A Dual-Mixed Approximation Method for a Three-Field Model of a Nonlinear Generalized Stokes Problem

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Abstract. In this work a dual-mixed approximation of a nonlinear generalized Stokes problem is studied. The problem is analyzed in Sobolev spaces which arise naturally in the problem formulation. Existence and uniqueness results are given and error estimates are derived. It is shown that both lowest-order and higher-order mixed finite elements are suitable for the approximation method. Numerical experiments that support the theoretical results are presented.

Key words. generalized Stokes problem, dual-mixed method, twofold saddle point problem, Sobolev spaces

1 Introduction

In this article we investigate the solution of a nonlinear generalized Stokes problem using a dual-mixed formulation. The nonlinear generalized Stokes problem arises in modeling flows of, for example, biological fluids, lubricants, paints, polymeric fluids, where the fluid viscosity is assumed to be a nonlinear function of the fluid's velocity gradient tensor. The generalized Stokes problem is given by: *Find* $(\mathbf{u}, p)$ *such that*

$$\begin{aligned} -\nabla \cdot (\nu(|\nabla \mathbf{u}|)\nabla \mathbf{u}) + \nabla p &= \mathbf{f} \quad \text{in } \Omega\,, & (1.1)\\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega\,, & (1.2)\\ \mathbf{u} &= \mathbf{u}_\Gamma \quad \text{on } \Gamma\,, & (1.3) \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary Γ. The fluid velocity is denoted by $\mathbf{u}$, and $\nabla \mathbf{u} := (\nabla \mathbf{u})_{ij} = \partial u_i / \partial x_j$ is the tensor gradient of $\mathbf{u}$. Here and throughout the paper we use the following notation: for tensors $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij})$, $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j} \sigma_{ij}\tau_{ij}$, $|\boldsymbol{\sigma}|^2 = \boldsymbol{\sigma} : \boldsymbol{\sigma}$. The pressure is denoted by p, and $\mathbf{f}$ describes the external forces on the fluid. The function ν describes the nonlinear kinematic viscosity of the fluid.

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Some classical examples of ν are given by:

Power Law

$$\nu(|\mathbf{d}(\mathbf{u})|) \;=\; \nu_0\,|\mathbf{d}(\mathbf{u})|^{r-2}\,, \quad \nu_0 > 0, \quad 1 < r < 2, \tag{1.4}$$

where $\mathbf{d}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$ denotes the fluid deformation tensor. The power law model has been used to model the viscosity of many polymeric solutions and melts over a considerable range of shear rates [18].

Ladyzhenskaya Law[21]:

$$\nu(|\nabla\mathbf{u}|) \;=\; (\nu_0 \,+\, \nu_1\,|\nabla\mathbf{u}|)^{r-2}\,, \quad \nu_0 \geq 0,\ \nu_1 > 0,\ r > 1\,, \tag{1.5}$$

which has been used in modeling fluids with large stresses.

Carreau Law:

$$\nu(|\mathbf{d}(\mathbf{u})|) \;=\; \nu_0\left(1 \,+\, |\mathbf{d}(\mathbf{u})|^2\right)^{(r-2)/2}\,, \quad \nu_0 > 0,\ r \geq 1\,, \tag{1.6}$$

used in modeling visco-plastic flows and creeping flow of metals.

General descriptions of (1.1) are often written in terms of the tensor $\boldsymbol{\sigma} = \nu(|\nabla\mathbf{u}|)\nabla\mathbf{u}$:

$$-\nabla\cdot\boldsymbol{\sigma} + \nabla p = \mathbf{f} \quad \text{ in } \Omega\,. \tag{1.7}$$

The work in this paper extends the investigations of [4, 22, 15]. In [4] Baranger, Najib, and Sandri provided an analysis for the existence and uniqueness of the modeling equations in appropriate Sobolev spaces and gave an error analysis of a finite element approximation method applied to the primitive variables ($\boldsymbol{\sigma}$, p, $\mathbf{u}$). Manouzi and Farhloul in [22] reformulated the modeling equations into a saddle point problem and used a *mixed formulation* to study the existence and uniqueness of the solution, again in appropriate Sobolev spaces. An error analysis for the finite element approximation was also given. In both [4] and [22] the analysis used the assumption that the equation describing $\boldsymbol{\sigma}$ in terms of $\mathbf{d}(\mathbf{u})$ or $\nabla\mathbf{u}$ was invertible to give $\mathbf{d}(\mathbf{u})$ or $\nabla\mathbf{u}$ as a function of $\boldsymbol{\sigma}$.

Recent work by Gatica in [13] and Gatica, Heuer, and Meddahi in [14] provided a general theory for solvability and Galerkin approximations of a class of nonlinear *twofold saddle point problems* posed in Hilbert spaces. In [15], Gatica, González, and Meddahi reformulated the modeling equations for a nonlinear generalized Stokes flow as a twofold saddle point problem, using the tensor $\boldsymbol{\psi}$ in place of $\boldsymbol{\sigma}$ ($\boldsymbol{\psi} = \boldsymbol{\sigma} - p\mathbf{I}$) and introducing an additional variable for $\nabla\mathbf{u}$. In doing so, their formulation used the constitutive equation for $\boldsymbol{\sigma}$ as a function of $\nabla\mathbf{u}$ and reduced the regulatity requirement for the velocity. Advantages of this approach include: (i) more flexibility in choosing the approximating finite element space for $\mathbf{u}$, (ii) Dirichlet boundary conditions for $\mathbf{u}$ become *natural* boundary conditions and are easily incorporated into the variational formulations, (iii) avoids the assumption of expressing $\nabla\mathbf{u}$ was a function of $\boldsymbol{\sigma}$. A disadvantage in this formulation is that additional unknowns are introduced. The analysis of this approach was only studied in a Hilbert space setting.

In this paper we recast the formulation described in [15] in appropriate Sobolev spaces. Because of the nonlinearity in (1.7), this problem is more appropriately studied in Sobolev spaces which should result in tighter error estimates for the approximate solution. This extends the work of [22] by avoiding the assumption of expressing $\nabla \mathbf{u}$ as a function of $\boldsymbol{\sigma}$. In addition, we show that higher-order approximating spaces can be used in the mixed finite element method for this formulation and give the associated a priori error estimates.

A description of the notation used in this paper, the mathematical problem, and the dual-mixed variational formulation is given in Section 2. Existence and uniqueness of the variational formulation is studied in Section 3. In Section 4 the finite element approximation is presented and analyzed. Numerical results are given in Section 5.

2 Mathematical Setting

Here and throughout the rest of this paper we consider the case where $1 < r < 2$. We denote the unitary conjugate of r by r', satisfying $r^{-1} + r'^{-1} = 1$. Used in the analysis below are the following function spaces and norms.

$$T := (L^r(\Omega))^{n\times n} \;=\; \{\boldsymbol{\tau} = (\tau_{ij});\; \tau_{ij} \in L^r(\Omega)\;;\; i,j = 1,\ldots,n\},$$

with norm $\|\boldsymbol{\tau}\|_T := \left(\int_\Omega |\boldsymbol{\tau}|^r \, d\Omega\right)^{1/r}$.

$$T' := \left(L^{r'}(\Omega)\right)^{n\times n} \quad \text{and} \quad T'_{div} := \left\{\boldsymbol{\tau} \in T'\;;\; div\,\boldsymbol{\tau} \in \left(L^{r'}(\Omega)\right)^n\right\},$$

with norm $\|\boldsymbol{\tau}\|_{T'_{div}} := \left(\int_\Omega (|\boldsymbol{\tau}|^{r'} + |div\,\boldsymbol{\tau}|^{r'})\, d\Omega\right)^{1/r'}$. Let $U := (L^r(\Omega))^n$, and $P := L^{r'}(\Omega)$.

For a Banach space X, X^* denotes its dual space with associated norm $\|\cdot\|_{X^*}$. Note that $T^* = T'$, and $\left(T'\right)^* = T$. The norm and seminorm associated with the Sobolev space $W^{m,r}(\Omega)$ will be denoted by $\|\cdot\|_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$, respectively, and the infinity norm will be denoted by $\|\cdot\|_\infty$.

Motivated by (1.4),(1.5),(1.6), we will assume that the extra stress tensor is a function of the velocity gradient, i.e.

$$\boldsymbol{\sigma} := \mathbf{g}(\nabla \mathbf{u}) = \nu(|\nabla \mathbf{u}|)\nabla \mathbf{u}\,. \tag{2.1}$$

Specifically, we assume

A1: $\quad \mathbf{g} : T \to T^*$ is a bounded, continuous, strictly monotone operator [24];

and that there exist constants $\hat{C}_1$ and $\hat{C}_2$ such that, for $\mathbf{s}, \mathbf{t}, \mathbf{w} \in T$,

$$\textbf{A2:}\; \int_\Omega (\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})) : (\mathbf{s} - \mathbf{t})\, d\Omega \geq \hat{C}_1 \left(\int_\Omega |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})||\mathbf{s} - \mathbf{t}|\, d\Omega + \frac{\|\mathbf{s} - \mathbf{t}\|_T^2}{\|\mathbf{s}\|_T^{2-r} + \|\mathbf{t}\|_T^{2-r}} \right), \tag{2.2}$$

$$\textbf{A3:}\; \int_\Omega (\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})) : \mathbf{w}\, d\Omega \leq \hat{C}_2 \left\| \frac{|\mathbf{s} - \mathbf{t}|}{|\mathbf{s}| + |\mathbf{t}|} \right\|_\infty^{\frac{2-r}{r}} \left(\int_\Omega |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})||\mathbf{s} - \mathbf{t}|\, d\Omega \right)^{1/r'} \|\mathbf{w}\|_T\,, \tag{2.3}$$

with the convention that $\mathbf{g}(\mathbf{s}) = \mathbf{0}$ if $\mathbf{s} = \mathbf{0}$ and $|\mathbf{s}(\mathbf{x}) - \mathbf{t}(\mathbf{x})|/(|\mathbf{s}(\mathbf{x})| + |\mathbf{t}(\mathbf{x})|) = 0$ if $\mathbf{s}(\mathbf{x}) = \mathbf{t}(\mathbf{x}) = \mathbf{0}$. Properties **A1**–**A3** have been established for power law and Carreau law fluids [3]. (For the case of a power law fluid monotonicity is also shown in [26, 7].) For Ladyzhenskaya law fluids, the analysis in [26] is easily extended to show that **A1**–**A3** hold.

Remark 2.1 *From (1.2) it follows that* $\mathbf{u}_\Gamma$ *must satisfy the compatibility condition*

$$\int_\Gamma \mathbf{u}_\Gamma \cdot \mathbf{n}\, d\Gamma \;=\; 0\,,$$

where $\mathbf{n}$ *denotes the outward pointing unit normal vector to* Ω*.*

In order to obtain the dual-mixed formulation, introduce two new variables, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

$$\begin{aligned}
\boldsymbol{\phi} \;&:=\; \nabla \mathbf{u}\,, && (2.4)\\
\boldsymbol{\psi} \;&:=\; \boldsymbol{\sigma} \,-\, p\mathbf{I}\,, \text{ the total stress tensor,} && (2.5)\\
&=\; \mathbf{g}(\boldsymbol{\phi}) \,-\, p\mathbf{I}\,, \text{ using (2.1)}\,. && (2.6)
\end{aligned}$$

With the definition of $\boldsymbol{\psi}$ a variational form for (1.1) can be written as

$$-\int_\Omega \mathbf{v} \cdot div\, \boldsymbol{\psi}\, d\Omega \;=\; \int_\Omega \mathbf{v} \cdot \mathbf{f}\, d\Omega\,, \text{ for } \mathbf{v} \in T\,. \tag{2.7}$$

Note that from the definition of $\boldsymbol{\phi}$ we have that, for sufficiently smooth functions,

$$\begin{aligned}
0 \;&=\; -\int_\Omega \boldsymbol{\phi} : \boldsymbol{\tau}\, d\Omega \;+\; \int_\Omega \nabla \mathbf{u} : \boldsymbol{\tau}\, d\Omega\\
&=\; -\int_\Omega \boldsymbol{\phi} : \boldsymbol{\tau}\, d\Omega \;+\; \int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma \;-\; \int_\Omega \mathbf{u} \cdot div\, \boldsymbol{\tau}\, d\Omega && (2.8)
\end{aligned}$$

where the integral over Γ is the duality pairing of $(W^{-1/r',r'}(\Gamma))^n$ and $(W^{1-1/r,r}(\Gamma))^n$ with respect to the $(L^2(\Omega))^n$ inner product. The incompressibility condition $div\, \mathbf{u} \;=\; 0$ is equivalent to

$$tr(\boldsymbol{\phi}) \;=\; 0\,, \tag{2.9}$$

where we use $tr(\boldsymbol{\phi})$ to denote the *trace* of $\boldsymbol{\phi}$.

