that $\mu_n > \lambda_c - \frac{\varepsilon}{4}$.

It follows from Theorem 2.10 and Remark 2.11 that $u_{\mu_n} = 0$ in $B_{\frac{\varepsilon}{4}}\left(x_0, \lambda_c - \frac{\varepsilon}{4}\right)$ for all $x_0 \in (-1,1)$ and for all $n \geq n_1$. Since $\{u_{\mu_n} > 0\}$ is connected by Theorem 2.8, we have contradicted the fact that $\operatorname{supp} u_{\mu_n}$ meets the line $y = \mu_n$.

Hence $\operatorname{supp} u^-$ not contained in $\{y < \lambda_c\}$. ■

Conjecture 3.5 *We conjecture that J_{λ_c} has a unique minimizer.*

Note that if the conjecture were true, then $u^+ = u^-$, and so the support of u^+ would touch the line $y = \lambda_c$ and be contained in the set $\{y \leq \lambda_c\}$. This would prove the existence of a *Stokes wave*. We have been unable to prove the conjecture.

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