High Order Fully Discrete Discontinuous Galerkin Methods For Miscible Displacement

Y.Epshteyn [a,1] B.Rivière [b,2]

[a] *Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, 15213, USA*

[b] *Department of Mathematics, University of Pittsburgh, 301 Thackeray, Pittsburgh, PA 15260, U.S.A.*

Abstract

We derive error estimates for a fully discrete scheme using primal discontinuous Galerkin discretization in space and backward Euler discretization in time. The estimates in the energy norm are optimal with respect to the mesh size and suboptimal with respect to the polynomial degree. The proposed scheme is of high order as polynomial approximations of pressure and concentration can take any value. In addition, the method can handle different types of boundary conditions and is well-suited for unstructured meshes.

Key words: flow, transport, porous media, miscible displacement, NIPG, SIPG, IIPG, h and p-version, fully discrete scheme

1 Introduction

A high order numerical method for solving miscible displacement is introduced and analyzed in this paper. Miscible displacement occurs in important applications such as remediation of contaminated groundwater and production of oil from petroleum reservoirs. The physical model that describes the miscible displacement phenomena arises from the natural law of conservation of mass. This law is applied to each component of the fluid mixture. Thus, the mathematical model consists of a coupled system of partial differential equations:

[1] The research of Y.Epshteyn is based upon work supported by the Center for Nonlinear Analysis (CNA) under the National Science Foundation Grant # DMS 0635983. This research was also supported by NSF grant DMS # 0506039 .

[2] The research of B.Riviere is funded by the NSF grant DMS 0506039.

a pressure equation and a concentration equation for each component. Since the components of the fluid mixture may react with each other, the numerical method must accurately solve the laws of conservation. In particular, it is important to solve the continuity equation that describes the flow phenomena with high accuracy. In this work, we propose a fully discrete scheme that is locally mass conservative. The approximations of pressure and concentration at each time step are discontinuous piecewise polynomials of different degrees. We show convergence of the numerical method with respect to both the mesh size and the polynomial degree. The flexibility inherent to discontinuous approximation spaces allows the use of complicated geometries and unstructured meshes. The primal discontinuous Galerkin method, analyzed in this paper, encompasses the nonsymmetric interior penalty Galerkin (NIPG) method, the symmetric interior penalty Galerkin (SIPG) and the incomplete interior penalty Galerkin (IIPG) method introduced for elliptic problems respectively by Rivière, Girault and Wheeler [16], Wheeler [24] and Dawson, Sun and Wheeler [3]. Discontinuous Galerkin methods have been recently popular in modeling complex flow and transport problems in porous media. The reader can refer to [20,5,8,12] for some applications.

Several methods for solving the miscible displacement have been analyzed. When classical continuous finite element approximations were used for both the pressure and the concentration equations, optimal convergence rates were proven in the dispersion-free case and nearly optimal convergence rates in the dispersion case, under somewhat idealized circumstances (Ewing and Wheeler [6]). However, this procedure did not handle the transport-dominated problem arising from the concentration equation. Strong improvement in the accuracy of the approximation of the concentration was obtained by considering interior penalty Galerkin methods that could be based on continuous piecewise polynomial spaces (Wheeler and Darlow [25]) or on discontinuous piecewise polynomial spaces (Douglas, Wheeler, Darlow and Kendall [9]): the pressure equation was solved with a standard Galerkin method and penalty terms involving the jumps in the normal derivative were introduced in the concentration equation.

Since only the velocity enters the equation for the concentration, a natural procedure for solving the pressure equation is the mixed finite element method which is locally mass conservative. The concentration equation can be handled in different ways. First, Douglas, Ewing and Wheeler [10,11] approximated the concentration by a standard continuous finite element method. Alternatively, the concentration equation can be solved by a modified method of characteristics (MMOC), which combines the time derivative and the advection terms as a directional derivative. Ewing, Russell and Wheeler [14] introduced and analyzed this method, optimal L^2 error estimates were proven and efficient time stepping techniques were used [22]. Improved error estimates for the MMOC were shown by Dawson, Russell and Wheeler [2]. Russell [21] used a combination of a continuous finite element method and the method of characteristics for the concentration equation and approximated the pressure with a standard continuous finite element method. As in the above cases, time stepping was done along the characteristics.

More recently, primal discontinuous Galerkin methods were applied and analyzed for solving the miscible displacement problem using a semi-discrete approach. The system of equations is discretized in space only. The work of Sun, Rivière and Wheeler [18] studies a combined mixed method for the pressure equation with NIPG for concentration equation. In [19], Sun and Wheeler analyze the NIPG method applied a discontinuous Galerkin scheme introduced in [15]: both pressure and concentration are approximated by the NIPG method. However, the convergence result is valid only for particular boundary conditions, namely Neumann boundary condition for the pressure. Our numerical scheme is fully discrete and valid for both Dirichlet and Neumann boundary conditions for the pressure and Dirichlet, Neuman and mixed boundary conditions for the concentration.

The outline of the paper is as follows. Section 2 contains the system of partial differential equations and assumptions on the data. The coupled discontinuous Galerkin scheme is formulated in Section 3. Existence and convergence of the numerical solution are obtained in Section 4. Extensions of the scheme to several types of boundary conditions are presented in Section 5. Concluding remarks end this paper.

2 Model Problem and Notation

Consider the miscible displacement of one incompressible fluid by another in a porous medium $\Omega \subset \mathbb{R}^2$ and over the time interval $(0,T)$. Let p denote the pressure in the fluid mixture and let c denote the concentration (fraction volume) of the displaced fluid in the fluid mixture. The partial differential equations describing this type of flow are:

$$-\nabla \cdot (\frac{K}{\mu(c)} \nabla p) = f_1, \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$u = -\frac{K}{\mu(c)} \nabla p, \quad \text{in} \quad \Omega \times (0,T), \tag{2}$$

$$\varphi \frac{\partial c}{\partial t} + \nabla \cdot (uc - D(u)\nabla c) = f_2, \quad \text{in} \quad \Omega \times (0,T), \tag{3}$$

subject to the following boundary conditions:

$$p = p_{\text{dir}} \quad \text{on} \quad \Gamma_{\text{D}} \times [0,T], \tag{4}$$
$$u \cdot n = u_{\text{dir}} \quad \text{on} \quad \Gamma_{\text{N}} \times [0,T], \tag{5}$$
$$c = c_{\text{dir}} \quad \text{on} \quad \partial\Omega \times [0,T], \tag{6}$$

where $\Gamma_{\text{D}} \cup \Gamma_{\text{N}}$ is a partition of the boundary $\partial\Omega$. Equation (1), refered to as the pressure equation, is coupled with equation (3) through the viscosity of the fluid mixture. Equation (3), refered to as the concentration equation, is coupled with equation (1) through the fluid velocity (2) and the dispersion-diffusion tensor $D(u)$:

$$D(u) = (\alpha_t \|u\|_2 + d_m)I + (\alpha_l - \alpha_t)\frac{uu^T}{\|u\|_2}.$$

The coefficient d_m is the molecular diffusivity, α_l and α_t are the longitudinal and transverse dispersivities, $\|u\|_2$ is the Euclidean norm of the velocity and I is the identity matrix. Assumptions on the coefficients are made below.

- Assumption H1. The function μ^{-1} is positive, bounded below and above by $\underline{\mu}$ and $\overline{\mu}$ respectively and it is also Lipschitz continuous.

$$\forall t_1, t_2 \in \mathbb{R}, \left|\frac{1}{\mu(t_1)} - \frac{1}{\mu(t_2)}\right| \leq \mu_L |t_1 - t_2|. \tag{7}$$

- Assumption H2. The matrix K is symmetric positive definite and uniformly bounded above and below. There are positive constants $\overline{k}, \underline{k}$ such that:

$$\forall x \in \mathbb{R}^2, \quad \underline{k} x^T x \leq x^T K x \leq \overline{k} x^T x. \tag{8}$$

- Assumption H3. The diffusion coefficient is strictly positive and the dispersivities are bounded.

$$\forall x \in \mathbb{R}^2, \quad 0 \leq \alpha_l(x) \leq \overline{\alpha_l}, \quad 0 \leq \alpha_t(x) \leq \overline{\alpha_t}, \text{ and } 0 < \underline{d} \leq d_m.$$

  Under assumption H3 it was shown that $D(u)$ is uniformly positive definite in Ω and Lipschitz continuous [19]:

$$\forall u \in \mathbb{R}^2, \quad \forall x \in \mathbb{R}^2, \quad \underline{d} x^T x \leq x^T D(u) x, \tag{9}$$
$$\forall u, v \in \mathbb{R}^2, \quad \|D(u) - D(v)\|_2 \leq k_2 \|u - v\|_2, \tag{10}$$

  where $k_2 = (7\overline{\alpha_t} + 6\overline{\alpha_l})2^{3/2}$.

- Assumption H4. The matrix $D(u)$ is uniformly bounded above.

$$\forall u \in \mathbb{R}^2, \quad \forall x \in \mathbb{R}^2, x^T D(u) x \leq \overline{d} x^T x. \tag{11}$$

We propose a discontinuous finite element discretization of (1)-(6). For this, we introduce a non-degenerate quasi-uniform subdivision of Ω, made of either triangles or quadrilaterals. The quasi-uniformity assumption is only needed for the p-version, i.e. for deriving error estimates in terms of the polynomial degree. As usual, the maximum diameter over all mesh elements is denoted by h. The set of interior edges is denoted by Γ_h. To each edge e in Γ_h, we associate a unit normal vector n_e. For a boundary edge, n_e is chosen so that it coincides with the outward normal. The discrete space of discontinuous piecewise polynomials of degree $r \geq 1$ is denoted by $\mathcal{D}_r(\mathcal{E}_h)$:

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h : v|_E \in \mathbb{P}_r(E)\}.$$

For any function $v \in \mathcal{D}_r(\mathcal{E}_h)$, we denote the jump and average over a given edge e by $[v]$ and $\{v\}$ respectively. Assuming that n_e is outward to E_e^1, we can write:

$$\forall e = \partial E_e^1 \cap \partial E_e^2, \quad [v]|_e = v|_{E_e^1} - v|_{E_e^2}, \quad \{v\}|_e = 0.5 v|_{E_e^1} + 0.5 v|_{E_e^2},$$
$$\forall e = \partial E_e^1 \cap \partial \Omega, \quad [v]|_e = v|_{E_e^1}, \quad \{v\}|_e = v|_{E_e^1}.$$

Let N be a positive integer and let $\Delta t = T/N$ be the time step. Denote $t^i = i\Delta t$ for $0 \leq i \leq N$. Define the space

$$\mathcal{D}_{r,h}^N = \{\boldsymbol{v} = (v^i)_{0 \leq i \leq N} : \quad \forall 0 \leq i \leq N \quad v^i \in \mathcal{D}_r(\mathcal{E}_h)\}.$$

We also denote by $\tilde{M}$ the constant that only depends on the maximum number of neighbors that one mesh element can have so that the following inequality holds. Let A be any quantity depending on E_e^1 or E_e^2:

$$\forall i = 1, 2, \quad (\sum_{e \in \Gamma_h} A(E_e^i))^{1/2} \leq \frac{\sqrt{\tilde{M}}}{2} (\sum_{E \in \mathcal{E}_h} A(E))^{1/2}. \tag{12}$$

$$(\sum_{e \in \Gamma_{\mathrm{D}}} A(E_e^1))^{1/2} \leq \sqrt{\tilde{M}} (\sum_{E \in \mathcal{E}_h} A(E))^{1/2}. \tag{13}$$

Let $H^k(\mathcal{O})$ be the usual Sobolev space on $\mathcal{O} \subset \mathbb{R}^d, d \geq 1$ with norm $\| \cdot \|_{k,\mathcal{O}}$. We also define the broken norm:

$$|||v|||_{k,\Omega} = \Big(\sum_{E \in \mathcal{E}_h} \|v\|_{k,E}^2 \Big)^{1/2}.$$

We now recall well-known trace results used in the error analysis in this paper.