Combining (1.4), (2.8), and (2.7) a variational formulation to (1.4), (2.8), and (2.7) is: *Given* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^n$, $\mathbf{u}_\Gamma \in \left(W^{1-1/r\,,\,r}(\Gamma)\right)^n$, *determine* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}) \in T \times T'_{div} \times P \times U$ *such that*

$$\begin{aligned}
\int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma}\, d\Omega \,-\, \int_\Omega \boldsymbol{\psi} : \boldsymbol{\varsigma}\, d\Omega \,-\, \int_\Omega p\, tr(\boldsymbol{\varsigma})\, d\Omega \;&=\; 0\,, \forall \boldsymbol{\varsigma} \in T\,, && (2.10)\\
-\int_\Omega \boldsymbol{\tau} : \boldsymbol{\phi}\, d\Omega \,-\, \int_\Omega q\, tr(\boldsymbol{\phi})\, d\Omega \,-\, \int_\Omega \mathbf{u} \cdot div\, \boldsymbol{\tau}\, d\Omega \;&=\; -\int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma\,,\\
&\qquad \forall (\boldsymbol{\tau}, q) \in T'_{div} \times P\,, && (2.11)\\
-\int_\Omega \mathbf{v} \cdot div\, \boldsymbol{\psi}\, d\Omega \;&=\; \int_\Omega \mathbf{v} \cdot \mathbf{f}\, d\Omega\,, \forall \mathbf{v} \in U\,. && (2.12)
\end{aligned}$$

Note that equations (2.10)-(2.12) do not uniquely define a solution; as adding $(0, c\mathbf{I}, -c, \mathbf{0})$ to a solution $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u})$, also satisfies (2.10)-(2.12) for any $c \in \mathbb{R}$. In order to guarantee uniqueness we proceed as in [2, 6, 15] and impose, via a Lagrange multiplier, the constraint $\int_\Omega tr(\boldsymbol{\psi})\, d\Omega \;=\; 0$.

The variational formulation may then be restated as: *Given* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^n$, $\mathbf{u}_\Gamma \in \left(W^{1-1/r,r}(\Gamma)\right)^n$, *determine* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T^{'}_{div} \times P \times U \times \mathbb{R}$ *such that*

$$\int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma} \, d\Omega \; - \; \int_\Omega \boldsymbol{\psi} : \boldsymbol{\varsigma} \, d\Omega \; - \; \int_\Omega p \, tr(\boldsymbol{\varsigma}) \, d\Omega \; = \; 0 \; , \forall \boldsymbol{\varsigma} \in T \, , \tag{2.13}$$

$$-\int_\Omega \boldsymbol{\tau} \, : \boldsymbol{\phi} \, d\Omega \; - \; \int_\Omega q \, tr(\boldsymbol{\phi}) \, d\Omega \; - \; \int_\Omega \mathbf{u} \cdot \, div \, \boldsymbol{\tau} \, d\Omega \; + \; \lambda \int_\Omega tr(\boldsymbol{\tau}) \, d\Omega$$

$$= - \int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \; , \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P \, , \tag{2.14}$$

$$-\int_\Omega \mathbf{v} \cdot \, div \, \boldsymbol{\psi} \, d\Omega \; + \; \eta \int_\Omega tr(\boldsymbol{\psi}) \, d\Omega \; = \; \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \, , \forall (\mathbf{v}, \eta) \in U \times \mathbb{R} \, . \tag{2.15}$$

Remark 2.2 *As commented in [15], the value of the Lagrange multiplier* λ *is* 0*, as can be seen from the choice of* $\boldsymbol{\tau} = \mathbf{I}$ *and* $q = -1$*. However, it is included in the variational formulation so that the formulation has a twofold saddle point structure.*

To formally rewrite (2.13)-(2.15) as a twofold saddle point problem define the following operators:

$$\mathbf{A} : T \longrightarrow T^{'} \, , \qquad \mathbf{B} : T \longrightarrow (T^{'}_{div} \times P)^* \, , \qquad \mathbf{C} : T^{'}_{div} \times P \longrightarrow (U \times \mathbb{R})^* \, .$$

$$[\mathbf{A}(\boldsymbol{\phi}) \, , \, \boldsymbol{\varsigma}] \; := \; \int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma} \, d\Omega \, , \tag{2.16}$$

$$[\mathbf{B}(\boldsymbol{\phi}) \, , \, (\boldsymbol{\tau}, q)] \; := \; -\int_\Omega \boldsymbol{\tau} \, : \boldsymbol{\phi} \, d\Omega \; - \; \int_\Omega q \, tr(\boldsymbol{\phi}) \, d\Omega \, , \tag{2.17}$$

$$[\mathbf{C}(\boldsymbol{\psi}, p) \, , \, (\mathbf{v}, \eta)] \; := \; -\int_\Omega \mathbf{v} \cdot \, div \, \boldsymbol{\psi} \, d\Omega \; + \; \eta \int_\Omega tr(\boldsymbol{\psi}) \, d\Omega \, . \tag{2.18}$$

The modeling equations can then be written in the form

$$[\mathbf{A}(\boldsymbol{\phi}) \, , \, \boldsymbol{\varsigma}] \, + \, [\boldsymbol{\varsigma} \, , \, \mathbf{B}^*(\boldsymbol{\psi}, p)] \; = \; 0 \; , \forall \boldsymbol{\varsigma} \in T \, , \tag{2.19}$$

$$[\mathbf{B}(\boldsymbol{\phi}) \, , \, (\boldsymbol{\tau}, q)] \, + \, [(\boldsymbol{\tau}, q) \, , \, \mathbf{C}^*(\mathbf{u}, \lambda)] \; = \; -\int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \; , \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P \, , \tag{2.20}$$

$$[\mathbf{C}(\boldsymbol{\psi}, p) \, , \, (\mathbf{v}, \eta)] \; = \; \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \, , \forall (\mathbf{v}, \eta) \in U \times \mathbb{R} \, , \tag{2.21}$$

where $\mathbf{B}^*$ and $\mathbf{C}^*$ denote the respective adjoint operators of $\mathbf{B}$ and $\mathbf{C}$, respectively.

3 Solvability of the Continuous Formulation

In this section we discuss the existence and uniqueness of a solution to (2.19)-(2.21). The proof of this result requires specific properties of the $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ operators (including suitable inf-sup conditions for $\mathbf{B}$ and $\mathbf{C}$), the general theory of saddle point problems, and monotone operator theory. We remark that direct applications of Hölder's inequality establishes that $[\mathbf{B}(\cdot), (\cdot, \cdot)] : T \times (T^{'}_{div} \times P) \to \mathbb{R}$ and $[\mathbf{C}(\cdot, \cdot), (\cdot, \cdot)] : (T^{'}_{div} \times P) \times (U \times \mathbb{R}) \to \mathbb{R}$ are bounded (componentwise) linear functionals.

Additionally, the assumptions **A1-A3** discussed in Section 2 imply that $\mathbf{A}(\boldsymbol{\phi})$ defines a bounded, continuous, strictly monotone operator on a reflexive Banach space. Before presenting the proof of solvability in Theorem 3.1, we present several technical lemmas that establish the appropriate inf-sup conditions for $\mathbf{B}$ and $\mathbf{C}$.

3.1 Inf-sup Condition for B

Define the null space for the operator $\mathbf{C}$, Z_1, as

$$\begin{aligned} Z_1 &:= \left\{(\boldsymbol{\tau}, q) \in T'_{div} \times P \,:\, [\mathbf{C}(\boldsymbol{\tau}, q)\,,\,(\mathbf{v}, \eta)] \,=\, 0\,,\, \forall (\mathbf{v}, \eta) \in U \times \mathbb{R}\right\}\,, \\ &= \left\{(\boldsymbol{\tau}, q) \in T'_{div} \times P \,:\, div\, \boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega, \text{ and } \int_\Omega tr(\boldsymbol{\tau})\, d\Omega = 0\right\}\,. \end{aligned} \tag{3.1}$$

Note that for $(\boldsymbol{\tau}, q) \in Z_1$, $\|\boldsymbol{\tau}\|_{T'_{div}} = \|\boldsymbol{\tau}\|_{T'}$. Helpful in establishing the inf-sup condition for $\mathbf{B}$ is the following lemma.

Lemma 3.1 *(See Lemma 3.1 in [2] for Hilbert space setting.) For $\boldsymbol{\tau} \in T'_{div}$ satisfying $\int_\Omega tr(\boldsymbol{\tau})\, d\Omega = 0$, let $\boldsymbol{\tau}^0 = \boldsymbol{\tau} - \frac{1}{n} tr(\boldsymbol{\tau})\mathbf{I}$. Then, there exists C, depending only Ω, such that*

$$\|\boldsymbol{\tau}\|_{L^{r'}} \leq C \left(\|\boldsymbol{\tau}^0\|_{L^{r'}} \,+\, \|div\, \boldsymbol{\tau}\|_{W^{-1,r'}}\right)\,. \tag{3.2}$$

Proof: Now, there exists a non-zero function $\varphi \in L^r(\Omega)$ such that

$$\|tr(\boldsymbol{\tau})\|_{L^{r'}(\Omega)}\, \|\varphi\|_{L^r(\Omega)} \;=\; \int_\Omega tr(\boldsymbol{\tau})\, \varphi\, d\Omega\,. \tag{3.3}$$

Since $\int_\Omega tr(\boldsymbol{\tau})\, d\Omega \;=\; 0$, we can assume $\int_\Omega \varphi\, d\Omega \;=\; 0$ (shift φ by its average). From [12], pg. 116, given $\varphi \in L^r(\Omega)$, $1 < r < \infty$ with $\int_\Omega \varphi\, d\Omega \;=\; 0$, then there exists $\mathbf{v} \in W_0^{1,r}(\Omega)$ and a constant C such that

$$div\, \mathbf{v} \;=\; \varphi \;\text{ in } \Omega \qquad \text{and} \qquad \|\mathbf{v}\|_{W^{1,r}(\Omega)} \;\leq\; C\|\varphi\|_{L^r(\Omega)}\,. \tag{3.4}$$

From (3.3) and (3.4),

$$\begin{aligned} \frac{1}{n\, C}\|tr(\boldsymbol{\tau})\|_{L^{r'}(\Omega)}\, \|\mathbf{v}\|_{W^{1,r}(\Omega)} &\leq \frac{1}{n}\int_\Omega tr(\boldsymbol{\tau})\, div\, \mathbf{v}\, d\Omega = \frac{1}{n}\int_\Omega tr(\boldsymbol{\tau})\, \mathbf{I} : \nabla \mathbf{v}\, d\Omega \\ &= \int_\Omega (\boldsymbol{\tau} - \boldsymbol{\tau}^0) : \nabla \mathbf{v}\, d\Omega \quad \text{(using the defn. of } \boldsymbol{\tau}^0) \\ &= -\int_\Omega (\boldsymbol{\tau}^0 : \nabla \mathbf{v} \,+\, div\, \boldsymbol{\tau} \cdot \mathbf{v})\, d\Omega \\ &\leq \left(\|\boldsymbol{\tau}^0\|_{L^{r'}(\Omega)} \,+\, \|div\, \boldsymbol{\tau}\|_{W^{-1,r'}(\Omega)}\right) \|\mathbf{v}\|_{W^{1,r}(\Omega)}\,. \end{aligned}$$

■

Lemma 3.2 *There exists a constant $c_1 > 0$ such that*

$$\inf_{(\boldsymbol{\tau}, q) \in Z_1} \sup_{\boldsymbol{\phi} \in T} \frac{[\mathbf{B}(\boldsymbol{\phi})\,,\,(\boldsymbol{\tau}, q)]}{\|\boldsymbol{\phi}\|_T\, \|(\boldsymbol{\tau}, q)\|_{T'_{div} \times P}} \;\geq\; c_1\,. \tag{3.5}$$

Proof: The inf-sup condition is established using the approach in [15] (and the references therein) for the Hilbert space case, in which two cases are considered and suitable choices of trial functions are constructed to form a lower bound on the supremum. We briefly illustrate the adjustments to the general Sobolev case and refer the reader to [9] for the complete proof.

Case 1.: $\|q\|_P \leq \|\boldsymbol{\tau}\|_{T'_{div}}$.

Let

$$\boldsymbol{\tau}^0 = \boldsymbol{\tau} - \frac{1}{n} tr(\boldsymbol{\tau})\mathbf{I}, \quad \text{and} \quad \boldsymbol{\phi} = -|\boldsymbol{\tau}^0|^{r'/r-1}\,\boldsymbol{\tau}^0/\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}. \tag{3.6}$$

Note that $\boldsymbol{\phi} \in T$, and $\|\boldsymbol{\phi}\|_T = 1$. Then, using Lemma 3.1 and the fact that $tr(\boldsymbol{\tau}^0) = 0$, there exists a constant $C > 0$ such that

$$\frac{[B(\boldsymbol{\phi}), (\boldsymbol{\tau}, q)]}{\|\boldsymbol{\phi}\|_T} \geq C\|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P} \tag{3.7}$$

for $(\boldsymbol{\tau}, q) \in Z_1$.

Case 2.: $\|q\|_P \geq \|\boldsymbol{\tau}\|_{T'_{div}}$.

Let

$$\boldsymbol{\phi} \;=\; \frac{-|q\mathbf{I} \,+\, \boldsymbol{\tau}|^{r'/r-1}\,(q\mathbf{I} \,+\, \boldsymbol{\tau})}{\|q\mathbf{I} \,+\, \boldsymbol{\tau}\|_{T'}^{r'-1}}. \tag{3.8}$$

Again, $\boldsymbol{\phi} \in T$, and $\|\boldsymbol{\phi}\|_T = 1$. This choice of $\boldsymbol{\phi}$ implies that there exists a $C > 0$ such that, for $(\boldsymbol{\tau}, q) \in Z_1$,

$$\frac{[B(\boldsymbol{\phi}), (\boldsymbol{\tau}, q)]}{\|\boldsymbol{\phi}\|_T} \geq C\|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P}.$$

■

3.2 Inf-sup Condition for C

The following lemma is an extension of Lemma 2.1 of [15] to the general Sobolev case and is helpful in establishing the inf-sup condition for **C**.

Lemma 3.3 *Let* ${}_0T'_{div} \;:=\; \left\{\boldsymbol{\tau} \in T'_{div} \;:\; \int_\Omega tr(\boldsymbol{\tau})\, d\Omega \;=\; 0\right\}$. *Then, there exists $C > 0$ such that for any $\mathbf{u} \in U$*

$$\sup_{\substack{\hat{\boldsymbol{\tau}} \in\ {}_0T'_{div} \\ \hat{\boldsymbol{\tau}} \neq 0}} \frac{\int_\Omega \mathbf{u} \cdot div\hat{\boldsymbol{\tau}}\, d\Omega}{\|\hat{\boldsymbol{\tau}}\|_{T'_{div}}} \;\geq\; C \sup_{\substack{\boldsymbol{\tau} \in T'_{div} \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{u} \cdot div\boldsymbol{\tau}\, d\Omega}{\|\boldsymbol{\tau}\|_{T'_{div}}}. \tag{3.9}$$

Proof: For $\boldsymbol{\tau} \in T'_{div}$, let $\boldsymbol{\tau}_0 \;=\; \boldsymbol{\tau} \;-\; \frac{1}{n|\Omega|}\left(\int_\Omega tr(\boldsymbol{\tau}))\, d\Omega\right)\mathbf{I}$. Then, $\boldsymbol{\tau}_0 \in\ {}_0T'_{div}$, and $div\boldsymbol{\tau} \;=\; div\boldsymbol{\tau}_0$. Let

$$\varsigma \;:=\; |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0 \;+\; \frac{sgn\left((\int_\Omega tr(\boldsymbol{\tau})\, d\Omega)\,(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\, d\Omega)\right)}{n\,|\Omega|}\left(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\, d\Omega\right)\mathbf{I}.$$

Note that as

$$\||\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\|_{L^r} \;=\; \left(\int_\Omega |\boldsymbol{\tau}_0|^{r'}\, d\Omega\right)^{1/r} \;=\; \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r},$$

and

$$\left|\int_{\Omega} |\boldsymbol{\tau}_0|^{r'/r\,-1} tr(\boldsymbol{\tau}_0)\, d\Omega\right| \le \sqrt{n} \int_{\Omega} |\boldsymbol{\tau}_0|^{r'/r}\, 1\, d\Omega \le C\, \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r} .$$

Thus

$$\|\boldsymbol{\varsigma}\|_{L^r} \;\le\; C \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r} . \tag{3.10}$$

We have that

$$\|\boldsymbol{\tau}\|_{L^{r'}} \;=\; \sup_{\boldsymbol{\sigma}\in L^r} \frac{(\boldsymbol{\tau}\,,\,\boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{L^r}} . \tag{3.11}$$

Now, using $\boldsymbol{\tau}_0 \in \ {}_0T^{'}_{div}$,

$$\begin{aligned}
(\boldsymbol{\tau}\,,\,\boldsymbol{\varsigma}) \;&=\; \int_{\Omega} |\boldsymbol{\tau}_0|^{r'}\, d\Omega \,+\, \frac{1}{n|\Omega|}\left(\int_{\Omega} tr(\boldsymbol{\tau})\, d\Omega\right)\left(\int_{\Omega} |\boldsymbol{\tau}_0|^{r'/r\,-1}\, tr(\boldsymbol{\tau}_0)\, d\Omega\right) \\
&\quad + \left|\frac{1}{n|\Omega|}\left(\int_{\Omega} tr(\boldsymbol{\tau})\, d\Omega\right)\left(\int_{\Omega} |\boldsymbol{\tau}_0|^{r'/r\,-1}\, tr(\boldsymbol{\tau}_0)\, d\Omega\right)\right| \\
&\ge\; \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'} .
\end{aligned} \tag{3.12}$$

Therefore, from (3.10), (3.11), and (3.12) we have that $\|\boldsymbol{\tau}\|_{L^{r'}} \;\ge\; C\, \|\boldsymbol{\tau}_0\|_{L^{r'}}$. Combining the above we obtain

$$\frac{\int_{\Omega} \mathbf{u}\cdot div\boldsymbol{\tau}\, d\Omega}{\|\boldsymbol{\tau}\|_{T^{'}_{div}}} \;=\; \frac{\int_{\Omega} \mathbf{u}\cdot div\boldsymbol{\tau}_0\, d\Omega}{\|\boldsymbol{\tau}\|_{T^{'}_{div}}} \;\le\; C\frac{\int_{\Omega} \mathbf{u}\cdot div\boldsymbol{\tau}_0\, d\Omega}{\|\boldsymbol{\tau}_0\|_{T^{'}_{div}}} ,$$

from which (3.9) then follows.

■

Lemma 3.4 *There exists a constant* $c_2 > 0$ *such that*

$$\inf_{(\mathbf{u},\lambda)\in U\times\mathbb{R}} \;\sup_{(\boldsymbol{\tau},q)\in T^{'}_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T^{'}_{div}\times P}\, \|(\mathbf{u},\lambda)\|_{U\times\mathbb{R}}} \ge\; c_2 . \tag{3.13}$$

Proof: As in the case of Lemma 3.2, the structure of the proof mirrors that in [15] and considers two cases:
<u>Case 1.</u>: $|\lambda| \;\ge\; \|\mathbf{u}\|_U$.
For this case we have

$$\sup_{(\boldsymbol{\tau},q)\in T^{'}_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T^{'}_{div}\times P}} \ge \frac{[\mathbf{C}(\lambda\mathbf{I},0)\,,\,(\mathbf{u},\lambda)]}{\|\lambda\mathbf{I}\|_{T^{'}_{div}}} \;=\; \frac{n\,\lambda^2\,|\Omega|}{|\lambda| n^{r'/2}\,|\Omega|^{1/r'}} \ge C\|(\mathbf{u},\lambda)\|_{U\times\mathbb{R}} . \tag{3.14}$$

<u>Case 2.</u>: $|\lambda| \;\le\; \|\mathbf{u}\|_U$.
Using Lemma 3.3,

$$\sup_{(\boldsymbol{\tau},q)\in T^{'}_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T^{'}_{div}\times P}} \ge \sup_{\boldsymbol{\tau}_0\in\ {}_0T^{'}_{div}} \frac{-\int_{\Omega} \mathbf{u}\cdot div\boldsymbol{\tau}_0\, d\Omega}{\|\boldsymbol{\tau}_0\|_{T^{'}_{div}}} \ge C \sup_{\boldsymbol{\tau}\in T^{'}_{div}} \frac{-\int_{\Omega} \mathbf{u}\cdot div\boldsymbol{\tau}\, d\Omega}{\|\boldsymbol{\tau}\|_{T^{'}_{div}}} . \tag{3.15}$$

Choose $\mathbf{w} \in (L^{r'}(\Omega))^n$ such that $\|\mathbf{u}\|_{L^r}\, \|\mathbf{w}\|_{L^{r'}} \;=\; \int_{\Omega} \mathbf{u}\cdot\mathbf{w}\, d\Omega$, and let $\boldsymbol{\tau}$ satisfy $div\,\boldsymbol{\tau} = \mathbf{w}$ in Ω with

$$\|\boldsymbol{\tau}\|_{T^{'}_{div}} \;\le\; C\, \|\mathbf{w}\|_{L^{r'}} ,$$

(see [12] pg. 116). Then,

$$\sup_{(\boldsymbol{\tau},q)\in T'_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}} \;\geq\; C\,\frac{-\int_\Omega \mathbf{u}\cdot div(-\boldsymbol{\tau})\,d\Omega}{\|-\boldsymbol{\tau}\|_{T'_{div}}} \;\geq\; C\,\frac{\int_\Omega \mathbf{u}\cdot\mathbf{w}\,d\Omega}{\|\mathbf{w}\|_{L^{r'}}}$$
$$\geq\; C\|\mathbf{u}\|_U \;\geq\; C\|(\mathbf{u},\lambda)\|_{U\times\mathbb{R}}\,. \qquad (3.16)$$

■

3.3 Existence, Uniqueness, and A Priori Estimates

Before proceeding to the proof of existence and uniqueness, we state two known results that will be utilized:

Lemma 3.5 *([17], Remark 4.2, pg. 61) Let $(X,\|\cdot\|_X)$ and $(M,\|\cdot\|_M)$ be two reflexive Banach spaces. Let $(X',\|\cdot\|_{X'})$ and $(M',\|\cdot\|_{M'})$ be their corresponding dual spaces. Let $B : X \to M'$ be a linear continuous operator and $B' : M'' \to X$ the dual operator of B. Let $V = ker(B)$ be the kernel of B; we denote by $V^o \subset X'$ the polar set of $V : V^o = \{x' \in X'\,,[x',v] = 0\,,\forall v \in V\}$ and $\dot{B} : (X/V) \to M'$ the quotient operator associated with B. The following three properties are equivalent:*
(i) $\exists\beta > 0$, such that

$$\inf_{q\in M}\sup_{v\in X}\frac{[Bv,q]}{\|q\|_M\,\|v\|_X} \geq \beta\,,$$

(ii) B' is an isomorphism from M'' onto V^o and

$$\|B'q\| \geq \beta\|q\|_{M''}\;\forall q \in M''\,,$$

(iii) $\dot{B}$ is an isomorphism from (X/V) onto M' and

$$\|\dot{B}\dot{v}\| \geq \beta\|\dot{v}\|_{(X/V)}\;\forall \dot{v} \in (X/V)\,.$$

■

Lemma 3.6 *([24], Theorem 9.45, pg. 361, **Browder-Minty**) Let X be a real, reflexive Banach space and let $T : X \to X'$ be bounded, continuous, coercive and monotone. Then for any $g \in X'$ there exists a solution u of the equation $T(u) = g$; i.e., $T(X) = X'$.* ■

The main result of this section is now presented.