Lemma 1 *There is a constant M_t independent of h such that if E is a triangle or quadrilateral:*

$$\forall v \in H^s(E), s \geq 1, \forall e \subset \partial E, \|v\|_{0,e} \leq M_t h^{-1/2} (\|v\|_{0,E} + h \|\nabla v\|_{0,E}), \tag{14}$$

$$\forall v \in H^s(E), s \geq 2, \forall e \subset \partial E, \|\nabla v \cdot n\|_{0,e} \leq M_t h^{-1/2} (\|\nabla v\|_{0,E} + h \left\|\nabla^2 v\right\|_{0,E}). \tag{15}$$

Lemma 2 *Let E be a mesh element. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined by $g(k) = (k+1)(k+2)$ if E is a triangle, and by $g(k) = k^2$ if E is a quadrilateral. There is a constant M_t independent of h and k such that:*

$$\forall v \in \mathbb{P}_k(E), \forall e \subset \partial E, \|v\|_{0,e} \leq M_t \sqrt{\frac{g(k)}{h}} \|v\|_{0,E} . \tag{16}$$

In the case of the triangle, if θ_E denotes the smallest angle and $|e|$ denotes the length of the edge e, an exact expression for M_t is given by:

$$M_t = \sqrt{2 \cot \theta_E \frac{h}{|e|}}.$$

The proofs of these results can be found in the literature: see Theorem 3.10 in [1] for Lemma 1, see Theorem 3 in [23] and the proof of Theorem 9 in [4] for the case of triangle for Lemma 2 and Lemma 2.1 in [17] for the case of quadrilateral for Lemma 2.

3 Scheme

At each discrete time t^i, we will approximate the pressure $p(t^i, \cdot)$ and concentration $c(t^i, \cdot)$ by discontinuous piecewise polynomials P^i and C^i of degree r_p and r_c respectively. For the p-version, we assume that the degrees are related in the following fashion. There exist positive constants δ_0, δ_1 such that

$$\delta_0 \leq \frac{r_c}{r_p} \leq \delta_1. \tag{17}$$

Before formulating the scheme, we introduce additional notation. Let ε be a parameter that takes the value -1, 0 or 1. By changing the value of ε, we will obtain the NIPG, SIPG or IIPG method. Let σ_p and σ_c be two positive parameters, called penalty parameters.
Our numerical method is the following: find $\boldsymbol{P} = (P^i)_{0 \leq i \leq N} \in \mathcal{D}^N_{r_p,h}$ and $\boldsymbol{C} = (C^i)_{0 \leq i \leq N} \in \mathcal{D}^N_{r_c,h}$ such that
<u>Initial Concentration</u>

$$\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega C^0 v = \int_\Omega c^0 v. \tag{18}$$

<u>Pressure Equation</u>

$$\forall 0 \leq i \leq N-1, \quad \forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \sum_{E \in \mathcal{E}_h} \int_E \frac{1}{\mu(C^{i+1})} K \nabla P^{i+1} \cdot \nabla z$$

$$+\sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|}\int_e [P^{i+1}][z] - \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot n_e\}[z] - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla P^{i+1}\cdot n_e z$$

$$+\varepsilon \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(C^{i+1})}K\nabla z\cdot n_e\}[P^{i+1}] + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e P^{i+1}$$

$$= \int_\Omega f_1 z + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e u_{\mathrm{dir}} z + \sigma_p \sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e p_{\mathrm{dir}} z + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e p_{\mathrm{dir}}. \tag{19}$$

Concentration Equation

$$\forall 0\le i\le N-1,\quad \forall v\in\mathcal{D}_{r_c}(\mathcal{E}_h),\quad \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1}-C^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot\nabla v$$

$$+\sum_{E\in\mathcal{E}_h}\int_E D(U^{i+1})\nabla C^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot n_e\}[v]$$

$$-\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla P^{i+1}\cdot n_e v$$

$$-\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e \{D(U^{i+1})\nabla C^{i+1}\cdot n_e\}[v] + \varepsilon\sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla v\cdot n_e\}[P^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e P^{i+1} + \varepsilon\sum_{e\in\Gamma_h}\int_e \{D(U^{i+1})\nabla v\cdot n_e\}[C^{i+1}]$$

$$+\sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e [C^{i+1}][v] = \varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e p_{\mathrm{dir}}$$

$$+\sigma_c \sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}} v + \int_\Omega f_2 v + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}} u_{\mathrm{dir}} v, \tag{20}$$

with the definition of the discrete velocity U^{i+1} given by

$$U^{i+1} = -\frac{K}{\mu(C^{i+1})}\nabla P^{i+1}. \tag{21}$$

We obtain a nonlinear system of equations that can writen in short as

$$\forall z\in\mathcal{D}_{r_p}(\mathcal{E}_h),\quad \forall v\in\mathcal{D}_{r_c}(\mathcal{E}_h),\quad 0\le i\le N,\quad \mathcal{L}(P^i,C^i;z,v)=0.$$

It is easy to check that the scheme (18)-(20) is consistent, i.e. if the solution of (1)-(6) is smooth enough, then it satisfies

$$\forall z\in\mathcal{D}_{r_p}(\mathcal{E}_h),\quad \forall v\in\mathcal{D}_{r_c}(\mathcal{E}_h),\quad 0\le i\le N,\quad \mathcal{L}(p^i,c^i;z,v)=0, \tag{22}$$

where $c^i = c(t^i,\cdot)$ and $p^i = p(t^i,\cdot)$.

4 Existence and Convergence of the Discrete Solution

In this section, we prove the existence and show convergence of the numerical solution by the use of the second Schauder's theorem (see theorem 6.44 in [7]). Let $\tilde{p}$ and $\tilde{c}$ be approximations of p and c. We assume that

$$\tilde{p} \in L^\infty(0,T,W^{1,\infty}(\Omega)), \quad \tilde{c} \in L^\infty(0,T,L^\infty(\Omega)), \quad \tilde{c}_{tt} \in L^\infty(0,T,L^2(\Omega)). \tag{23}$$

We will denote $\tilde{p}^i(\cdot) = \tilde{p}(t^i,\cdot)$ and $\tilde{c}^i(\cdot) = \tilde{c}(t^i,\cdot)$. We assume that there are constants $\kappa_p, \kappa_c \geq 2$ such that

$$\forall 0 \leq i \leq N, \quad \forall t > 0, \quad p^i(t) \in H^{\kappa_p}(\Omega), \quad c^i(t) \in H^{\kappa_c}(\Omega).$$

We also assume that the following hp-type approximation results hold

$$\forall 0 \leq i \leq N, \quad \|\tilde{p}^i - p^i\|_{H^s(\Omega)} \leq M \frac{h^{\min(r_p+1,\kappa_p)-s}}{r_p^{\kappa_p-s}} \|p^i\|_{H^{\kappa_p}(\Omega)}, \tag{24}$$

$$\forall 0 \leq i \leq N, \quad \|\tilde{c}^i - c^i\|_{H^s(\Omega)} \leq M \frac{h^{\min(r_c+1,\kappa_c)-s}}{r_c^{\kappa_c-s}} \|c^i\|_{H^{\kappa_c}(\Omega)}, \tag{25}$$

where M is a constant independent of h, r_p, r_c and Δt. In addition, in the case of the p-version, we assume that $\kappa_p, \kappa_c \geq 3$. In th rest of the paper, the variable M will denote a generic constant independent of h, r_c, r_p and Δt, that takes different values at different places.

Next we prove existence and convergence of the solution using a technique found in [13]. Let us define the following subset of the broken Sobolev space:

$$\mathcal{W} = \Big\{ (\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{D}^N_{r_p,h} \times \mathcal{D}^N_{r_c,h} : \phi^0 = \tilde{c}^0, \text{ and there exist positive constants}$$

$K_1, K_2, \ldots, K_6, \Delta t_0$ independent of h such that for $\Delta t \leq \Delta t_0$ and $0 \leq i \leq N-1$:

$$(\frac{1}{\Delta t} - K_1) \left\| \phi^{i+1} - \tilde{c}^{i+1} \right\|^2_{0,\Omega} - \frac{1}{\Delta t} \left\| \phi^i - \tilde{c}^i \right\|^2_{0,\Omega}$$

$$+ |||\phi^{i+1} - \tilde{c}^{i+1}|||^2_1 \leq K_2 \frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + K_3 \frac{h^{2r_c}}{r_c^{2\kappa_c-4}} + K_4 \Delta t^2,$$

$$|||\psi^{i+1} - \tilde{p}^{i+1}|||^2_1 \leq K_5 \frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + K_6 \frac{h^{2r_c}}{r_c^{2\kappa_c-4}} \Big\}.$$

Clearly the subset $\mathcal{W}$ is closed, convex and non-empty since it contains the element $(\tilde{p}^i, \tilde{c}^i)_{0 \leq i \leq N}$.