Theorem 3.1 *There exists a unique solution $(\boldsymbol{\phi},\boldsymbol{\psi},p,\mathbf{u},\lambda) \in T \times T'_{div} \times P \times U \times \mathbb{R}$ satisfying (2.19)–(2.21).*

Proof: Following the approach in [10], from Lemmas 3.4 and 3.5 (i) and (iii), with the associations $X = T'_{div} \times P$, $M = U \times \mathbb{R}$, $B : X \to M'$ defined by $B((\boldsymbol{\tau},q)) := [\mathbf{C}(\boldsymbol{\tau},q),(\cdot,\cdot)]$, $V = ker\,B = Z_1$, we have that there exists $(\dot{\boldsymbol{\psi}},\dot{p}) \in (T'_{div} \times P)/Z_1$ such that

$$[\mathbf{C}(\dot{\boldsymbol{\psi}},\dot{p}),(\mathbf{v},\eta)] = \int_\Omega \mathbf{v}\cdot\mathbf{f}\,d\Omega, \qquad \forall\,(\mathbf{v},\eta) \in U \times \mathbb{R}$$

with $\|(\dot{\boldsymbol{\psi}}, \dot{p})\|_{(T'_{div}\times P)/Z_1} \leq (1/c_2)\|\mathbf{f}\|_{0,r'}$. As the cosets in $(T'_{div} \times P)/Z_1$ are closed, we can choose $(\boldsymbol{\psi}_0, p_0) \in (\dot{\boldsymbol{\psi}}, \dot{p})$ such that

$$\|\boldsymbol{\psi}_0\|_{T'_{div}} + \|p_0\|_P = \|(\boldsymbol{\psi}_0, p_0)\|_{T'_{div}\times P} = \|(\dot{\boldsymbol{\psi}}, \dot{p})\|_{(T'_{div}\times P)/Z_1} \leq (1/c_2)\|\mathbf{f}\|_{0,r'} . \tag{3.17}$$

Let $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_0$ and $p = \tilde{p} + p_0$. Then solving (2.19)–(2.21) is equivalent to: find $(\boldsymbol{\phi}, \tilde{\boldsymbol{\psi}}, \tilde{p}) \in T \times Z_1$ such that

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\phi})\,,\,\boldsymbol{\varsigma}] + [\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\tilde{\boldsymbol{\psi}}, \tilde{p})] &= -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0, p_0)]\,, \qquad \forall \boldsymbol{\varsigma} \in T\,, & (3.18)\\ [\mathbf{B}(\boldsymbol{\phi})\,,\,(\boldsymbol{\tau}, q)] &= -\int_\Gamma (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\, d\Gamma\,, \qquad \forall (\boldsymbol{\tau}, q) \in Z_1\,. & (3.19)\end{aligned}$$

Introduce a subspace of T defined by

$$Z_2 \;:=\; \{\boldsymbol{\varsigma} \in T\,:\,[\boldsymbol{\varsigma}, \mathbf{B}^*(\boldsymbol{\tau}, q)] \,=\, 0\,, \forall (\boldsymbol{\tau}, q) \in Z_1\} = \{\boldsymbol{\varsigma} \in T\,:\,[\mathbf{B}(\boldsymbol{\varsigma}), (\boldsymbol{\tau}, q)] \,=\, 0\,, \forall (\boldsymbol{\tau}, q) \in Z_1\}\,.$$

Now, from Lemmas 3.2 and 3.5 (i) and (iii), through the same argument as above now with associations $X = T$, $M = Z_1$, $B : X \to M'$ defined by $B(\boldsymbol{\varsigma}) := [\mathbf{B}(\boldsymbol{\varsigma}), (\cdot, \cdot)]$, $V = ker\, B = Z_2$, there is a $\boldsymbol{\phi}_0 \in T$ such that

$$[\mathbf{B}(\boldsymbol{\phi}_0)\,,\,(\boldsymbol{\tau}, q)] \;=\; -\int_\Gamma (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\, d\Gamma\;, \forall (\boldsymbol{\tau}, q) \in Z_1\,,$$

with

$$\|\boldsymbol{\phi}_0\|_T \leq \frac{1}{c_1}\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma}\,. \tag{3.20}$$

Then, solving (3.18)–(3.19) is equivalent to: find $\tilde{\boldsymbol{\phi}} \in Z_2$ such that

$$[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\,,\,\boldsymbol{\varsigma}] \;=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0, p_0)]\,, \quad \forall \boldsymbol{\varsigma} \in Z_2\,. \tag{3.21}$$

Lemma 3.6 and the assumptions **A1**–**A3** guarantee the existence of a $\tilde{\boldsymbol{\phi}}$ satisfying (3.21). Uniqueness of $\tilde{\boldsymbol{\phi}}$ is implied by assumption **A2**, and this uniquely determines $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0$. Thus, Lemma 3.2 and (2.19) imply there exist unique $(\tilde{\boldsymbol{\psi}}, \tilde{p}) \in Z_1$ that satisfies

$$[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\tilde{\boldsymbol{\psi}}, \tilde{p})] \;=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0, p_0)] - [\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\,,\,\boldsymbol{\varsigma}]\;, \quad \forall \boldsymbol{\varsigma} \in T\,. \tag{3.22}$$

This uniquely determines $\boldsymbol{\psi}$ and p. Then Lemma 3.4 and (2.20) imply there exists a unique $(\mathbf{u}, \lambda) \in U \times \mathbb{R}$ such that

$$[(\boldsymbol{\tau}, q)\,,\,\mathbf{C}^*(\mathbf{u}, \lambda)] \;=\; -\int_\Gamma (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\, d\Gamma - [\mathbf{B}(\boldsymbol{\phi})\,,\,(\boldsymbol{\tau}, q)]\,, \quad \forall (\boldsymbol{\tau}, q) \in T'_{div} \times P\,, \tag{3.23}$$

which completes the proof. ■

Corollary 3.1 *The solution $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T'_{div} \times P \times U \times \mathbb{R}$ to (2.19)–(2.21) satisfies*

$$\begin{aligned} \|\boldsymbol{\phi}\|_T + \|\mathbf{u}\|_U + |\lambda| &\leq C\left(\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma} + \|\mathbf{f}\|_{0,r',\Omega}^{r'/r}\right)\,, & (3.24)\\ \|\boldsymbol{\psi}\|_{T'_{div}} + \|p\|_P &\leq C\left(\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma}^{1/r'} + \|\mathbf{f}\|_{0,r',\Omega} + \|\mathbf{f}\|_{0,r',\Omega}^{1/r}\right)\,, & (3.25)\end{aligned}$$

for some constant $C > 0$.

Proof: Let $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \tilde{\boldsymbol{\phi}}, \boldsymbol{\psi} = \boldsymbol{\psi}_0 + \tilde{\boldsymbol{\psi}}, p = p_0 + \tilde{p}, \mathbf{u}$, and λ be as in the proof of Theorem 3.1. From (2.2) we have that

$$\hat{C}_1 \left(\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0| \, d\Omega \right) \leq \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : (\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) \, d\Omega \,. \tag{3.26}$$

Using (2.16) and (3.26) with $\boldsymbol{\varsigma} = \tilde{\boldsymbol{\phi}}$ we have

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\phi}), \tilde{\boldsymbol{\phi}}] &= \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : \tilde{\boldsymbol{\phi}} \, d\Omega = \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : (\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) \, d\Omega + \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : \boldsymbol{\phi}_0 \, d\Omega \\
&\geq \hat{C}_1 \left(\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0| \, d\Omega \right) \\
&\quad - \hat{C}_2 \left(\int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0| \, d\Omega \right)^{1/r'} \|\boldsymbol{\phi}_0\|_T \\
&\geq \hat{C}_1 \|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \left(\hat{C}_1 - \frac{\epsilon_1 \hat{C}_2}{r'} \right) \int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0| \, d\Omega - \frac{\hat{C}_2}{r\epsilon_1} \|\boldsymbol{\phi}_0\|_T^r \,.
\end{aligned} \tag{3.27}$$

Now we also have from (3.21), using Young's inequality and the triangle inequality,

$$\begin{aligned}
[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0), \tilde{\boldsymbol{\phi}}] &= -[\mathbf{B}(\tilde{\boldsymbol{\phi}}), (\boldsymbol{\psi}_0, p_0)] = \int_\Omega \boldsymbol{\psi}_0 : \tilde{\boldsymbol{\phi}} \, d\Omega + \int_\Omega p_0 \, tr(\tilde{\boldsymbol{\phi}}) \, d\Omega \\
&\leq \|\boldsymbol{\psi}_0\|_{T'} \|\tilde{\boldsymbol{\phi}}\|_T + \sqrt{n} \, \|p_0\|_P \|\tilde{\boldsymbol{\phi}}\|_T \\
&\leq \frac{2\epsilon_2}{r} \|\tilde{\boldsymbol{\phi}}\|_T^r + \frac{1}{r'\epsilon_2} \left(\|\boldsymbol{\psi}_0\|_{T'}^{r'} + \sqrt{n} \, \|p_0\|_P^{r'} \right) \\
&\leq \frac{2\epsilon_2}{r} \left(\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \|\boldsymbol{\phi}_0\|_T^r \right) + \frac{1}{r'\epsilon_2} \left(\|\boldsymbol{\psi}_0\|_{T'}^{r'} + \sqrt{n} \, \|p_0\|_P^{r'} \right) .
\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28), we have

$$\begin{aligned}
&\left(\hat{C}_1 - \frac{2\epsilon_2}{r} \right) \|\boldsymbol{\phi}\|_T^r + \left(\hat{C}_1 - \frac{\epsilon_1 \hat{C}_2}{r'} \right) \int_\Omega |\mathbf{g}(\boldsymbol{\phi})||\boldsymbol{\phi}| \, d\Omega \\
&\qquad \leq \left(\frac{\hat{C}_2}{r\epsilon_1} + \frac{2\epsilon_2}{r} \right) \|\boldsymbol{\phi}_0\|_T^r + \frac{1}{r'\epsilon_2} \left(\|\boldsymbol{\psi}_0\|_{T'}^{r'} + \sqrt{n} \, \|p_0\|_P^{r'} \right) .
\end{aligned} \tag{3.29}$$

Together with (3.17), (3.20) and choices for ϵ_1, ϵ_2 that ensure

$$\hat{C}_1 - \frac{2\epsilon_2}{r} > 0 \,, \quad \text{and} \quad \hat{C}_1 - \frac{\epsilon_1 \hat{C}_2}{r'} > 0 \,,$$

we have

$$\|\boldsymbol{\phi}\|_T + \int_\Omega |\mathbf{g}(\boldsymbol{\phi})||\boldsymbol{\phi}| \, d\Omega \leq C \left(\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma} + \|\mathbf{f}\|_{0,r',\Omega}^{r'/r} \right) , \tag{3.30}$$

for some $C > 0$. From **A3**, (3.22), Lemma 3.2 and Lemma 3.5 (i) and (ii), we have that

$$\|\tilde{\boldsymbol{\psi}}\|_{T'_{div}} + \|\tilde{p}\|_P = \|(\tilde{\boldsymbol{\psi}}, \tilde{p})\|_{T'_{div} \times P} \leq C \left(\|\boldsymbol{\psi}_0\|_{T'} + \|p_0\|_P + \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi})||\boldsymbol{\phi}| \, d\Omega \right)^{1/r'} \right) . \tag{3.31}$$

Combining (3.17), (3.30), and (3.31) we obtain (3.25). Finally, (3.23), Lemma 3.4 and Lemma 3.5 (i) and (ii) complete the estimate (3.24) by bounding $\|(\mathbf{u}, \lambda)\|_{U \times \mathbb{R}}$ with $\|\boldsymbol{\phi}\|_T$ and $\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma}$. ■

4 Finite Element Approximation

Let $\Omega \subset \mathbb{R}^n$ be a polygonal domain and let $\mathcal{T}_h$ be a triangulation of Ω into triangles ($n = 2$) or tetrahedrals ($n = 3$). Thus

$$\Omega = \cup K\,, \quad K \in \mathcal{T}_h\,,$$

and assume that there exist constants γ_1, γ_2 such that

$$\gamma_1 h \leq h_K \leq \gamma_2 \rho_K \tag{4.1}$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K\in\mathcal{T}_h} h_K$. Define the finite-dimensional subspaces $T_h \subseteq T$, $T'_{div,\,h} \subseteq T'_{div}$, $P_h \subseteq P$, and $U_h \subseteq U$. Then the discrete formulation of (2.13)-(2.15) is defined as:

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\phi}_h)\,,\,\boldsymbol{\varsigma}_h]\,+\,[\boldsymbol{\varsigma}_h\,,\,\mathbf{B}^*(\boldsymbol{\psi}_h, p_h)] \;&=\; 0\;, \forall \boldsymbol{\varsigma}_h \in T_h\,, & (4.2)\\
[\mathbf{B}(\boldsymbol{\phi}_h), (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h), \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \;&=\; -\int_\Gamma (\boldsymbol{\tau}_h\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\, d\Gamma\;, &\\
&\qquad\qquad \forall(\boldsymbol{\tau}_h, q_h) \in T'_{div,h}\times P_h\,, & (4.3)\\
[\mathbf{C}(\boldsymbol{\psi}_h, p_h)\,,\,(\mathbf{v}_h, \eta_h)] \;&=\; \int_\Omega \mathbf{v}_h\cdot\mathbf{f}\, d\Omega\,, \forall(\mathbf{v}_h, \eta_h) \in U_h\times\mathbb{R}\,. & (4.4)
\end{aligned}$$

The corresponding discrete kernels of $\mathbf{B}$ and $\mathbf{C}$ are defined similarly. We have

$$Z_{1h} \;:=\; \Big\{(\boldsymbol{\tau}_h, q_h) \in T'_{div,h}\times P_h\;:\;[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)] \,=\, 0\,, \forall(\mathbf{v}_h, \eta_h) \in U_h\times\mathbb{R}\Big\}\,,$$

and

$$Z_{2h} \;:=\; \{\boldsymbol{\varsigma}_h \in T_h\;:\;[\mathbf{B}(\boldsymbol{\varsigma}_h),\,(\boldsymbol{\tau}_h, q_h)] \,=\, 0\,, \forall(\boldsymbol{\tau}_h, q_h) \in Z_{1h}\}\,.$$

4.1 Existence, Uniqueness, and A Priori Estimates

Theorem 4.1 *Let $\mathbf{g}$ satisfy (2.2) and (2.3). Let $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T\times T'_{div}\times P\times U\times\mathbb{R}$ solve (2.13)-(2.15). Assume that*
(1) There exists a positive constant c_1 such that

$$\inf_{(\boldsymbol{\tau}_h, q_h)\in Z_{1h}} \sup_{\boldsymbol{\varsigma}_h\in T_h} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h)\,,\,(\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\varsigma}_h\|_T\,\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \;\geq\; c_1\;. \tag{4.5}$$

(2) There exists a positive constant c_2 such that

$$\inf_{(\mathbf{u}_h,\lambda_h)\in U_h\times\mathbb{R}} \sup_{(\boldsymbol{\tau}_h, q_h)\in T'_{div,h}\times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{u}_h, \lambda_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}\,\|(\mathbf{u}_h, \lambda_h)\|_{U\times\mathbb{R}}} \;\geq\; c_2\,. \tag{4.6}$$

Then, for $\mathbf{f} \in \Big(L^{r'}(\Omega)\Big)^n$ and $\mathbf{u}_\Gamma \in \big(W^{1-1/r\,,\,r}(\Gamma)\big)^n$, there exists a unique solution $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h\times T'_{div,\,h}\times P_h\times U_h\times\mathbb{R}$ to the problem (4.2)-(4.4).

Proof: With the assumptions as stated above, existence and uniqueness of $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T'_{div,\,h} \times P_h \times U_h \times \mathbb{R}$ solving (4.2)-(4.4) follows directly from the continuous solution approach outlined in Section 3 and summarized in Theorem 3.1. ■

It should be noted that the stability estimates shown in Corollary 3.1 carry over to the discrete case as well. We now give the abstract a priori error estimate.

Theorem 4.2 *Let*

$$\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) = \left\| \frac{|\boldsymbol{\phi} - \boldsymbol{\phi}_h|}{|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|} \right\|_\infty^{(2-r)/r} . \tag{4.7}$$

Assume the hypotheses of Theorem 4.1 are satisfied. Also assume that for h sufficiently small, there is a constant $c_3 > 0$ such that

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h} \sup_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h) \in T_h \times U_h \times \mathbb{R}} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}\, \|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \geq\ c_3 \,. \tag{4.8}$$

where $\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}} = \|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + \|\eta_h\|_{\mathbb{R}}$. Then

$$\begin{aligned} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 &+ \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \\ &\leq C \Big\{ \inf_{\boldsymbol{\varsigma}_h \in T_h} \left(\|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r\, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \right) + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U^2 \\ &\qquad + \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|^2_{T'_{div}} + \inf_{q_h \in P_h} \|p - q_h\|_P^2 \Big\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P &\leq C \Big\{ \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'_{div}} + \inf_{q_h \in P_h} \|p - q_h\|_P \Big\} \\ &\quad + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} , \end{aligned} \tag{4.10}$$

and

$$\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \leq C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U \,, \tag{4.11}$$

for some constant $C > 0$.

Proof: Let $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h)$ satisfy (4.2)–(4.4), and note that the continuous solution $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda)$ also satisfies (4.2)–(4.4). Define the following subspaces:

$$\tilde{Z}_{1h} \ := \ \left\{ (\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h \ :\ [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)] \ = \ \int_\Omega \mathbf{v}_h \cdot \mathbf{f}\, d\Omega\,, \forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R} \right\},$$

and

$$\tilde{Z}_{2h} \ := \ \left\{ \boldsymbol{\varsigma}_h \in T_h \ :\ [\mathbf{B}(\boldsymbol{\varsigma}_h),\, (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h),\, \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \ = \ -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma\,, \forall (\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h} \right\}.$$

Note that $\mathbf{u}_h \in \tilde{Z}_{2h}$ and $(\boldsymbol{\psi}_h, p_h) \in \tilde{Z}_{1h}$. From (2.2) and the definition of $\mathbf{A}$ (2.16), we have ,

$$\hat{C}_1 \frac{\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} + \hat{C}_1 \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \; d\Omega \\ \leq \int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\, d\Omega\,, \quad (4.12)$$

and

$$\begin{aligned} \int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\, d\Omega &= [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\phi}_h] \\ &= [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\varsigma}_h] \\ &\quad + [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h]\,. \end{aligned} \quad (4.13)$$

We examine the first term on the RHS of (4.13). For $\mathcal{E}$ given by (4.7), note that $\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \leq 1$. From (2.3) and Young's inequality, we have

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\varsigma}_h] &= \int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\varsigma}_h)\, d\Omega \\ &\leq \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \\ &\leq \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega + \frac{1}{r\epsilon_1}\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r\, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r\,. \end{aligned} \quad (4.14)$$

For the second term on the RHS of (4.13), if $\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}$, we have

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] &= [\mathbf{A}(\boldsymbol{\phi}),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] - [\mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] \\ &= -[\mathbf{B}(\boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h),\, (\boldsymbol{\psi}, p)] + [\mathbf{B}(\boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h),\, (\boldsymbol{\psi}_h, p_h)] \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi}, p)] \qquad (\text{as } \boldsymbol{\varsigma}_h, \boldsymbol{\phi}_h \in \tilde{Z}_{2h}) \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi}, p)] - [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\tau}_h, q_h)] \qquad (\text{for } (\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}) \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\phi}),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] + [\mathbf{B}(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] \\ &= -\int_\Omega (\boldsymbol{\phi}_h - \boldsymbol{\phi}) : (\boldsymbol{\psi} - \boldsymbol{\tau}_h)\, d\Omega - \int_\Omega (p - q_h) tr(\boldsymbol{\phi}_h - \boldsymbol{\phi})\, d\Omega \\ &\quad - \int_\Omega (\boldsymbol{\phi} - \boldsymbol{\varsigma}_h) : (\boldsymbol{\psi} - \boldsymbol{\tau}_h)\, d\Omega - \int_\Omega (p - q_h) tr(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h)\, d\Omega \\ &\leq \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n} \|p - q_h\|_P \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \\ &\quad + \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n} \|p - q_h\|_P \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \\ &\leq \frac{\epsilon_2 + \epsilon_3}{2} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 + \frac{\epsilon_4 + \epsilon_5}{2} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 \\ &\quad + \left(\frac{1}{2\epsilon_2} + \frac{1}{2\epsilon_4} \right) \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \sqrt{n} \left(\frac{1}{2\epsilon_3} + \frac{1}{2\epsilon_5} \right) \|p - q_h\|_P^2\,. \end{aligned} \quad (4.15)$$

Combining (4.12)-(4.15) with $\epsilon_4 = \epsilon_5 = 1$ we have

$$\begin{aligned}
\left(\frac{\hat{C}_1}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} - \frac{\epsilon_2 + \epsilon_3}{2} \right) \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 & \\
+ \left(\hat{C}_1 - \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \right) \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \; d\Omega & \\
\leq \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \frac{1}{r\epsilon_1} \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r & \\
+ \left(\frac{1}{2\epsilon_2} + \frac{1}{2} \right) \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \sqrt{n} \left(\frac{1}{2\epsilon_3} + \frac{1}{2} \right) \|p - q_h\|_P^2 \, . & \quad (4.16)
\end{aligned}$$

Choosing $\epsilon_1, \epsilon_2, \epsilon_3$ small enough to ensure

$$\left(\frac{\hat{C}_1}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} - \frac{\epsilon_2 + \epsilon_3}{2} \right) > 0 \, ,$$

and

$$\left(\hat{C}_1 - \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \right) > 0 \, ,$$

we have

$$\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 + \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \, d\Omega \leq C \Bigg\{ & \inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \left(\|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \right) \\
& + \inf_{(\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}} \left(\|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \|p - q_h\|_P^2 \right) \Bigg\} . \quad (4.17)
\end{aligned}$$

The estimate (4.17) holds for $(\boldsymbol{\varsigma}_h, \boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{2h} \times \tilde{Z}_{1h} \subseteq T_h \times T^{'}_{div,\,h} \times P_h$. In order to show that this estimate holds in all of $T_h \times T^{'}_{div,\,h} \times P_h$, we employ a lifting argument similar to that in [10]. Define the subspace

$$\begin{aligned}
\tilde{W}_h := \Bigg\{ \boldsymbol{\varsigma}_h \in T_h : & [\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h), \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \\
& = - \int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \quad \forall \, (\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h \Bigg\} .
\end{aligned}$$

We first show that (4.17) holds for all $\boldsymbol{\varsigma}_h \in T_h$. Then we show that (4.17) holds for all $(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h$.

Note that $\boldsymbol{\varsigma}_h \in \tilde{W}_h \Rightarrow \boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}$. Thus, for $\mathbf{v}_h \in U_h$,

$$\inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \leq \inf_{\boldsymbol{\varsigma}_h \in \tilde{W}_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T \times U} \, . \quad (4.18)$$

From the inf-sup condition (4.8), there exist operators $\Pi_T : T \to T_h$ and $\Pi_U : U \to U_h$ such that

$$[\mathbf{B}(\boldsymbol{\varsigma} - \Pi_T \boldsymbol{\varsigma}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{v} - \Pi_U \mathbf{v}, \lambda_h)] = 0 \, , \quad \forall \, (\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h \, , \quad (4.19)$$

and

$$\|(\Pi_T\boldsymbol{\varsigma}, \Pi_U\mathbf{v})\|_{T\times U} \leq \tilde{C}\|(\boldsymbol{\varsigma}, \mathbf{v})\|_{T\times U}\,, \quad \forall\,(\boldsymbol{\varsigma}, \mathbf{v}) \in T \times U\,. \tag{4.20}$$

Now, let $(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in T_h \times U_h$ and set $\tilde{\boldsymbol{\phi}} := \boldsymbol{\varsigma}_h - \Pi_T(\boldsymbol{\varsigma}_h - \boldsymbol{\phi})$ and $\tilde{\mathbf{u}} := \mathbf{v}_h - \Pi_U(\mathbf{v}_h - \mathbf{u})$. Note that $(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) \in T_h \times U_h$. Then for all $(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h$,

$$\begin{aligned}
[\mathbf{B}(\tilde{\boldsymbol{\phi}}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\tilde{\mathbf{u}}, \lambda_h)] &= [\mathbf{B}(\boldsymbol{\phi}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}, \lambda_h)] \\
&= -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma, . \qquad (4.21)
\end{aligned}$$

Thus $\tilde{\boldsymbol{\phi}} \in \tilde{W}_h$. Now, using (4.20), we have

$$\begin{aligned}
\|(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} &= \|(\Pi_T(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h), \Pi_U(\mathbf{u} - \mathbf{v}_h))\|_{T\times U} \\
&\leq \tilde{C}\|(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h, \mathbf{u} - \mathbf{v}_h)\|_{T\times U}\,. \qquad (4.22)
\end{aligned}$$

Thus we have

$$\begin{aligned}
\inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T &\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h)\in \tilde{W}_h\times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} \\
&\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h)\in T_h\times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}})\|_{T\times U} \\
&\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h)\in T_h\times U_h} \Big(\|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} + \|(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U}\Big) \\
&\leq (1 + \tilde{C}) \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h)\in T_h\times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U}\,, \qquad (4.23)
\end{aligned}$$

which lifts the best approximation of $\boldsymbol{\phi}$ from $\tilde{Z}_{2h}$ to T_h.
Now, we must also show

$$\inf_{(\boldsymbol{\tau}_h, q_h)\in \tilde{Z}_{1h}} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div}\times P} \leq C \inf_{(\boldsymbol{\tau}_h, q_h)\in T^{'}_{div,\,h}\times P_h} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div}\times P}\,. \tag{4.24}$$

From (4.6), we have the existence of operators $\Pi_{T'} : T^{'}_{div} \to T^{'}_{div,\,h}$ and $\Pi_P : P \to P_h$ such that

$$[\mathbf{C}(\boldsymbol{\tau} - \Pi_{T'}\boldsymbol{\tau}, q - \Pi_P q), (\mathbf{v}_h, \eta_h)] = 0\,, \quad \forall\,(\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}\,, \tag{4.25}$$

and

$$\|(\Pi_{T'}\boldsymbol{\tau}, \Pi_P q)\|_{T^{'}_{div}\times P} \leq \tilde{C}\|(\boldsymbol{\tau}, q)\|_{T^{'}_{div}\times P}\,. \tag{4.26}$$

Now for $(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h$, let $\tilde{\boldsymbol{\psi}} := \boldsymbol{\tau}_h - \Pi_{T'}(\boldsymbol{\tau}_h - \boldsymbol{\psi})$ and $\tilde{p} := q_h - \Pi_P(q_h - p)$. Note that $(\tilde{\boldsymbol{\psi}}, \tilde{p}) \in T^{'}_{div,\,h} \times P_h$. Then for all $(\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}$ we have

$$[\mathbf{C}(\tilde{\boldsymbol{\psi}}, \tilde{p}), (\mathbf{v}_h, \eta_h)] = [\mathbf{C}(\boldsymbol{\psi}, p), (\mathbf{v}_h, \eta_h)] = \int_\Omega \mathbf{v}_h \cdot \mathbf{f}\,, d\Omega\,. \tag{4.27}$$