Lemma 3 *For any $(\boldsymbol{\psi}, \boldsymbol{\phi})$ in $\mathcal{W}$, if Δt is small enough (namely $\Delta t = \mathcal{O}(h/r_c) <$*

$1/K_1$), there exist positive constant M_1, M_2, M_3 for any $1 \le i \le N$

$$\left\|\phi^i - \tilde{c}^i\right\|_{0,\Omega} \le M_1\Big(\frac{h^{r_p}}{r_p^{\kappa_p-2}} + \frac{h^{r_c}}{r_c^{\kappa_c-2}} + \Delta t\Big), \tag{26}$$

$$||\phi^i||_{\infty,\Omega} \le M_2, \tag{27}$$

$$|||\psi^i|||_1 \le M_3. \tag{28}$$

The constants M_1, M_2 are independent of h, r_p, r_c and Δt but depend on $K_1, \ldots, K_4$. The constant M_3 is independent of h, r_p, r_c and Δt but depends on K_5, K_6. In addition, the constant M_2 depends on δ_1.

PROOF. We remark that (28) is a simple consequence of the definition of $\mathcal{W}$. We now prove (26), which will yield (27). From the definition of the space $\mathcal{W}$, we have for $0 \le i \le N-1$:

$$\begin{aligned}&\left\|\phi^{i+1} - \tilde{c}^{i+1}\right\|^2_{0,\Omega} - \left\|\phi^i - \tilde{c}^i\right\|^2_{0,\Omega} + \Delta t|||\phi^{i+1} - \tilde{c}^{i+1}|||^2_1\\ &\le \Delta t K_2 \frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + \Delta t K_3 \frac{h^{2r_c}}{r_c^{2\kappa_c-4}} + K_4 \Delta t^3 + \Delta t K_1 \left\|\phi^{i+1} - \tilde{c}^{i+1}\right\|^2_{0,\Omega}.\end{aligned}$$

We fix $n \in \{1, \ldots, N\}$. Summing from $i = 0$ to $i = n-1$ and noting that $\sum_{i=0}^{n-1} \Delta t \le T$, we obtain:

$$\begin{aligned}&\|\phi^n - \tilde{c}^n\|^2_{0,\Omega} - \left\|\phi^0 - \tilde{c}^0\right\|^2_{0,\Omega} + \Delta t \sum_{i=0}^{n-1} |||\phi^{i+1} - \tilde{c}^{i+1}|||^2_1\\ &\le K_2 T \frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + K_3 T \frac{h^{2r_c}}{r_c^{2\kappa_c-4}} + K_4 T \Delta t^2 + \Delta t K_1 \sum_{i=0}^{n-1} \left\|\phi^{i+1} - \tilde{c}^{i+1}\right\|^2_{0,\Omega}.\end{aligned}$$

From Gronwall's lemma, if $\Delta t < 1/K_1$, there is a constant M independent of h and Δt such that

$$\begin{aligned}&\|\phi^n - \tilde{c}^n\|^2_{0,\Omega} + \Delta t \sum_{i=0}^{N-1} |||\phi^{i+1} - \tilde{c}^{i+1}|||^2_1\\ &\le \left\|\phi^0 - \tilde{c}^0\right\|^2_{0,\Omega} + M\frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + M\frac{h^{2r_c}}{r_c^{2\kappa_c-4}} + M\Delta t^2.\end{aligned}$$

Since $\phi^0 = \tilde{c}^0$, we obtain (26). Besides, from (17) and choosing $\Delta t = \mathcal{O}(\frac{h}{r_c})$, we conclude that

$$\|\phi^n - \tilde{c}^n\|_{0,\Omega} \le M\frac{h}{r_c}.$$

Using an inverse inequality, we have

$$\|\phi^n - \tilde{c}^n\|_{\infty,\Omega} \le M r_c h^{-1} \|\phi^n - \tilde{c}^n\|_{0,\Omega} \le M.$$

This implies that

$$\|\phi^n\|_{\infty,\Omega} \le M + \|\tilde{c}^n\|_{\infty,\Omega} \le M_2,$$

which with (23) yields gives the result (27). $\square$

We now define an operator $\mathcal{F}$ that acts on elements in $\mathcal{W}$.

$$\forall(\boldsymbol{\psi},\boldsymbol{\phi}) \in \mathcal{W}, \quad \mathcal{F}(\boldsymbol{\psi},\boldsymbol{\phi}) = (\boldsymbol{\psi}_L,\boldsymbol{\phi}_L),$$

where

$$(\psi_L^0,\phi_L^0) = (\psi^0,\phi^0), \tag{29}$$

and for $0 \le i \le N-1$, $\psi_L^{i+1} \in \mathcal{D}_{r_p}(\mathcal{E}_h)$ and $\phi_L^{i+1} \in \mathcal{D}_{r_c}(\mathcal{E}_h)$ such that

$$\begin{aligned}
\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad & \sum_{E\in\mathcal{E}_h}\int_E \frac{1}{\mu(\phi^{i+1})}K\nabla\psi_L^{i+1}\cdot\nabla z + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e [\psi_L^{i+1}][z] \\
& - \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(\phi^{i+1})}K\nabla\psi_L^{i+1}\cdot n_e\}[z] - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\psi_L^{i+1}\cdot n_e z \\
& +\varepsilon \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(\phi^{i+1})}K\nabla z\cdot n_e\}[\psi_L^{i+1}] + \varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e\psi_L^{i+1} \\
= \int_\Omega f_1 z + & \sum_{e\in\Gamma_{\mathrm{N}}}\int_e u_{\mathrm{dir}}z + \sigma_p\sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e p_{\mathrm{dir}}z + \varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e p_{\mathrm{dir}}.
\end{aligned} \tag{30}$$

$$\begin{aligned}
\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad & \int_\Omega \frac{\varphi}{\Delta t}(\phi_L^{i+1}-\phi_L^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\psi_L^{i+1}\cdot\nabla v \\
& + \sum_{E\in\mathcal{E}_h}\int_E D(\zeta^{i+1})\nabla\phi_L^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\psi_L^{i+1}\cdot n_e\}[v] \\
& - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\psi_L^{i+1}\cdot n_e v \\
& - \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e\{D(\zeta^{i+1})\nabla\phi_L^{i+1}\cdot n_e\}[v] + \varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla v\cdot n_e\}[\psi_L^{i+1}] \\
& +\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e\psi_L^{i+1} + \varepsilon\sum_{e\in\Gamma_h}\int_e\{D(\zeta^{i+1})\nabla v\cdot n_e\}[\phi_L^{i+1}] \\
& +\sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e[\phi_L^{i+1}][v] = \varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e p_{\mathrm{dir}} \\
& +\sigma_c\sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}}v + \int_\Omega f_2 v + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}}u_{\mathrm{dir}}v,
\end{aligned} \tag{31}$$

where

$$\zeta^{i+1} = -\frac{K}{\mu(\phi^{i+1})}\nabla\psi^{i+1}. \tag{32}$$

First, we show that $\mathcal{F}$ is well-defined by proving existence and uniqueness of $(\boldsymbol{\psi}_L,\boldsymbol{\phi}_L)$.

Lemma 4 *There exists a unique solution* $(\boldsymbol{\psi}_L, \boldsymbol{\phi}_L) \in \mathcal{D}^N_{r_p,h} \times \mathcal{D}^N_{r_c}$ *that satisfies (29)-(31).*

PROOF. Since the problem (29)-(31) is linear and finite-dimensional, it suffices to show uniqueness of the solution. Let $(\boldsymbol{\psi}_{L1}, \boldsymbol{\phi}_{L1})$ and $(\boldsymbol{\psi}_{L2}, \boldsymbol{\phi}_{L2})$ be two solutions and let $(\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}})$ denote their differences. Then, the pair $(\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}})$ satisfies (29)-(31) with zero data $f_1 = p_{\text{dir}} = u_{\text{dir}} = c_{\text{dir}} = f_2 = 0$ and $\phi_L^i = 0$. Clearly, we have $(\bar{\psi}^0, \bar{\phi}^0) = (0, 0)$. Fix $i \in \{0, \ldots, N-1\}$ and choose the test function $z = \bar{\psi}^{i+1}$ in (30).

$$|||\frac{1}{\mu(\phi^{i+1})^{1/2}} K^{1/2} \nabla \bar{\psi}^{i+1}|||^2_{0,\Omega} + \sigma_p \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \|[\bar{\psi}^{i+1}]\|^2_{0,e}$$

$$-(1-\epsilon) \sum_{e \in \Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K \nabla \bar{\psi}^{i+1} \cdot n_e\}[\bar{\psi}^{i+1}] - (1-\epsilon) \sum_{e \in \Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\text{dir}})} K \nabla \bar{\psi}^{i+1} \cdot n_e \bar{\psi}^{i+1} = 0.$$

If $\epsilon = 1$, we directly have that $\bar{\psi}^{i+1} = 0$. Otherwise, using assumption H1 and trace and inverse inequalities, we can bound the last two terms of the left-hand side of the equation above by

$$\frac{1}{2} |||\frac{1}{\mu(\phi^{i+1})^{1/2}} K^{1/2} \nabla \bar{\psi}^{i+1}|||^2_{0,\Omega} + M \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}}} \frac{1}{|e|} \|[\bar{\psi}^{i+1}]\|^2_{0,e},$$

which implies that $\bar{\psi}^{i+1} = 0$ if the penalty value σ_p is large enough. Next, we choose the test function $v = \bar{\phi}^{i+1}$ in (31). The equation reduces to:

$$\|\frac{\varphi^{1/2}}{\Delta t^{1/2}} \bar{\phi}^{i+1}\|^2_{0,\Omega} + |||D(\zeta^{i+1})^{1/2} \nabla \bar{\phi}^{i+1}|||^2_{0,\Omega} + \sigma_c \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|} \|[\bar{\phi}^{i+1}]\|^2_{0,e}$$

$$- \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}} \int_e \{D(\zeta^{i+1}) \nabla \bar{\phi}^{i+1} \cdot n_e\}[\bar{\phi}^{i+1}] + \varepsilon \sum_{e \in \Gamma_h} \int_e \{D(\zeta^{i+1}) \nabla \bar{\phi}^{i+1} \cdot n_e\}[\bar{\phi}^{i+1}] = 0.$$

As above, the last two terms in the left-hand side of the equation above can be bounded by

$$\frac{1}{2} |||D(\zeta^{i+1})^{1/2} \nabla \bar{\phi}^{i+1}|||^2_{0,\Omega} + M \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}} \frac{1}{|e|} \|[\bar{\phi}^{i+1}]\|^2_{0,e}.$$

Therefore, if the penalty value σ_c is chosen large enough, we obtain $\bar{\phi}^{i+1} = 0$. □

We now show that the range of $\mathcal{F}$ is included in the space $\mathcal{W}$. The same technique can be used to show that $\mathcal{F}$ is continuous.

Theorem 5

$$\forall (\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}, \quad \mathcal{F}(\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}.$$

PROOF. Let $(\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}$, $(\boldsymbol{\psi}_L, \boldsymbol{\phi}_L) = \mathcal{F}(\boldsymbol{\psi}, \boldsymbol{\phi})$ and denote

$$\forall 0 \leq i \leq N, \quad \tau^i = \psi_L^i - \tilde{p}^i, \theta^i = p^i - \tilde{p}^i, \xi^i = \phi_L^i - \tilde{c}^i, \chi^i = c^i - \tilde{c}^i.$$

From the consistency equations (22), we have

$$\sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(\phi^{i+1})} K\nabla\tilde{p}^{i+1} \cdot \nabla z + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e [\tilde{p}^{i+1}][z]$$

$$- \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K\nabla\tilde{p}^{i+1} \cdot n_e\}[z] - \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla\tilde{p}^{i+1} \cdot n_e z - \sum_{e\in\Gamma_{\mathrm{N}}} \int_e u_{\mathrm{dir}} z$$

$$+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K\nabla z \cdot n_e\}[\tilde{p}^{i+1}] + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla z \cdot n_e \tilde{p}^{i+1}$$

$$- \int_\Omega f_1 z - \sigma_p \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e p_{\mathrm{dir}} z - \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla z \cdot n_e p_{\mathrm{dir}}$$

$$= - \sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(c^{i+1})} K\nabla\theta^{i+1} \cdot \nabla z - \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e [\theta^{i+1}][z]$$

$$+ \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(c^{i+1})} K\nabla\theta^{i+1} \cdot n_e\}[z] + \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla\theta^{i+1} \cdot n_e z$$

$$-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(c^{i+1})} K\nabla z \cdot n_e\}[\theta^{i+1}] - \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla z \cdot n_e \theta^{i+1}$$

$$+ \sum_{E\in\mathcal{E}_h} \int_E (\frac{1}{\mu(\phi^{i+1})} - \frac{1}{\mu(c^{i+1})}) K\nabla\tilde{p}^{i+1} \cdot \nabla z - \sum_{e\in\Gamma_h} \int_e \{(\frac{1}{\mu(\phi^{i+1})} - \frac{1}{\mu(c^{i+1})}) K\nabla\tilde{p}^{i+1} \cdot n_e\}[z]$$

$$+\varepsilon \sum_{e\in\Gamma_h} \int_e \{(\frac{1}{\mu(\phi^{i+1})} - \frac{1}{\mu(c^{i+1})}) K\nabla z \cdot n_e\}[\tilde{p}^{i+1}]. \qquad (33)$$

Subtracting equation (33) from (30) and choosing $z = \tau^{i+1}$, we obtain:

$$|||\frac{1}{\mu(\phi^{i+1})^{1/2}} K^{1/2} \nabla\tau^{i+1}|||_{0,\Omega}^2 + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \|[\tau^{i+1}]\|_{0,e}^2$$

$$= (1-\varepsilon) \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K\nabla\tau^{i+1} \cdot n_e\}[\tau^{i+1}] + (1-\varepsilon) \sum_{e\in\Gamma_{\mathrm{D}}} \int_e (\frac{1}{\mu(c_{\mathrm{dir}})} K\nabla\tau^{i+1} \cdot n_e)\tau^{i+1}$$

$$- \sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(c^{i+1})} K\nabla\theta^{i+1} \cdot \nabla\tau^{i+1} - \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e [\theta^{i+1}][\tau^{i+1}]$$

$$+ \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(c^{i+1})} K\nabla\theta^{i+1} \cdot n_e\}[\tau^{i+1}] + \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla\theta^{i+1} \cdot n_e \tau^{i+1}$$

$$-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(c^{i+1})} K\nabla\tau^{i+1} \cdot n_e\}[\theta^{i+1}] - \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla\tau^{i+1} \cdot n_e \theta^{i+1}$$

$$+\sum_{E\in\mathcal{E}_h}\int_E(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot\nabla\tau^{i+1}-\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot n_e\}[\tau^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla z\cdot n_e\}[\widetilde{p}^{i+1}].$$

$$= T_1+\cdots+T_{11}. \quad (34)$$

Next, we bound each term in the right-hand side of (34) using techniques standard for discontinuous Galerkin methods. In what follows, the quantities ε_i are positive real numbers to be defined later. Using Assumptions $H1$ and $H2$ and Cauchy-Schwarz inequality, we have

$$|T_1|\le(1-\varepsilon)\overline{\mu}\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e}\left\|[\tau^{i+1}]\right\|_{0,e}$$

We now fix an interior edge e and denote E_e^1 and E_e^2 two elements sharing the edge e. Using (12) and the trace inequality (16), we have:

$$\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e}\left\|[\tau^{i+1}]\right\|_{0,e}\le\sum_{e\in\Gamma_h}\frac{1}{2}(\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}|_{E_e^1}\right\|_{0,e}+\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}|_{E_e^2}\right\|_{0,e})\left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\le\frac{1}{2}M_t\sqrt{\frac{g(r_p)}{h}}\sum_{e\in\Gamma_h}(\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^1}+\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^2})\left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\le(\sum_{e\in\Gamma_h}\frac{M_t^2g(r_p)}{4h}\left\|[\tau^{i+1}]\right\|_{0,e}^2)^{\frac{1}{2}}\Big((\sum_{e\in\Gamma_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^1}^2)^{\frac{1}{2}}+(\sum_{e\in\Gamma_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^2}^2)^{\frac{1}{2}}\Big)$$

$$\le(\sum_{e\in\Gamma_h}\frac{\widetilde{M}M_t^2g(r_p)}{4h}\left\|[\tau^{i+1}]\right\|_{0,e}^2)^{\frac{1}{2}}(\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2)^{\frac{1}{2}}.$$

Therefore, we have the following bound on T_1:

$$|T_1|\le\frac{\mu}{24}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2+(1-\varepsilon)^2\frac{3(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{2\underline{\mu}}\sum_{e\in\Gamma_h}\frac{g(r_p)}{h}\left\|[\tau^{i+1}]\right\|_{0,e}^2. \quad (35)$$

Similarly, using (13) and (16), we have for T_2:

$$|T_2|\le\frac{\mu}{24}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2+(1-\varepsilon)^2\frac{6(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{\underline{\mu}}\sum_{e\in\Gamma_{\rm D}}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|_{0,e}^2. \quad (36)$$

The term T_3 is bounded using assumption H1 and (8), Cauchy-Schwarz and Young's inequalities:

$$|T_3|\le\overline{\mu}(\overline{k})^{\frac{1}{2}}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}\left\|\nabla\theta^{i+1}\right\|_{0,E}$$

$$\le\frac{\mu}{12}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2+M|||\nabla\theta^{i+1}|||_{0,\Omega}^2.$$

Using the trace inequality (14), we have for the term T_4:

$$|T_4| \leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\theta^{i+1}\right\|_{0,E}^2 + \left\|\nabla\theta^{i+1}\right\|_{0,E}^2). \tag{37}$$

The terms T_5 and T_6 are bounded in a similar way as the terms T_1 and T_2, except that the trace inequality (15) is used instead of (16).

$$|T_5| \leq \overline{\mu}\overline{k} \sum_{e\in\Gamma_h} \left\|\{\nabla\theta^{i+1}\}\right\|_{0,e} \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\leq (\sum_{e\in\Gamma_h} \frac{M_t^2\widetilde{M}\overline{\mu}^2\overline{k}^2}{4h} \left\|[\tau^{i+1}]\right\|_{0,e}^2)^{1/2} \Big((\sum_{E\in\mathcal{E}_h} \left\|\nabla\theta^{i+1}\right\|_{0,E}^2)^{1/2} + (\sum_{E\in\mathcal{E}_h} h^2 \left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2)^{1/2} \Big)$$

$$\leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{M}{g(r_p)} \sum_{E\in\mathcal{E}_h} (\left\|\nabla\theta^{i+1}\right\|_{0,E}^2 + h^2 \left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2). \tag{38}$$

Similarly we have for term T_6 we have:

$$|T_6| \leq \overline{\mu}\overline{k} \sum_{e\in\Gamma_h} \left\|\{\nabla\theta^{i+1}\}\right\|_{0,e} \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\leq (\sum_{e\in\Gamma_h} \frac{M_t^2\widetilde{M}\overline{\mu}^2\overline{k}^2}{4h} \left\|[\tau^{i+1}]\right\|_{0,e}^2)^{1/2} \Big((\sum_{E\in\mathcal{E}_h} \left\|\nabla\theta^{i+1}\right\|_{0,E}^2)^{1/2} + (\sum_{E\in\mathcal{E}_h} h^2 \left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2)^{1/2} \Big)$$

$$\leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{M}{g(r_p)} \sum_{E\in\mathcal{E}_h} (\left\|\nabla\theta^{i+1}\right\|_{0,E}^2 + h^2 \left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2). \tag{39}$$

The terms T_7 and T_8 are handled in the same way as terms T_1 and T_2, with the exception that the trace inequality (14) is used to handle the approximation error term.

$$|T_7| \leq \overline{\mu}(\overline{k})^{\frac{1}{2}} \sum_{e\in\Gamma_h} \left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e} \left\|[\theta^{i+1}]\right\|_{0,e}$$

$$\leq \frac{\mu}{12} |||K^{1/2}\nabla\tau^{i+1}|||_{0,\Omega}^2 + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\theta^{i+1}\right\|_{0,E}^2 + \left\|\nabla\theta^{i+1}\right\|_{0,E}^2). \tag{40}$$

Similarly, for term T_8, we have:

$$|T_8| \leq \frac{\mu}{12} |||K^{1/2}\nabla\tau^{i+1}|||_{0,\Omega}^2 + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\theta^{i+1}\right\|_{0,E}^2 + \left\|\nabla\theta^{i+1}\right\|_{0,E}^2). \tag{41}$$

Using (7), (8), Cauchy-Schwarz inequality and assumption on $\widetilde{p}^{i+1}$ (23), we have:

$$|T_9| \leq \mu_L \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty (\overline{k})^{\frac{1}{2}} \sum_{E\in\mathcal{E}_h} \left\|\phi^{i+1} - c^{i+1}\right\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}$$

$$\leq \mu_L \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty (\overline{k})^{\frac{1}{2}} \sum_{E\in\mathcal{E}_h} (\left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_0 + \left\|\chi^{i+1}\right\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E})$$

$$\leq \frac{\underline{\mu}}{12} |||K^{\frac{1}{2}}\nabla\tau^{i+1}|||^2_{0,\Omega} + M\|\nabla\widetilde{p}^{i+1}\|^2_\infty \|\phi^{i+1} - \widetilde{c}^{i+1}\|^2_{0,\Omega} + M\|\nabla\widetilde{p}^{i+1}\|^2_\infty \left\|\chi^{i+1}\right\|^2_{0,\Omega}. \tag{42}$$