So $(\tilde{\boldsymbol{\psi}}, \tilde{p}) \in \tilde{Z}_{1h}$. Now, using (4.26) we have

$$\begin{aligned}
\|(\tilde{\boldsymbol{\psi}}, \tilde{p}) - (\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div}\times P} &= \|(\Pi_{T'}(\boldsymbol{\psi} - \boldsymbol{\tau}_h), \Pi_P(p - q_h)\|_{T^{'}_{div}\times P} \\
&\leq \tilde{C}\|(\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)\|_{T^{'}_{div}\times P}\,. \qquad (4.28)
\end{aligned}$$

Thus

$$\begin{aligned}\inf_{(\boldsymbol{\tau}_h,q_h)\in\tilde{Z}_{1h}} \|(\boldsymbol{\psi},p)-(\boldsymbol{\tau}_h,q_h)\|_{T^{'}_{div}\times P} \;&\leq\; \inf_{(\boldsymbol{\tau}_h,q_h)\in T^{'}_{div,\,h}\times P_h} \|(\boldsymbol{\psi},p)-(\tilde{\boldsymbol{\psi}},\tilde{p})\|_{T^{'}_{div}\times P} \\ &\leq \inf_{(\boldsymbol{\tau}_h,q_h)\in T^{'}_{div,\,h}\times P_h} \Big(\|(\boldsymbol{\psi},p)-(\boldsymbol{\tau}_h,q_h)\|_{T^{'}_{div}\times P} + \|(\tilde{\boldsymbol{\psi}},\tilde{p})-(\boldsymbol{\tau}_h,q_h)\|_{T^{'}_{div}\times P}\Big) \\ &\leq (1+\tilde{C}) \inf_{(\boldsymbol{\tau}_h,q_h)\in T^{'}_{div,\,h}\times P_h} \|(\boldsymbol{\psi},p)-(\boldsymbol{\tau}_h,q_h)\|_{T^{'}_{div}\times P}\,. \end{aligned} \tag{4.29}$$

This lifts the best approximation of $(\boldsymbol{\psi},p)$ from $\tilde{Z}_{1h}$ to $T^{'}_{div}\times P$. Thus, from (4.17), (4.23), and (4.29) we have

$$\begin{aligned}&\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T^2 + \int_\Omega |\mathbf{g}(\boldsymbol{\phi})-\mathbf{g}(\boldsymbol{\phi}_h)|\,|\boldsymbol{\phi}-\boldsymbol{\phi}_h|\,d\Omega \\ &\qquad\leq C\Big\{\inf_{\boldsymbol{\varsigma}_h\in T_h}\big(\|\boldsymbol{\phi}-\boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h)^r\,\|\boldsymbol{\phi}-\boldsymbol{\varsigma}_h\|_T^r\big) + \inf_{\mathbf{v}_h\in U_h}\|\mathbf{u}-\mathbf{v}_h\|_U^2 \\ &\qquad\qquad + \inf_{\boldsymbol{\tau}_h\in T^{'}_{div,\,h}}\|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|^2_{T^{'}_{div}} + \inf_{q_h\in P_h}\|p-q_h\|_P^2\Big\}\,. \end{aligned} \tag{4.30}$$

The proof of the remaining estimates will be outlined below, the reader is referred to [9] for complete details. To obtain the a priori estimate for $\boldsymbol{\psi}$ and p, we use with the discrete inf-sup condition satisfied by $\mathbf{B}$. It can be shown that, for $(\tau_h,q_h)\in\tilde{Z}_{1h}$,

$$\begin{aligned}&c_1\Big(\|\boldsymbol{\psi}_h-\boldsymbol{\tau}_h\|_{T^{'}_{div}} + \|p_h-q_h\|_P\Big) \\ &\qquad\leq \sup_{\boldsymbol{\varsigma}_h\in T_h}\frac{\int_\Omega(\mathbf{g}(\boldsymbol{\phi}_h)-\mathbf{g}(\boldsymbol{\phi})):\boldsymbol{\varsigma}_h\,d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} + \|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p-q_h\|_P\,. \end{aligned} \tag{4.31}$$

The first term on the RHS of (4.31) can be handled using (2.3) and the definition of $\mathcal{E}$:

$$\sup_{\boldsymbol{\varsigma}_h\in T_h}\frac{\int_\Omega(\mathbf{g}(\boldsymbol{\phi}_h)-\mathbf{g}(\boldsymbol{\phi})):\boldsymbol{\varsigma}_h\,d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \leq \hat{C}_2\,\mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h)\,\left(\int_\Omega|\mathbf{g}(\boldsymbol{\phi})-\mathbf{g}(\boldsymbol{\phi}_h)|\,|\boldsymbol{\phi}-\boldsymbol{\phi}_h|\,d\Omega\right)^{1/r'}\,. \tag{4.32}$$

Combining (4.31), (4.32), and an application of the triangle inequality imply

$$\begin{aligned}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{T^{'}_{div}} + \|p-p_h\|_P &\leq C\Big\{\inf_{(\boldsymbol{\tau}_h,q_h)\in\tilde{Z}_{1h}}\Big(\|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|_{T'} + \|p-q_h\|_P\Big)\Big\} \\ &\qquad + \hat{C}_2\,\mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h)\,\left(\int_\Omega|\mathbf{g}(\boldsymbol{\phi})-\mathbf{g}(\boldsymbol{\phi}_h)|\,|\boldsymbol{\phi}-\boldsymbol{\phi}_h|\,d\Omega\right)^{1/r'}\,. \end{aligned} \tag{4.33}$$

Now the previously described argument to lift the best approximations of $(\boldsymbol{\tau}_h,q_h)$ from $\tilde{Z}_{1h}$ to $T^{'}_{div,\,h}\times P_h$ can be applied here. Thus we have, from (4.33)

$$\begin{aligned}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{T^{'}_{div}} + \|p-p_h\|_P &\leq C\Big\{\inf_{\boldsymbol{\tau}_h\in T^{'}_{div,\,h}}\|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|_{T^{'}_{div}} + \inf_{q_h\in P_h}\|p-q_h\|_P\Big\} \\ &\qquad + \hat{C}_2\,\mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h)\,\left(\int_\Omega|\mathbf{g}(\boldsymbol{\phi})-\mathbf{g}(\boldsymbol{\phi}_h)|\,|\boldsymbol{\phi}-\boldsymbol{\phi}_h|\,d\Omega\right)^{1/r'}\,. \end{aligned} \tag{4.34}$$

From the discrete inf-sup condition for $\mathbf{C}$, (4.30), and the triangle inequality we have

$$\|\mathbf{u}-\mathbf{u}_h\|_U + |\lambda-\lambda_h| \le C\,\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T + \inf_{\mathbf{v}_h\in U_h} \|\mathbf{u}-\mathbf{v}_h\|_U\,.$$

Thus the estimates (4.9)–(4.11) are proven. ■

Remark 4.1 *Note that $\mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h)\le 1$. In addition, if $1/(|\boldsymbol{\phi}|+|\boldsymbol{\phi}_h|)\le C$ for some constant $C>0$, then*

$$\mathcal{E}(\boldsymbol{\phi},\boldsymbol{\phi}_h) \le \min\left\{1, C\,\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_\infty^{(2-r)/r}\right\}.$$

Furthermore, if $\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_\infty \sim \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T$, the estimates (4.9)–(4.11) may be written as

$$\begin{aligned}\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T + \|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{T'_{div}} &+ \|p-p_h\|_P + \|\mathbf{u}-\mathbf{u}_h\|_U + |\lambda-\lambda_h| \\ &\le C\Bigg\{\inf_{\boldsymbol{\varsigma}_h\in T_h}\|\boldsymbol{\phi}-\boldsymbol{\varsigma}_h\|_T + \inf_{\mathbf{v}_h\in U_h}\|\mathbf{u}-\mathbf{v}_h\|_U \\ &\qquad + \inf_{\boldsymbol{\tau}_h\in T'_{div,\,h}}\|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|_{T'_{div}} + \inf_{q_h\in P_h}\|p-q_h\|_P\Bigg\}. \qquad (4.35)\end{aligned}$$

4.2 Approximation Using Raviart-Thomas Elements and Discontinuous Piecewise Polynomials

In this section we consider $\Omega\subset\mathbb{R}^2$ and show that the approximating spaces of discontinuous piecewise polynomials and Raviart-Thomas elements are suitable for problem (2.13)-(2.15). Specifically, we show that these spaces satisfy the inf-sup conditions (4.5) and (4.6) and then show that the error estimate given in Theorem 4.2 holds.

4.2.1 Discrete Inf-Sup Conditions for B and C

Let $K\in\mathcal{T}_h$ and let $\mathbb{P}_k(K)$ be the set of all polynomials in the variables x_1, x_2 of degree less than or equal to k defined on the triangle K. Let $\mathbb{RT}_k(K)$ be the 2-vector of Raviart-Thomas elements [23, 25] on K defined by

$$\mathbb{RT}_k(K) = (\mathbb{P}_k(K))^2 + \begin{bmatrix} x_1\\ x_2\end{bmatrix}\mathbb{P}_k(K)\,.$$

For $k\ge 0$, define the following discrete spaces:

$$\begin{aligned} T_h &:= \left\{\boldsymbol{\phi}\in T : \boldsymbol{\phi}|_K \in (\mathbb{P}_k(K))^{2\times 2}, \quad \forall K\in\mathcal{T}_h\right\}, \\ T'_{div,\,h} &:= \Big\{\boldsymbol{\psi}\in T'_{div} : \boldsymbol{\psi} = (\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2)^{\mathrm{T}}|_K \in (\mathbb{RT}_k(K))^2\,, \\ &\qquad (\psi_{i1}\quad \psi_{i2})^{\mathrm{T}}|_K \in \mathbb{RT}_k(K)\,,\quad \forall i\in\{1,2\},\quad \forall K\in\mathcal{T}_h\Big\}, \\ P_h &:= \left\{p\in P : p|_K\in\mathbb{P}_k(K)\,,\quad \forall K\in\mathcal{T}_h\right\}, \\ U_h &:= \left\{\mathbf{u}\in U : \mathbf{u}|_K\in(\mathbb{P}_k(K))^2\,,\quad \forall K\in\mathcal{T}_h\right\}. \end{aligned}$$

Remark 4.2 *There is no interelement continuity requirement on the spaces T_h, U_h, and P_h.*

Let $s > 1$ and let $\mathcal{I}_h^k : \left(W^{1,s}(\Omega)\right)^{2\times 2} \longrightarrow T^{'}_{div,\,h}$ be the k-th order Raviart-Thomas interpolation operator [23, 6, 8], defined by, for row $j = 1, 2$ of $\boldsymbol{\tau} \in T^{'}_{div}$,

$$\begin{aligned}
\int_{e_i} (\boldsymbol{\tau}_j - \mathcal{I}_h^k \boldsymbol{\tau}_j) \cdot \mathbf{n}_{e_i} v_k \, ds &= 0, \quad \forall v_k \in \mathbb{P}_k(K)\,, \quad \forall e_i \in \partial K, \quad i = 1,2,3, \quad \forall K \in \mathcal{T}_h\,, \\
\int_K (\boldsymbol{\tau}_j - \mathcal{I}_h^k \boldsymbol{\tau}_j) \cdot \mathbf{v}_{k-1} \, dK &= 0, \quad \forall \mathbf{v}_{k-1} \in (\mathbb{P}_{k-1}(K))^2\,, \quad \forall K \in \mathcal{T}_h\,,
\end{aligned}$$

where $\mathbf{n}_{e_i}$ denotes the outer unit normal vector to edge e_i of K. Then, for $0 \le m \le k+1$, we have

$$\|\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau}\|_{0,r',\Omega} \le C h^m |\boldsymbol{\tau}|_{m,r',\Omega}\,, \tag{4.36}$$

$$\|div\,(\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau})\|_{0,r',\Omega} \le C h^m |div\,\boldsymbol{\tau}|_{m,r',\Omega}\,, \tag{4.37}$$

and, for $\mathbf{v} \in U$,

$$\int_\Omega \mathbf{v} \cdot div(\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau}) \, d\Omega = 0, \quad \forall \boldsymbol{\tau} \in T^{'}_{div}\,. \tag{4.38}$$

In the lowest-order case, i.e., $k = 0$, for $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$,

$$\boldsymbol{\phi}^* \;=\; \frac{-|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h|^{r'/r-1}\,(q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h)}{\|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h\|_{T'}^{r'-1}} \in T_h\,. \tag{4.39}$$