The term T_{10} is a summation term over interior edges. We assume that the edge e is shared by the elements E_e^1 and E_e^2. Thus, we have using (7), (8), (23) and Cauchy-Schwarz inequality:

$$|T_{10}| \leq \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty \overline{k}\frac{\mu_L}{2} \sum_{e\in\Gamma_h} \Big((\left\|(\phi^{i+1} - \widetilde{c}^{i+1})|_{E_e^1}\right\|_{0,e} + \left\|(\phi^{i+1} - \widetilde{c}^{i+1})|_{E_e^2}\right\|_{0,e}) \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$+ (\left\|\chi^{i+1}|_{E_e^1}\right\|_{0,e} + \left\|\chi^{i+1}|_{E_e^2}\right\|_{0,e}) \left\|[\tau^{i+1}]\right\|_{0,e} \Big).$$

Using the trace inequality (14), (16), we have:

$$|T_{10}| \leq \frac{\mu_L}{2}\overline{k} \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty M_t \sqrt{\frac{g(r_c)}{h}} \sum_{e\in\Gamma_h} (\left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,E_e^1} + \left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,E_e^2}) \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$+ \frac{\mu_L}{2}\overline{k} \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty M_t h^{-1/2} \sum_{e\in\Gamma_h} (\left\|\chi^{i+1}\right\|_{0,E_e^1} + \left\|\chi^{i+1}\right\|_{0,E_e^2} + h\left\|\nabla\chi^{i+1}\right\|_{0,E_e^1} + h\left\|\nabla\chi^{i+1}\right\|_{0,E_e^2}) \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h} \frac{g(r_c)}{|e|} \left\|[\tau^{i+1}]\right\|^2_{0,e} + M\|\nabla\widetilde{p}^{i+1}\|^2_\infty \|\phi^{i+1} - \widetilde{c}^{i+1}\|^2_{0,\Omega}$$

$$+ \frac{M\,\|\nabla\widetilde{p}^{i+1}\|^2_\infty}{g(r_c)} \sum_{E\in\mathcal{E}_h} (\left\|\chi^{i+1}\right\|^2_{0,E} + h^2\left\|\nabla\chi^{i+1}\right\|^2_{0,E}). \tag{43}$$

The term T_{11} vanishes if the approximation $\widetilde{p}$ is continuous. Otherwise, we can bound exactly like the term T_5.

$$|T_{11}| \leq \sum_{e\in\Gamma_h} |\int_e \{(\frac{1}{\mu(C^{i+1})} - \frac{1}{\mu(c^{i+1})}) K\nabla\tau^{i+1}\cdot n_e\}[\theta^{i+1}]|$$

$$\leq \frac{\underline{\mu}}{12} |||K^{1/2}\nabla\tau^{i+1}|||^2_{0,\Omega} + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2}\left\|\theta^{i+1}\right\|^2_{0,E} + \left\|\nabla\theta^{i+1}\right\|^2_{0,E}). \tag{44}$$

Combining all the bounds (35)-(44) we have the following estimate for the velocity equation:

$$\frac{\underline{\mu}}{2} |||K^{\frac{1}{2}}\nabla\tau^{i+1}|||^2_{0,\Omega} + (\frac{\sigma_p}{2} - (1-\varepsilon)^2 \frac{3(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{2\underline{\mu}}) \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|^2_{0,e}$$

$$+(\frac{7}{8}\sigma_p - (1-\varepsilon)^2 \frac{6(\overline{\mu})^2 \overline{k}\widetilde{M} M_t^2}{\underline{\mu}}) \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2$$

$$\leq M\frac{g(r_p)}{h^2}\left\|\theta^{i+1}\right\|_{0,\Omega}^2 + M(1+g(r_p)+\frac{1}{g(r_p)})|||\nabla\theta^{i+1}|||_{0,\Omega}^2$$

$$+M\|\nabla\widetilde{p}^{i+1}\|_\infty^2(1+\frac{1}{g(r_p)})\left\|\chi^{i+1}\right\|_{0,\Omega}^2 + \frac{M\left\|\nabla\widetilde{p}^{i+1}\right\|_\infty^2 h^2}{g(r_c)}|||\nabla\chi^{i+1}|||_{0,\Omega}^2$$

$$+M\left\|\nabla\widetilde{p}^{i+1}\right\|_\infty^2 \|\phi^{i+1}-\widetilde{c}^{i+1}\|_{0,\Omega}^2.$$

Define the limiting value of the penalty parameter:

$$\sigma_p^* = (1-\varepsilon)^2\frac{48(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{7\underline{\mu}}.$$

Assuming that $\sigma_p > \sigma_p^*$, using the approximation results, the fact that $\boldsymbol{\phi}$ belongs to $\mathcal{W}$ and the fact that $1 \leq r^2 \leq g(r) \leq 6r^2$, we obtain:

$$|||\nabla\tau^{i+1}|||_{0,\Omega}^2 + \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|_{0,e}^2$$

$$\leq M\max(\frac{1}{\underline{\mu}\underline{k}},\frac{1}{\sigma_p-\sigma_p^*})\Big(\frac{h^{2\min(r_p+1,\kappa_p)-2}}{r_p^{2\kappa_p-4}}\|p^{i+1}\|_{H^{\kappa_p}(\Omega)}^2 + \|\nabla\widetilde{p}^{i+1}\|_\infty^2\frac{h^{2\min(r_c+1,\kappa_c)-2}}{r_c^{2\kappa_c-2}}\|c^{i+1}\|_{H^{\kappa_c}(\Omega)}^2\Big)$$

$$+MM_1\max(\frac{1}{\underline{\mu}\underline{k}},\frac{1}{\sigma_p-\sigma_p^*})\left\|\nabla\widetilde{p}^{i+1}\right\|_\infty^2(\frac{h^{2r_p}}{r_p^{2\kappa_p-4}}+\frac{h^{2r_c}}{r_c^{2\kappa_c-4}}+\Delta t^2). \qquad (45)$$

Next, we consider the concentration equation in the system (22). The same way as for the pressure equation, the concentration equation can be rewritten as:

$$\int_\Omega \frac{\phi}{\Delta t}(\widetilde{c}^{i+1}-\widetilde{c}^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\widetilde{p}^{i+1}\cdot\nabla v$$

$$+\sum_{E\in\mathcal{E}_h}\int_E D(\zeta^{i+1})\nabla\widetilde{c}^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\widetilde{p}^{i+1}\cdot n_e\}[v]$$

$$-\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\widetilde{p}^{i+1}\cdot n_e v - \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}}u_{\mathrm{dir}}v$$

$$-\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e\{D(\zeta^{i+1})\nabla\widetilde{c}^{i+1}\cdot n_e\}[v] + \varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla v\cdot n_e\}[\widetilde{p}^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e\widetilde{p}^{i+1} + \varepsilon\sum_{e\in\Gamma_h}\int_e\{D(\zeta^{i+1})\nabla v\cdot n_e\}[\widetilde{c}^{i+1}]$$

$$+\sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e[\widetilde{c}^{i+1}][v] - \varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla v\cdot n_e p_{\mathrm{dir}} \qquad (46)$$

$$-\sigma_c\sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}}v - \int_\Omega f_2 v$$

$$
\begin{aligned}
&= \int_\Omega \phi \Delta t \rho^{i+1} v - \int_\Omega \phi \chi_t^{i+1} v - \sum_{E\in\mathcal{E}_h} \int_E \frac{c^{i+1}}{\mu(c^{i+1})} K\nabla\theta^{i+1}\cdot\nabla v \\
&- \sum_{E\in\mathcal{E}_h} \int_E D(u^{i+1})\nabla\chi^{i+1}\cdot\nabla v + \sum_{e\in\Gamma_h} \int_e \{\frac{c^{i+1}}{\mu(c^{i+1})} K\nabla\theta^{i+1}\cdot n_e\}[v] \\
&+ \sum_{e\in\Gamma_\mathrm{D}} \int_e \frac{c_\mathrm{dir}}{\mu(c_\mathrm{dir})} K\nabla\theta^{i+1}\cdot n_e v + \sum_{e\in\Gamma_h\cup\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N}} \int_e \{D(u^{i+1})\nabla\chi^{i+1}\cdot n_e\}[v] \\
&-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{c^{i+1}}{\mu(c^{i+1})} K\nabla v\cdot n_e\}[\theta^{i+1}] - \varepsilon \sum_{e\in\Gamma_\mathrm{D}} \int_e \frac{c_\mathrm{dir}}{\mu(c_\mathrm{dir})} K\nabla v\cdot n_e \theta^{i+1} \\
&-\varepsilon \sum_{e\in\Gamma_h} \int_e \{D(u^{i+1})\nabla v\cdot n_e\}[\chi^{i+1}] \\
&-\sigma_c \sum_{e\in\Gamma_h\cup\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N}} \frac{g(r_c)}{|e|} \int_e [\chi^{i+1}][v] + \sum_{E\in\mathcal{E}_h} \int_E (D(\zeta^{i+1}) - D(u^{i+1}))\nabla\widetilde{c}^{i+1}\cdot\nabla v \\
&+ \sum_{E\in\mathcal{E}_h} \int_E \Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla\widetilde{p}^{i+1}\cdot\nabla v \\
&- \sum_{e\in\Gamma_h} \int_e \{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla\widetilde{p}^{i+1}\cdot n_e\}[v] \\
&- \sum_{e\in\Gamma_h\cup\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N}} \int_e \{\Big(D(\zeta^{i+1}) - D(u^{i+1})\Big)\nabla\widetilde{c}^{i+1}\cdot n_e\}[v] \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla v\cdot n_e\}[\widetilde{p}^{i+1}] \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\Big(D(\zeta^{i+1}) - D(u^{i+1})\Big)\nabla v\cdot n_e\}[\widetilde{c}^{i+1}]. \qquad (47)
\end{aligned}
$$

Substracting equation (47) from (31), using (32) and choosing $z = \xi^{i+1}$, we obtain:

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(\xi^{i+1} - \xi^i)\xi^{i+1} + |||D(\zeta^{i+1})^{1/2}\nabla\xi^{i+1}|||_{0,\Omega}^2 + \sigma_c \sum_{e\in\Gamma_h\cup\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N}} \frac{g(r_c)}{|e|} \|[\xi^{i+1}]\|_{0,e}^2 \\
&= \sum_{E\in\mathcal{E}_h} \int_E \frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla\tau^{i+1}\cdot\nabla\xi^{i+1} - \sum_{e\in\Gamma_h} \int_e \{\frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla\tau^{i+1}\cdot n_e\}[\xi^{i+1}] \\
&- \sum_{e\in\Gamma_\mathrm{D}} \int_e \frac{c_\mathrm{dir}}{\mu(c_\mathrm{dir})} K\nabla\tau^{i+1}\cdot n_e\xi^{i+1} \\
&+(1-\varepsilon) \sum_{e\in\Gamma_h} \int_e \{D(\zeta^{i+1})\nabla\xi^{i+1}\cdot n_e\}[\xi^{i+1}] + \sum_{e\in\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N}} \int_e D(\zeta^{i+1})\nabla\xi^{i+1}\cdot n_e\xi^{i+1} \\
&-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla\xi^{i+1}\cdot n_e\}[\tau^{i+1}] - \varepsilon \sum_{e\in\Gamma_\mathrm{D}} \int_e \frac{c_\mathrm{dir}}{\mu(c_\mathrm{dir})} K\nabla\xi^{i+1}\cdot n_e\tau^{i+1} \\
&- \int_\Omega \phi\Delta t\rho^{i+1}\xi^{i+1} + \int_\Omega \phi\chi_t^{i+1}\xi^{i+1} + \sum_{E\in\mathcal{E}_h} \int_E \frac{c^{i+1}}{\mu(c^{i+1})} K\nabla\theta^{i+1}\cdot\nabla\xi^{i+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{E\in\mathcal{E}_h}\int_E D(u^{i+1})\nabla\chi^{i+1}\cdot\nabla\xi^{i+1}-\sum_{e\in\Gamma_h}\int_e\{\frac{c^{i+1}}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&-\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\theta^{i+1}\cdot n_e\xi^{i+1}-\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e\{D(u^{i+1})\nabla\chi^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{c^{i+1}}{\mu(c^{i+1})}K\nabla\xi^{i+1}\cdot n_e\}[\theta^{i+1}]+\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\xi^{i+1}\cdot n_e\theta^{i+1}\\
&+\varepsilon\sum_{e\in\Gamma_h}\int_e\{D(u^{i+1})\nabla\xi^{i+1}\cdot n_e\}[\chi^{i+1}]\\
&+\sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e[\chi^{i+1}][\xi^{i+1}]-\sum_{E\in\mathcal{E}_h}\int_E(D(\zeta^{i+1})-D(u^{i+1}))\nabla\tilde{c}^{i+1}\cdot\nabla\xi^{i+1}\\
&-\sum_{E\in\mathcal{E}_(}\int_E\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\tilde{p}^{i+1}\cdot\nabla\xi^{i+1}\\
&+\sum_{e\in\Gamma_h}\int_e\{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\tilde{p}^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&+\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e\{\Big(D(\zeta^{i+1})-D(u^{i+1})\Big)\nabla\tilde{c}^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\xi^{i+1}\cdot n_e\}[\tilde{p}^{i+1}]\\
&-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\Big(D(\zeta^{i+1})-D(u^{i+1})\Big)\nabla\xi^{i+1}\cdot n_e\}[\tilde{c}^{i+1}]\\
&=S_1+\dots S_{24}.
\end{aligned}
\tag{48}
$$

The term S_8 contains the numerical error in the time discretization:

$$
\rho^{i+1}=\frac{1}{\Delta t}(\frac{\tilde{c}^{i+1}-\tilde{c}^i}{\Delta t}-\frac{\partial\tilde{c}^{i+1}}{\partial t}).
$$

We now bound each term in the right-hand side of (48). The term S_1 is bounded using assumption H1, (8) and (27):

$$
|S_1|\leq\frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2+MM_2^2|||\nabla\tau^{i+1}|||_{0,\Omega}^2. \tag{49}
$$

Using H1, (8) and (27), the term S_2 is bounded in a similar way as for the term T_1:

$$
|S_2|\leq\frac{\sigma_c}{18}\sum_{e\in\Gamma_h}\frac{g(r_c)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2+\frac{Mg(r_p)}{g(r_c)}M_2^2|||\nabla\tau^{i+1}|||_{0,\Omega}^2 \tag{50}
$$

Using H1 and (8) and similarly as for T_2, we obtain:

$$
|S_3|\leq\frac{\sigma_c}{12}\sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_c)}{|e|}\left\|\xi^{i+1}\right\|_{0,e}^2+\frac{Mg(r_p)}{g(r_c)}\|c_{\mathrm{dir}}\|_\infty^2|||\nabla\tau^{i+1}|||_{0,\Omega}^2. \tag{51}
$$

Using assumption (11), the term S_4 is bounded in a similar way as for the term T_1:

$$|S_4| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + (1-\varepsilon)^2 \frac{7(\overline{d})^2 \widetilde{M} M_t^2}{4\underline{d}} \sum_{e\in\Gamma_h} \frac{g(r_c)}{|e|} \left\|[\xi^{i+1}]\right\|_{0,e}^2. \tag{52}$$

The term S_5 is bounded in a similar way as for the term T_2:

$$|S_5| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + \frac{7(\overline{d})^2 \widetilde{M} M_t^2}{\underline{d}} \sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|} \left\|\xi^{i+1}\right\|_{0,e}^2. \tag{53}$$

The term S_6 is bounded in a similar way as for the term T_1:

$$|S_6| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + M\frac{g(r_c)}{g(r_p)} M_2^2 \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2. \tag{54}$$

The term S_7 is bounded in a similar way as for the term T_2:

$$|S_7| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + M\frac{g(r_c)}{g(r_p)}\|c_{\mathrm{dir}}\|_\infty^2 \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|\tau^{i+1}\right\|_{0,e}^2. \tag{55}$$

The terms S_8 and S_9 are simply bounded using Cauchy-Schwarz's inequality.

$$|S_8| \leq \frac{1}{4}\left\|\xi^{i+1}\right\|_{0,\Omega}^2 + \overline{\phi}^2 \Delta t^2 \left\|\rho^{i+1}\right\|_{0,\Omega}^2. \tag{56}$$

$$|S_9| \leq \frac{1}{4}\left\|\xi^{i+1}\right\|_{0,\Omega}^2 + \overline{\phi}^2 \left\|\chi_t^{i+1}\right\|_{0,\Omega}^2. \tag{57}$$

The term S_{10} is bounded in a similar way as for the term T_3:

$$|S_{10}| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + M\|c^{i+1}\|_\infty^2 |||\nabla\theta^{i+1}|||_{0,\Omega}^2. \tag{58}$$

The term S_{11} is bounded like the term T_3:

$$|S_{11}| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_{0,\Omega}^2 + M\overline{d}^2 |||\nabla\chi^{i+1}|||_{0,\Omega}^2. \tag{59}$$

The term S_{12} is bounded exactly like the term T_5:

$$|S_{12}| \leq \frac{\sigma_c}{18} \sum_{e\in\Gamma_h} \frac{g(r_c)}{|e|} \left\|[\xi^{i+1}]\right\|_{0,e}^2 + \frac{M\|c^{i+1}\|_\infty^2}{g(r_c)} \left(\sum_{E\in\mathcal{E}_h} \left(\left\|\nabla\theta^{i+1}\right\|_{0,E}^2 + h^2\left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2\right)\right). \tag{60}$$

The term S_{13} is bounded like the term T_6:

$$|S_{13}| \leq \frac{\sigma_c}{12} \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_c)}{|e|} \left\|\xi^{i+1}\right\|_{0,e}^2 + \frac{M}{g(r_c)}\|c_{\mathrm{dir}}\|_\infty^2 \left(\sum_{E\in\mathcal{E}_h} \left(\left\|\nabla\theta^{i+1}\right\|_{0,E}^2 + h^2\left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2\right)\right). \tag{61}$$

The term S_{14} is bounded like the terms T_5 and T_6 with assumption H4:

$$|S_{14}| \leq \frac{\sigma_c}{18} \sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}} \frac{g(r_c)}{|e|} \left\|[\xi^{i+1}]\right\|^2_{0,e} + M\frac{\overline{d}^2}{g(r_c)} (\sum_{E\in\mathcal{E}_h} (\left\|\nabla\chi^{i+1}\right\|^2_{0,E} + h^2 \left\|\nabla^2\chi^{i+1}\right\|^2_{0,E})). \tag{62}$$

The term S_{15} is bounded similarly as the term T_7:

$$|S_{15}| \leq \frac{d}{28} |||\nabla\xi^{i+1}|||^2_{0,\Omega} + Mg(r_c)\|c^{i+1}\|^2_\infty (\sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\theta^{i+1}\right\|^2_{0,E} + \left\|\nabla\theta^{i+1}\right\|^2_{0,E})). \tag{63}$$

The term S_{16} is bounded exactly like the term T_8:

$$|S_{16}| \leq \frac{d}{28} |||\nabla\xi^{i+1}|||^2_{0,\Omega} + Mg(r_c)\|c_{\rm dir}\|^2_\infty (\sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\theta^{i+1}\right\|^2_{0,E} + \left\|\nabla\theta^{i+1}\right\|^2_{0,E})). \tag{64}$$

The term S_{17} is bounded like the term T_7:

$$|S_{17}| \leq \frac{d}{28} |||\nabla\xi^{i+1}|||^2_{0,\Omega} + Mg(r_c)\overline{d}^2 (\sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\chi^{i+1}\right\|^2_{0,E} + \left\|\nabla\chi^{i+1}\right\|^2_{0,E})). \tag{65}$$

The term S_{18} is bounded like the term T_4:

$$|S_{18}| \leq \frac{\sigma_c}{18} \sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}} \frac{g(r_c)}{|e|} \left\|[\xi^{i+1}]\right\|^2_{0,e} + Mg(r_c)(\sum_{E\in\mathcal{E}_h} (h^{-2} \left\|\chi^{i+1}\right\|^2_{0,E} + \left\|\nabla\chi^{i+1}\right\|^2_{0,E})). \tag{66}$$

Consider the term S_{19} using the assumptions H1, H3 and that $(\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}$ we have:

$$\begin{aligned}
S_{19} \leq{}& \bar{k}k_2\bar{\mu}\|\nabla\tilde{c}^{i+1}\|_\infty \sum_{E\in\mathcal{E}_h} \|\nabla\xi^{i+1}\|_{0,E}\Big(\|\nabla(\psi^{i+1} - \tilde{p}^{i+1})\|_{0,E} + \|\nabla\theta^{i+1}\|_{0,E}\Big) \\
&+\bar{k}k_2\|\nabla\tilde{c}^{i+1}\|_\infty\mu_L\|\nabla p^{i+1}\|_\infty \sum_{E\in\mathcal{E}_h} \|\nabla\xi^{i+1}\|_{0,E}\Big(\|\phi^{i+1} - \tilde{c}^{i+1}\|_{0,E} + \|\chi^{i+1}\|_{0,E}\Big) \\
\leq{}& \frac{d}{28} |||\nabla\xi^{i+1}|||^2_0 + M\|\nabla\tilde{c}^{i+1}\|^2_\infty (|||\nabla(\psi^{i+1} - \tilde{p}^{i+1})|||^2_0 + |||\nabla\theta^{i+1}|||^2_0) \\
&+M\|\nabla\tilde{c}^{i+1}\|^2_\infty\|\nabla p^{i+1}\|^2_\infty (\|\phi^{i+1} - \tilde{c}^{i+1}\|^2_0 + \|\chi^{i+1}\|^2_0).
\end{aligned}$$