The proof of the discrete inf-sup condition for $\mathbf{B}$ then follows as in the continuous case. However, for higher-order approximations, $\boldsymbol{\phi}^*$ defined by (4.39) for $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$ is not a polynomial and hence not in T_h. In these cases a suitable projection of $\boldsymbol{\phi}^*$ is required. Let $\Pi : T \to T_h = (\mathcal{P}_k)^{2\times 2}$ denote the L^2 projection operator, defined by $\Pi(\boldsymbol{\phi}^*) := \boldsymbol{\phi}_h$, where

$$\int_\Omega \boldsymbol{\phi}^* : \boldsymbol{\tau}_h \, d\Omega = \int_\Omega \boldsymbol{\phi}_h : \boldsymbol{\tau}_h \, d\Omega \qquad \forall \boldsymbol{\tau}_h \in T_h.$$

Lemma 4.1 *Let $\boldsymbol{\phi} \in T$ and $\boldsymbol{\phi}_h = \Pi\boldsymbol{\phi}$. Then there is a constant $C_* > 0$ such that*

$$\|\boldsymbol{\phi}_h\|_T \le C_* \|\boldsymbol{\phi}\|_T. \tag{4.40}$$

Proof: Note that, since T_h is the space of 2×2 tensors whose components are discontinuous piecewise polynomials of degree k on each $K \in \mathcal{T}_h$, we have that,

$$\phi_h = \Pi\boldsymbol{\phi} = \sum_{K \in \mathcal{T}_h} (\Pi\boldsymbol{\phi})|_K = \sum_{K \in \mathcal{T}_h} \Pi(\boldsymbol{\phi}|_K)\,, \tag{4.41}$$

where $\boldsymbol{\phi}|_K$ is the restriction of $\boldsymbol{\phi}$ to K. Let $\boldsymbol{\phi}_K = \boldsymbol{\phi}|_K$. Let $K \in \mathcal{T}_h$, and let $\widehat{K}$ denote the reference element in $\mathcal{T}_h$. Let χ represent the affine map from $\widehat{K}$ to K. Then $\widehat{\boldsymbol{\phi}} = \boldsymbol{\phi}_K \circ \chi$ is the representation of $\boldsymbol{\phi}_K$ on the reference element $\widehat{K}$.

Let $m = \dim((\mathcal{P}_k(\widehat{K}))^{2\times 2})$ and let $\{\widehat{\boldsymbol{\Phi}}_i\}_{i=1}^m$ be an L^2 orthonormal basis for $(\mathcal{P}_k(\widehat{K}))^{2\times 2}$. Then we can write

$$\widehat{\boldsymbol{\phi}}_h(\boldsymbol{\xi}) = \sum_{i=1}^m \phi_i \widehat{\boldsymbol{\Phi}}_i(\boldsymbol{\xi})$$

where the coefficients ϕ_i are given by

$$\phi_i = (\widehat{\boldsymbol{\phi}}, \widehat{\mathbf{\Phi}}_i)_{\widehat{K}} \tag{4.42}$$

where $(\cdot,\cdot)_{\widehat{K}}$ represents the L^2 inner product over $\widehat{K}$.

Now we have

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{0,r,K} &= \left(\int_K |\boldsymbol{\phi}_h|^r \, dK\right)^{1/r} \\
&= \left(\int_{\widehat{K}} |\widehat{\boldsymbol{\phi}}_h|^r \frac{|K|}{|\widehat{K}|}\, d\widehat{K}\right)^{1/r} \\
&= \left(\int_{\widehat{K}} \left|\sum_{i=1}^m \phi_i \widehat{\mathbf{\Phi}}_i\right|^r \, d\widehat{K}\right)^{1/r} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&\le m^{(r-1)/r} \sum_{i=1}^m |\phi_i| \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} .
\end{aligned} \tag{4.43}$$

Now (4.42) implies

$$|\phi_i| \le \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}}. \tag{4.44}$$

We also have

$$\|\boldsymbol{\phi}\|_{0,r,K} = \left(\int_K |\boldsymbol{\phi}|^r \, dK\right)^{1/r} = \left(\int_{\widehat{K}} |\widehat{\boldsymbol{\phi}}|^r \frac{|K|}{|\widehat{K}|}\, d\widehat{K}\right)^{1/r} = \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r},$$

which implies

$$\|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} = \left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \|\boldsymbol{\phi}\|_{0,r,K} . \tag{4.45}$$

Combining (4.43)–(4.45), we have

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{0,r,K} &\le m^{(r-1)/r} \sum_{i=1}^m |\phi_i| \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&\le m^{1/r'} \sum_{i=1}^m \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&= m^{1/r'} \sum_{i=1}^m \left(\left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \|\boldsymbol{\phi}\|_{0,r,K}\right) \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&= m^{1/r'} \|\boldsymbol{\phi}\|_{0,r,K} \left(\sum_{i=1}^m \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}}\right) \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \\
&= m^{1/r'} \|\boldsymbol{\phi}\|_{0,r,K} \left(\sum_{i=1}^m \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}}\right) .
\end{aligned} \tag{4.46}$$

Now $\|\widehat{\mathbf{\Phi}}_i\|_{0,2,\widehat{K}} = 1$ and since $\mathcal{P}_k(\widehat{K})$ is finite-dimensional, the equivalence of finite dimensional norms implies there exist constants c_r and $c_{r'}$ such that

$$\|\widehat{\mathbf{\Phi}}_i\|_{0,r,\widehat{K}} \le c_r \|\widehat{\mathbf{\Phi}}_i\|_{0,2,\widehat{K}} = c_r \quad \text{and} \quad \|\widehat{\mathbf{\Phi}}_i\|_{0,r',\widehat{K}} \le c_{r'} \|\widehat{\mathbf{\Phi}}_i\|_{0,2,\widehat{K}} = c_{r'}$$

Thus (4.46) implies

$$\|\boldsymbol{\phi}_h\|_{0,r,K} \le C_* \, \|\boldsymbol{\phi}\|_{0,r,K} \tag{4.47}$$

for $C_* = m^{1+1/r'} c_r c_{r'}$, which is independent of K. Therefore

$$\begin{aligned} \|\boldsymbol{\phi}_h\|_T = \left(\sum_{K \in \mathcal{T}_h} \|\boldsymbol{\phi}_h\|^r_{0,r,K} \right)^{1/r} &\le \left(\sum_{K \in \mathcal{T}_h} C_*^r \|\boldsymbol{\phi}\|^r_{0,r,K} \right)^{1/r} \\ &= C_* \left(\sum_{K \in \mathcal{T}_h} \|\boldsymbol{\phi}\|^r_{0,r,K} \right)^{1/r} = C_* \, \|\boldsymbol{\phi}\|_T \,, \end{aligned} \tag{4.48}$$

and thus the result is shown. ■

The constant C_* in Lemma 4.1 depends only on the constants c_r and $c_{r'}$, as the dimension m of $(\mathcal{P}_k)^{2\times 2}$ is fixed for k. The constants c_r and $c_{r'}$ that arise in the norm equivalences depend only on the dimension of the space (which is m as well) and not on the size of the domain. A result analogous to Lemma 4.1 holds for the L^2 projection from U onto U_h. Let $\Pi_U : U \to U_h$ be denoted by $\Pi_U \mathbf{u}^* := \mathbf{u}_h$, where

$$\int_\Omega \mathbf{u}^* \cdot \mathbf{w}_h \, d\Omega = \int_\Omega \mathbf{u}_h \cdot \mathbf{w}_h \, d\Omega \qquad \forall \mathbf{w}_h \in U_h.$$

Corollary 4.1 *Let $\mathbf{u} \in U$ and $\mathbf{u}_h = \Pi_U \mathbf{u}$. Then there is a constant $C_{**} > 0$ such that*

$$\|\mathbf{u}_h\|_U \le C_{**} \|\mathbf{u}\|_U \,. \tag{4.49}$$

Lemma 4.2 *For the choices of T_h, $T'_{div,\,h}$, P_h, and U_h above, there exists a positive constant c_1 such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in Z_{1h}} \; \sup_{\boldsymbol{\phi}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\phi}_h) \,,\, (\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\phi}_h\|_T \, \|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P}} \ge \; c_1 \;.$$

Proof: Note that for $(\boldsymbol{\phi}_h, q_h) \in Z_{1h}$, $div\, \boldsymbol{\tau}_h = \mathbf{0}$ implies $\boldsymbol{\tau}_h|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. We also have that $(\boldsymbol{\tau}_h + q_h \mathbf{I})|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. Thus $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$ implies $\boldsymbol{\tau}_h \in T_h$ and $(\boldsymbol{\tau}_h + q_h \mathbf{I}) \in T_h$.
Assume that $\|q_h\|_P \le \|\boldsymbol{\tau}_h\|_{T'_{div}}$. Let $\boldsymbol{\tau}_h^0 = \boldsymbol{\tau}_h - \frac{1}{n} tr(\boldsymbol{\tau}_h)\mathbf{I}$, and

$$\boldsymbol{\phi}^* = -|\boldsymbol{\tau}_h^0|^{r'/r-1} \, \boldsymbol{\tau}_h^0 / \|\boldsymbol{\tau}_h^0\|_{T'}^{r'-1}.$$

Then $\|\boldsymbol{\phi}^*\|_T = 1$, and let $\boldsymbol{\varsigma}_h = \Pi \boldsymbol{\phi}^*$. From Lemma 4.1,

$$\|\boldsymbol{\varsigma}_h\|_T \le C_* \|\boldsymbol{\phi}^*\|_T = C_* \,.$$

Also $[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, p_h)] = [\mathbf{B}(\boldsymbol{\phi}^*), (\boldsymbol{\tau}_h, p_h)]$ for all $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$. Continuing as in (3.7), the result is shown as in Case 1 of Lemma 3.2, with the inclusion of the constant $1/C_*$.
Now assume $\|q_h\|_P \geq \|\boldsymbol{\tau}_h\|_{T'_{div}}$. Let

$$\boldsymbol{\phi}^* \;=\; \frac{-|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h|^{r'/r-1}\,(q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h)}{\|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h\|_{T'}^{r'-1}}\,.$$

Again let $\boldsymbol{\varsigma}_h = \Pi\boldsymbol{\phi}^*$ and note that $\|\boldsymbol{\varsigma}_h\|_{0,r} \leq C_*\|\boldsymbol{\phi}^*\|_T = C_*$. Continuing as in the proof of Case 2 of Lemma 3.2, the result is shown. ■

Lemma 4.3 *For the choices of T_h, $T'_{div,\,h}$, P_h, and U_h above, there exists a positive constant c_2 such that*

$$\inf_{(\mathbf{u}_h,\lambda_h)\in U_h\times\mathbb{R}}\;\sup_{(\boldsymbol{\tau}_h,q_h)\in T'_{div,h}\times P_h}\frac{[\mathbf{C}(\boldsymbol{\tau}_h,q_h)\,,\,(\mathbf{u}_h,\lambda_h)]}{\|(\boldsymbol{\tau}_h,q_h)\|_{T'_{div}\times P}\;\|(\mathbf{u}_h,\lambda_h)\|_{U\times\mathbb{R}}} \;\geq\; c_2\,. \tag{4.50}$$

Proof: As in the approach to the proof of Lemma 3.4 and of Theorem 3.1 of [15], we consider two cases:
Case 1: $|\lambda_h| \geq \|\mathbf{u}_h\|_U$.
The choice $(\boldsymbol{\tau}_h, q_h) = (\lambda_h\mathbf{I}, 0) \in T'_{div,\,h} \times P_h$ shows the result as in Case 1 of the proof of Lemma 3.4.
Case 2: $|\lambda_h| \leq \|\mathbf{u}_h\|_U$.
Note that Lemma 3.3 applies to the subspace $T'_{div,\,h} \subset T'_{div}$, thus we have

$$\sup_{(\boldsymbol{\tau}_h,q_h)\in T'_{div,h}\times P_h}\frac{[\mathbf{C}(\boldsymbol{\tau}_h,q_h)\,,\,(\mathbf{u}_h,\lambda_h)]}{\|(\boldsymbol{\tau}_h,q_h)\|_{T'_{div}\times P}} \geq C\sup_{\boldsymbol{\tau}\in T'_{div,h}}\frac{-\int_\Omega \mathbf{u}_h\cdot div\,\boldsymbol{\tau}_h\,d\Omega}{\|\boldsymbol{\tau}_h\|_{T'_{div,h}}}\,. \tag{4.51}$$

The proof then proceeds in a manner similar to that of Proposition 5 of [22] (as well as Proposition 3.1 of [11]), in which an auxiliary Laplacian problem is solved and the properties (4.36)-(4.38) are used to bound the supremum in (4.51). See [9] for complete details. ■

4.2.2 Error Estimate

To apply Theorem 4.2, we must show that the inf-sup condition (4.8) holds for the chosen approximation spaces. To accomplish this, some properties of the Raviart-Thomas elements must be presented. Let $K \in \mathcal{T}_h$ and let $\mathbf{r} \in \mathbb{RT}_k(K)$. Then $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{r}^k + \mathbf{r}^*$, where $\mathbf{r}^k \in (\mathbb{P}_k(K))^2$ and the components of $\mathbf{r}^*$ consist of polynomial terms of degree $k+1$ only. In fact, $\mathbf{r}^*$ can be written as

$$\mathbf{r}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sum_{j=0}^{k} \gamma_j\, x_1^{k-j}\, x_2^j = \begin{bmatrix} \displaystyle\sum_{j=0}^{k} \gamma_j\, x_1^{k-j+1}\, x_2^j \\ \displaystyle\sum_{j=0}^{k} \gamma_j\, x_1^{k-j}\, x_2^{j+1} \end{bmatrix}\,,$$

for some constants $\gamma_j, j = 0, \ldots, k$. We can also write $div\,\mathbf{r} = div\,\mathbf{r}^k + div\,\mathbf{r}^*$, where $div\,\mathbf{r}^k$ is a polynomial of degree at most $k-1$ and $div\,\mathbf{r}^*$ is a polynomial with terms of degree k only. It is important to note that if $div\,\mathbf{r} = 0$, then $div\,\mathbf{r}^* = 0$.