Before bounding the term S_{20} we remark that

$$|\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}| \leq \bar{\mu}^2(\frac{1}{\bar{\mu}} + \mu_L\|c^{i+1}\|_\infty)|\phi^{i+1} - c^{i+1}|.$$

Therefore we have

$$S_{20} \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_0^2$$
$$+M\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2(\|\phi^{i+1}-\tilde{c}^{i+1}\|_0^2+\|\chi^{i+1}\|_0^2).$$

The term S_{21} is bounded similarly to the term T_{10}:

$$|S_{21}| \leq \frac{\sigma_c}{18}\sum_{e\in\Gamma_h}\frac{g(r_c)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2 + M\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2\|\phi^{i+1}-\tilde{c}^{i+1}\|_0^2$$
$$+\frac{M}{g(r_c)}\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2\sum_{E\in\mathcal{E}_h}(\left\|\chi^{i+1}\right\|_{0,E}^2+h^2\left\|\nabla\chi^{i+1}\right\|_{0,E}^2).$$

Consider the term S_{22}, using the assumptions H1 and H3 we have:

$$S_{22} \leq \frac{\bar{k}k_2}{2}\|\nabla\tilde{c}^{i+1}\|_\infty\bar{\mu}\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\|[\xi^{i+1}]\|_{0,e}\Big(\|\nabla(\psi^{i+1}-\tilde{p}^{i+1})|_{E_1}\|_{0,e}+\|\nabla\theta^{i+1}|_{E_1}\|_{0,e}\Big)$$
$$+\frac{\bar{k}k_2}{2}\|\nabla\tilde{c}^{i+1}\|_\infty\bar{\mu}\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\|[\xi^{i+1}]\|_{0,e}\Big(\|\nabla(\psi^{i+1}-\tilde{p}^{i+1})|_{E_2}\|_{0,e}+\|\nabla\theta^{i+1}|_{E_2}\|_{0,e}\Big)$$
$$+\frac{\bar{k}k_2}{2}\|\nabla\tilde{c}^{i+1}\|_\infty\mu_L\|\nabla p^{i+1}\|_\infty\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\|[\xi^{i+1}]\|_{0,e}\Big(\|(\phi^{i+1}-\tilde{c}^{i+1})|_{E_1}\|_{0,e}+\|\chi^{i+1}|_{E_1}\|_{0,e}\Big)$$
$$+\frac{\bar{k}k_2}{2}\|\nabla\tilde{c}^{i+1}\|_\infty\mu_L\|\nabla p^{i+1}\|_\infty\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\|[\xi^{i+1}]\|_{0,e}\Big(\|(\phi^{i+1}-\tilde{c}^{i+1})|_{E_2}\|_{0,e}+\|\chi^{i+1}|_{E_2}\|_{0,e}\Big)$$
$$\leq \frac{\sigma_c}{18}\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\frac{g(r_c)}{|e|}\|[\xi^{i+1}]\|_{0,e}^2+\frac{Mg(r_p)\|\nabla\tilde{c}^{i+1}\|_\infty^2}{g(r_c)}|||\nabla(\psi^{i+1}-\tilde{p}^{i+1})|||_1^2$$
$$+\frac{M\|\nabla\tilde{c}^{i+1}\|_\infty^2}{g(r_c)}\sum_{E\in\mathcal{E}_h}(\|\nabla\theta^{i+1}\|_E^2+h^2\|\nabla^2\theta^{i+1}\|_E^2)+M\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2\|\phi^{i+1}-\tilde{c}^{i+1}\|_0^2$$
$$+\frac{M\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2}{g(r_c)}\sum_{E\in\mathcal{E}_h}(\|\chi^{i+1}\|_E^2+h^2\|\nabla\chi^{i+1}\|_E^2).$$

The term S_{23} is bounded like the term T_{11} :

$$|S_{23}| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_0^2+Mg(r_c)(M_2+\|c^{i+1}\|_\infty)^2\sum_{E\in\mathcal{E}_h}(h^{-2}\left\|\theta^{i+1}\right\|_{0,E}^2+\left\|\nabla\theta^{i+1}\right\|_{0,E}^2). \tag{67}$$

The term S_{24} is bounded like the term T_{11} :

$$|S_{24}| \leq \frac{d}{28}|||\nabla\xi^{i+1}|||_0^2+M\bar{d}^2g(r_c)\sum_{E\in\mathcal{E}_h}(h^{-2}\left\|\chi^{i+1}\right\|_{0,E}^2+\left\|\nabla\chi^{i+1}\right\|_{0,E}^2). \tag{68}$$

Combining the bounds (50)-(68), we obtain:

$$\frac{1}{2\Delta t}(\left\|\phi^{1/2}\xi^{i+1}\right\|_{0,\Omega}^2 - \left\|\phi^{1/2}\xi^{i}\right\|_{0,\Omega}^2) + \frac{d}{2}|||\nabla\xi^{i+1}|||_{0,\Omega}^2$$

$$+\Big(\frac{\sigma_c}{3}-(1-\varepsilon)^2\frac{7(\overline{d})^2\widetilde{M}M_t^2}{4\underline{d}}\Big)\sum_{e\in\Gamma_h}\frac{g(r_c)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2+\Big(\frac{\sigma_c}{3}-\frac{7(\overline{d})^2\widetilde{M}M_t^2}{\underline{d}}\Big)\sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{h}\left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\le M(M_2^2(1+\frac{g(r_p)}{g(r_c)})+\|c_{\mathrm{dir}}\|_\infty^2\frac{g(r_p)}{g(r_c)})|||\nabla\tau^{i+1}|||_0^2+M\frac{g(r_c)}{g(r_p)}(\|c_{\mathrm{dir}}\|_\infty^2+M_2^2)\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|_{0,e}^2$$

$$+\frac{1}{2}\left\|\xi^{i+1}\right\|_{0,\Omega}^2+M\Delta t^2\left\|\rho^{i+1}\right\|_{0,\Omega}^2+M\left\|\chi_t^{i+1}\right\|_{0,\Omega}^2$$

$$+M\Big(\frac{g(r_c)}{h^2}(1+\overline{d}^2)+(1+\frac{1}{g(r_c)})(\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2+\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2)\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$

$$+M\frac{g(r_c)}{h^2}\Big(\|c^{i+1}\|_\infty^2+\|c_{\mathrm{dir}}\|_\infty^2+M_2^2\Big)\|\theta^{i+1}\|_{0,\Omega}^2$$

$$+M\Big(\overline{d}^2+g(r_c)(1+\overline{d}^2)+\frac{1}{g(r_c)}(\overline{d}^2+\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2h^2+\|\nabla p^{i+1}\|_\infty^2\|\nabla\tilde{c}^{i+1}\|_\infty^2h^2\Big)|||\nabla\chi^{i+1}|||_{0,\Omega}^2$$

$$+M\Big((\|c^{i+1}\|_\infty^2+(g(r_c)+\frac{1}{g(r_c)})(\|c^{i+1}\|_\infty^2+\|c_{\mathrm{dir}}\|_\infty^2)+(1+\frac{1}{g(r_c)})\|\nabla\tilde{c}^{i+1}\|_\infty^2\Big)|||\nabla\theta^{i+1}|||_{0,\Omega}^2$$

$$+M\overline{d}^2\frac{h^2}{g(r_c)}|||\nabla^2\chi^{i+1}|||_{0,\Omega}^2$$

$$+M\frac{h^2}{g(r_c)}\Big(\|c^{i+1}\|_\infty^2+\|c_{\mathrm{dir}}\|_\infty^2+\|\nabla\tilde{c}^{i+1}\|_\infty^2\Big)|||\nabla^2\theta^{i+1}|||_{0,\Omega}^2$$

$$+M\|\nabla\tilde{c}^{i+1}\|_\infty^2(1+\frac{g(r_p)}{g(r_c)})|||\nabla(\tilde{\psi}^{i+1}-\tilde{p}^{i+1})|||_{0,\Omega}^2$$

$$+M\Big(\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2+\|\nabla\tilde{p}^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2\Big)\|\phi^{i+1}-\tilde{c}^{i+1}\|_{0,\Omega}^2$$

The error $\|\rho^{i+1}\|_{0,\Omega}$ is bounded using a Taylor expansion with integral remainder:

$$\tilde{c}^i = \tilde{c}^{i+1} - \Delta t\frac{\partial\tilde{c}^{i+1}}{\partial t} + \frac{1}{2}\int_{t^i}^{t^{i+1}}(t-t^i)\frac{\partial^2\tilde{c}^{i+1}}{\partial t^2}dt,$$

which yields

$$\|\rho^{i+1}\|_{0,\Omega} \le M\|\tilde{c}_{tt}\|_{L^\infty(t^i,t^{i+1},L^2(\Omega))}.$$

Define

$$\sigma_c^* = \max\Big((1-\varepsilon)^2\frac{21(\overline{d})^2\widetilde{M}M_t^2}{4\underline{d}},\frac{21(\overline{d})^2\widetilde{M}M_t^2}{\underline{d}}\Big).$$

Under the condition $\sigma_c > \sigma_c^*$, using the approximation result, the bound (45), the fact that $1 \le r^1 \le g(r) \le 6r^2$, we obtain the following estimate:

$$\frac{\left\|\varphi^{1/2}\xi^{i+1}\right\|_{0,\Omega}^2}{\Delta t} - \frac{\left\|\varphi^{1/2}\xi^{i}\right\|_{0,\Omega}^2}{\Delta t} + |||\nabla\xi^{i+1}|||_{\Omega}^2 + \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}$$

$$\leq \max(1, \frac{1}{\underline{d}}, \frac{3}{2(\sigma_c - \sigma_*)})\Big(\left\|\xi^{i+1}\right\|^2_{0,\Omega} + K_2 \frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + K_3 \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} + K_4 \Delta t\Big). \quad (69)$$

Equations (45) and (69) imply that $(\boldsymbol{\psi}, \boldsymbol{\phi})$ belongs to $\mathcal{W}$.

From Lemma 3 and Theorem 5, we have that the set $\mathcal{F}(\mathcal{W})$ is bounded. Using similar techniques as in Lemma 4 and Theorem 5 it can be show that operator $\mathcal{F}$ is continuous. Since we are in finite dimension, it follows that the operator $\mathcal{F}$ is compact. Therefore, by Schauder's second fixed point theorem there is a solution $\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}$ such that

$$(\boldsymbol{\psi}, \boldsymbol{\phi}) = \mathcal{F}(\boldsymbol{\psi}, \boldsymbol{\phi}).$$

This fixed point solution is the solution to (18)-(20). Using the definition of the space $\mathcal{W}$, the approximation results (24), (25) and Lemma 3, we obtain the following *a priori* error estimates.

Theorem 6 *Let $(\boldsymbol{P}, \boldsymbol{C})$ be a solution to (18)-(20). Assume that the solution (p, c) to (1)-(6) belongs to $L^\infty(0, T; H^{\kappa_p}(\Omega)) \times L^\infty(0, T; H^{\kappa_c}(\Omega))$. Assume that the penalty parameters satisfy:*

$$\sigma_p \geq \sigma_p^*, \quad \sigma_p^* = (1-\varepsilon)^2 \frac{48(\overline{\mu})^2 \overline{k} \widetilde{M} M_t^2}{7\underline{\mu}}$$

$$\sigma_c \geq \sigma_c^*, \quad \sigma_c^* = \max\left((1-\varepsilon)^2 \frac{21(\overline{d})^2 \widetilde{M} M_t^2}{4\underline{d}}, \frac{21(\overline{d})^2 \widetilde{M} M_t^2}{\underline{d}}\right).$$

Then, there exists a constant M independent of h, r_p, r_c and Δt such that

$$1 \leq i \leq N, \quad \left\|C^i - c^i\right\|_{0,\Omega} + (\Delta t \sum_{j=1}^{i} |||C^j - c^j|||_1^2)^{1/2} + |||P^i - p^i|||_1$$
$$\leq M_1 \Big(\frac{h^{r_p}}{r_p^{\kappa_p - 2}} + \frac{h^{r_c}}{r_c^{\kappa_c - 2}} + \Delta t \Big). \quad (70)$$

5 Extensions

The method can be slightly modified to consider several other boundary conditions. For instance, we may have
Case 1:

$$c = c_{\text{dir}} \quad \text{on} \quad \forall (x,t) \in \partial\Omega \times \bar{J},$$
$$u \cdot n = u_{\text{dir}} \quad \forall (x,t) \in \partial\Omega \times \bar{J},$$

Case 2:

$$c = c_{\text{dir}} \quad \text{on} \quad \Gamma_{\text{D}} \times \bar{J},$$

$$u \cdot n = u_{\text{dir}} \quad \forall (x,t) \in \partial\Omega \times \bar{J},$$
$$D(u)\nabla c \cdot n = 0, \quad \Gamma_{\text{N}} \times \bar{J}$$

In Case 1, the pressure and concentration equations become:
<u>Pressure Equation</u>

$$\forall 0 \le i \le N{-}1, \quad \forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot \nabla z + \sigma_p \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \int_e [P^{i+1}][z]$$
$$- \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot n_e\}[z] + \varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(C^{i+1})} K\nabla z \cdot n_e\}[P^{i+1}]$$
$$= \int_\Omega f_1 z + \sum_{e\in\partial\Omega} \int_e u_{\text{dir}} z. \tag{71}$$

<u>Concentration Equation</u>

$$\forall 0 \le i \le N{-}1, \quad \forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1} - C^i)v + \sum_{E\in\mathcal{E}_h} \int_E \frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot \nabla v$$
$$+ \sum_{E\in\mathcal{E}_h} \int_E D(U^{i+1})\nabla C^{i+1} \cdot \nabla v - \sum_{e\in\Gamma_h} \int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot n_e\}[v]$$
$$- \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{D(U^{i+1})\nabla C^{i+1} \cdot n_e\}[v] + \varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla v \cdot n_e\}[P^{i+1}]$$
$$+\varepsilon \sum_{e\in\Gamma_h} \int_e \{D(U^{i+1})\nabla v \cdot n_e\}[C^{i+1}] + \sigma_c \sum_{e\in\Gamma_h\cup\partial\Omega} \frac{g(r_c)}{|e|} \int_e [C^{i+1}][v] =$$
$$+\sigma_c \sum_{e\in\partial\Omega} \frac{g(r_c)}{|e|} \int_e c_{\text{dir}} v + \int_\Omega f_2 v + \sum_{e\in\partial\Omega} \int_e c_{\text{dir}} u_{\text{dir}} v, \tag{72}$$

In Case 2, the pressure equation is the same as in case 1, but the concentration equation becomes:
<u>Concentration Equation</u>

$$\forall 0 \le i \le N{-}1, \quad \forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1} - C^i)v + \sum_{E\in\mathcal{E}_h} \int_E \frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot \nabla v$$
$$+ \sum_{E\in\mathcal{E}_h} \int_E D(U^{i+1})\nabla C^{i+1} \cdot \nabla v - \sum_{e\in\Gamma_h} \int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot n_e\}[v]$$
$$- \sum_{e\in\Gamma_{\text{N}}} \int_e C^{i+1} u_{\text{dir}} v$$
$$- \sum_{e\in\Gamma_h\cup\Gamma_{\text{D}}} \int_e \{D(U^{i+1})\nabla C^{i+1} \cdot n_e\}[v] + \varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla v \cdot n_e\}[P^{i+1}]$$
$$+\varepsilon \sum_{e\in\Gamma_h} \int_e \{D(U^{i+1})\nabla v \cdot n_e\}[C^{i+1}]$$

$$+\sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e [C^{i+1}][v] =$$

$$+\sigma_c \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}} v + \int_\Omega f_2 v + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}} u_{\mathrm{dir}} v, \tag{73}$$

The analysis can be modified to accomodate those different boundary conditions.

6 Conclusions

We study the application of primal discontinuous Galerkin methods, namely NIPG, IIPG, SIPG, and backward Euler discretization to solve the miscible displacement problem. We give explicit expressions of the limiting values of the penalty parameters above which the method is stable and convergent. They depend in particular of the trace constants. In the case of NIPG, any penalty values can be used for the pressure equation whereas a minimum penalty value is required for the concentration equation.

References

[1] S. Agmon. *Lectures on Elliptic Boundary Value Problems.* Van Nostrand, Princeton, NJ, 1965.

[2] C.N.Dawson, T.F.Russell, and M.F.Wheeler. Some improved error estimates for the modified method of characteristics. *SIAM J.Numer. Anal.*, 26(6):1487–1512, 1989.

[3] C. Dawson, S. Sun, and M.F. Wheeler. Compatible algorithms for coupled flow and transport. *Comput. Meth. Appl. Mech. Engng*, 193:2565–2580, 2004.

[4] Y. Epshteyn and B. Rivière. Estimation of penalty parameters for symmetric interior penalty Galerkin methods. *Journal of Computational and Applied Mathematics*, 206:843–872, 2007.

[5] Y. Epshteyn and B. Rivière. Fully implicit discontinuous finite element methods for two-phase flow. *Applied Numerical Mathematics*, 57:383–401, 2007.

[6] R.E. Ewing and M.F. Wheeler. Galerkin methods for miscible displacement problems in porous media. *SIAM J. Numer. Anal.*, 17(3):351–365, June 1980.

[7] D.H. Griffel, editor. *Applied Functional Analysis.* Dover, 2002.

[8] H. Hoteit and A. Firoozabadi. Compositional modeling by the combined discontinuous Galerkin and mixed methods. *SPE Journal*, pages 19–34, March 2006.

[9] J.Douglas, M.F.Wheeler, B.L.Darlow, and R.P.Kendall. Self-adaptive finite element simulation of miscible displacement in porous media. *Computer methods in applied mechanics and engineering*, 47:131–159, 1984.

[10] J.Douglas, R.E.Ewing, and M.F.Wheeler. The approximation of the pressure by a mixed method in the simulation of miscible displacement. *R.A.I.R.O. Numerical Analysis*, 17(1):17–33, 1983.

[11] J.Douglas, R.E.Ewing, and M.F.Wheeler. A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media. *R.A.I.R.O. Numerical Analysis*, 17(3):249–265, 1983.

[12] E.J. Kubatko, J.J. Westerink, and C. Dawson. hp discontinous Galerkin methods for advection-dominated problems in shallow water. *Computer Methods in Applied Mechanics and Engineering*, 196:437–451, 2006.

[13] C. Ortner and E. Süli. Discontinuous Galerkin finite element approximation of nonlinear second-order elliptic and hyperbolic systems. *SIAM Journal on Numerical Analysis*, 45(4):1370–1397, 2007.

[14] R.E.Ewing, T.F.Russell, and M.F.Wheeler. Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics. *Comput.Methods Appl.Mech.Engrg*, 47:73–92, 1984.

[15] B. Rivière and M.F. Wheeler. Discontinuous Galerkin methods for flow and transport problems in porous media. *Communications in Numerical Methods in Engineering*, 18:63–68, 2002.

[16] B. Rivière, M.F. Wheeler, and V. Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I. *Computational Geosciences*, 3:337–360, April 1999.

[17] B. Rivière, M.F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM Journal on Numerical Analysis*, 39(3):902–931, 2001.

[18] S. Sun, B. Rivière, and M. Wheeler. A combined mixed finite element and discontinuous Galerkin method for miscible displacement problem in porous media. In *Recent Progress in Computational and Applied PDEs*, pages 323–348. Kluwer/Plenum, 2002.

[19] S. Sun and M.F. Wheeler. Discontinuous Galerkin methods for coupled flow and reactive transport problems. *Applied Numerical Mathematics*, 52:273–298, 2005.

[20] S. Sun and M.F. Wheeler. Analysis of discontinuous galerkin methods for multicomponent reactive transport problems. *Computers and Mathematics with Applications*, 52:637–650, 2006.

[21] T.F.Russell. Time-stepping along characteristics with incomplete iteration for a Galerkin approximation of miscible displacement in porous media. *SIAM J.Numer.Anal.*, 22:970–1013, 1985.

[22] T.Russell, M.F.Wheeler, and C.Chiang. Large-scale simulation of miscible displacement by mixed and characteristics finite element methods. *Mathematical and computational methods in seismic exploration and reservoir modeling*, pages 85–107, 1985.

[23] T. Warburton and J.S. Hesthaven. On the constants in hp-finite element trace inverse inequalities. *Comput. Methods Appl. Mech. Engrg.*, 192:2765–2773, 2003.

[24] M.F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM Journal on Numerical Analysis*, 15(1):152–161, 1978.

[25] M.F. Wheeler and B.L.Darlow. Interior penalty Galerkin procedures for miscible displacement problems in porous media. *Computational Methods in Nonlinear Mechanics*, pages 485–506, 1980.