The following lemma is a result from the general theory of finite-dimensional normed spaces (see [20]).

Lemma 4.4 *Let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a normed space $\mathbf{X}$ of dimension at least $n+1$. Then, there is a constant $C_* > 0$ such that for every choice of scalars $\gamma_0, \dots, \gamma_n$, we have*

$$\|\gamma_0 \mathbf{v}_0 + \cdots + \gamma_n \mathbf{v}_n\| \geq C_* (|\gamma_0| + \cdots + |\gamma_n|) .$$

■

For Raviart-Thomas elements we have that the norm of the gradient of the highest-degree terms can be bounded by the norm of the divergence.

Lemma 4.5 *Let $K \in \mathcal{T}_h$, $\mathbf{r} := \mathbf{r}^k + \mathbf{r}^* \in \mathbb{RT}_k(K)$ where the components of $\mathbf{r}^*$ consist of polynomial terms of degree $k+1$ only. Then there exists a constant $\tilde{C} > 0$, independent of K, such that*

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq \tilde{C}\, \|div\, \mathbf{r}\|_{0,r',K} . \tag{4.52}$$

Proof: Let the finite-dimensional vector space $\mathbf{X}$ be defined by

$$\mathbf{X} = \text{span} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} x_1^{k-j}\, x_2^j , \quad j = 0, \dots, k \right\} = \text{span}\, \{\mathbf{v}_j\, , \quad j = 0, \dots, k\} \, ,$$

and $\mathbf{v} \in \mathbf{X}$ be represented as $\mathbf{v} = (v_1 \quad v_2)^{\mathrm{T}} := \gamma_0 \mathbf{v}_0 + \cdots + \gamma_k \mathbf{v}_k$. Define the norms $\|\cdot\|_{grad}$ and $\|\cdot\|_{div}$ on $\mathbf{X}$ by

$$\|\mathbf{v}\|_{grad} := \int_K \left|\frac{\partial v_1}{\partial x_1}\right| + \left|\frac{\partial v_1}{\partial x_2}\right| + \left|\frac{\partial v_2}{\partial x_1}\right| + \left|\frac{\partial v_2}{\partial x_2}\right| \, dK \, , \qquad \|\mathbf{v}\|_{div} = \int_K \left|\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right| \, dK \, .$$

Note that $\|\mathbf{v}\|_{grad} = \|\nabla \mathbf{v}\|_{0,1,K}$ and $\|\mathbf{v}\|_{div} = \|div\, \mathbf{v}\|_{0,1,K}$. By the equivalence of norms on a finite-dimensional vector space there exist constants C_1, C_2, and C_3 such that

$$\|\nabla \mathbf{v}\|_{0,r',K} \leq C_1 \|\nabla \mathbf{v}\|_{0,1,K} \leq C_2 \|div\, \mathbf{v}\|_{0,1,K} \leq C_3 \|div\, \mathbf{v}\|_{0,r',K} .$$

Thus for $\mathbf{r}^*$ as defined above, there is a $C_K > 0$ such that

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq C_K \, \|div\, \mathbf{r}^*\|_{0,r',K} \tag{4.53}$$

for all $K \in \mathcal{T}_h$. The dependence of C_K on $K \in \mathcal{T}_h$ is due to the integral over K. The condition (4.1) guarantees that $\mathcal{T}_h$ is a quasi-uniform triangulation of Ω, thus we can find a global constant C, independent of K, such that

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq C \, \|div\, \mathbf{r}^*\|_{0,r',K} \tag{4.54}$$

for all $K \in \mathcal{T}_h$.
Now, let $\mathbf{X}^k$ be the finite dimensional vector space spanned by the polynomials of degree k only, and let $\overline{\mathbf{X}} = \mathbb{P}_k(K)$. Note that $\overline{\mathbf{X}} = \mathcal{P}_{k-1}(K) \oplus \mathbf{X}^k$, and that $div\, \mathbf{r} \in \overline{\mathbf{X}}$, $div\, \mathbf{r}^k \in \mathcal{P}_{k-1}(K)$, and

$div\,\mathbf{r}^* \in \mathbf{X}^k$. Let $\{\mathbf{v}_0, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_n\}$ be a basis for $\overline{\mathbf{X}}$ where $\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$ is also a basis for $\mathbf{X}^k$. From Lemma 4.4, there is a constant $C_* > 0$ such that, for all $\mathbf{v} = \gamma_0\mathbf{v}_0 + \cdots + \gamma_n\mathbf{v}_n \in \overline{\mathbf{X}}$,

$$\|\mathbf{v}\|_{0,r',K} \geq C_*(|\gamma_0| + \cdots + |\gamma_n|)\,.$$

Define the norm $\|\cdot\|_* : \overline{\mathbf{X}} \to \mathbb{R}$ as $\|\mathbf{v}\|_* := C_*(|\gamma_0| + \cdots + |\gamma_n|)$. By the definition of $\mathbf{r}$, $\mathbf{r}^*$, the equivalence of norms on a finite-dimensional space, and the quasi-uniform triangulation $\mathcal{T}_h$, we have that there is a constant C_4 such that

$$\begin{aligned}\|div\,\mathbf{r}^*\|_{0,r',K} &\leq C_4\|div\,\mathbf{r}^*\|_* = C_4C_*(|\gamma_0| + \cdots + |\gamma_k|)\\ &\leq C_4C_*(|\gamma_0| + \cdots + |\gamma_k| + \cdots + |\gamma_n|) = C_4\|div\,\mathbf{r}\|_* \leq C_4\|div\,\mathbf{r}\|_{0,r',K}\,. \end{aligned} \tag{4.55}$$

Combining (4.54) and (4.55) the result is shown. ■

The above result can be applied to the tensor space $T'_{div,\,h}$ to obtain, for $\boldsymbol{\tau}_h = \boldsymbol{\tau}^k + \boldsymbol{\tau}^*$ where the components of $\boldsymbol{\tau}^*$ consist of polynomial terms of degree $k+1$ only,

$$\|\nabla\boldsymbol{\tau}^*\|_{0,r',K} \leq \tilde{C}\,\|div\,\boldsymbol{\tau}_h\|_{0,r',K}\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{4.56}$$

Let $\Pi_k : T'_{div,\,h} \longrightarrow T_h$ be the classical Lagrangian $\mathcal{P}_k$ interpolation operator ([8]) and define

$$\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}^k + \Pi_k\boldsymbol{\tau}^*\,. \tag{4.57}$$

Note that $\hat{\boldsymbol{\tau}}|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$, and $div\,\boldsymbol{\tau}_h = \mathbf{0}$ implies $\boldsymbol{\tau}^* = \mathbf{0}$ and $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}_h$. Then, using (4.56) and standard polynomial approximation properties [5, 8], the error associated in the approximation of $\boldsymbol{\tau}_h$ by $\hat{\boldsymbol{\tau}}$ is given by

$$\begin{aligned}\|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{0,r',\Omega} = \|\boldsymbol{\tau}^* - \Pi_k\boldsymbol{\tau}^*\|_{0,r',\Omega} \;&\leq\; C\,h\left(\sum_{K\in\mathcal{T}_h}\|\nabla\boldsymbol{\tau}^*\|^{r'}_{0,r',K}\right)^{1/r'}\\ &\leq C\,h\left(\sum_{K\in\mathcal{T}_h}\tilde{C}\,\|div\,\boldsymbol{\tau}_h\|^{r'}_{0,r',K}\right)^{1/r'} \;\leq\; C\tilde{C}\,h\,\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} \;=\; \hat{C}\,h\,\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,. \end{aligned}\tag{4.58}$$

Lemma 4.6 *For h sufficiently small, there is a constant $c_3 > 0$ such that*

$$\inf_{(\boldsymbol{\tau}_h,q_h)\in T'_{div,h}\times P_h}\;\sup_{(\boldsymbol{\varsigma}_h,\mathbf{v}_h,\eta_h)\in T_h\times U_h\times\mathbb{R}} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h),(\boldsymbol{\tau}_h,q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h,q_h)\,,\,(\mathbf{v}_h,\eta_h)]}{\|(\boldsymbol{\varsigma}_h,\mathbf{v}_h,\eta_h)\|_{T\times U\times\mathbb{R}}\,\|(\boldsymbol{\tau}_h,q_h)\|_{T'_{div}\times P}} \geq\; c_3\,. \tag{4.59}$$

where $\|(\boldsymbol{\varsigma}_h,\mathbf{v}_h,\eta_h)\|_{T\times U\times\mathbb{R}} = \|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + \|\lambda_h\|_{\mathbb{R}}$.

Proof: The usual approach of considering two cases (as in Theorem 3.1 of [15] and Lemmas 3.2, 3.4, 4.2, 4.3 here) and constructing particular functions that lie in the appropriate finite element spaces (using Lemma 4.1 and Corollary 4.1) is used, along with the property (4.58), to give the proof of (4.59) (see [9] for details). ■

From [6, 25] we have the standard approximation properties: for all $(\boldsymbol{\varsigma}, \boldsymbol{\tau}, q, \mathbf{v}) \in (W^{m,r}(\Omega))^{2\times 2} \times \left(W^{m,r'}(\Omega)\right)^{2\times 2} \times W^{m,r'}(\Omega) \times (W^{m,r}(\Omega))^2$ with $div\, \boldsymbol{\tau} \in \left(W^{m,r'}(\Omega)\right)^2$, there exists $(\boldsymbol{\varsigma}_h, \boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in T_h \times T^{'}_{div,\,h} \times P_h \times U_h$ satisfying

$$
\begin{aligned}
\|\boldsymbol{\varsigma} - \boldsymbol{\varsigma}_h\|_T &\leq Ch^m \|\boldsymbol{\varsigma}\|_{m,r,\Omega}\,, \qquad \forall \boldsymbol{\varsigma} \in (W^{m,r}(\Omega))^{2\times 2}\,, &(4.60)\\
\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{T'} &\leq Ch^m \|\boldsymbol{\tau}\|_{m,r',\Omega}\,, \qquad \forall \boldsymbol{\tau} \in \left(W^{m,r'}(\Omega)\right)^{2\times 2}\,, &(4.61)\\
\|div\,(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\|_{T'} &\leq Ch^m \|div\,\boldsymbol{\tau}\|_{m,r',\Omega}\,, \quad \forall\,(div\,\boldsymbol{\tau}) \in \left(W^{m,r'}(\Omega)\right)^{2}\,, &(4.62)\\
\|q - q_h\|_P &\leq Ch^m \|q\|_{m,r',\Omega}\,, \qquad \forall q \in W^{m,r'}(\Omega)\,, &(4.63)\\
\|\mathbf{v} - \mathbf{v}_h\|_U &\leq Ch^m \|\mathbf{v}\|_{m,r,\Omega}\,, \qquad \forall \mathbf{v} \in (W^{m,r}(\Omega))^{2}\,. &(4.64)
\end{aligned}
$$

Theorem 4.3 *Let* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^2$ *and* $\mathbf{u}_\Gamma \in \left(W^{1-1/r\,,\,r}(\Gamma)\right)^2$. *Let* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T^{'}_{div} \times P \times U \times \mathbb{R}$ *solve (2.13)-(2.15) and let* $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T^{'}_{div,\,h} \times P_h \times U_h \times \mathbb{R}$ *solve (4.2)-(4.4). Assume* $1 \leq m \leq k+1$ *and* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}) \in (W^{m,r}(\Omega))^{2\times 2} \times \left(W^{m,r'}(\Omega)\right)^{2\times 2} \times W^{m,r'}(\Omega) \times (W^{m,r}(\Omega))^2$ *with* $div\,\boldsymbol{\psi} \in \left(W^{m,r'}(\Omega)\right)^2$. *Then there exists a positive constant* C *such that*

$$
\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 \leq C\Bigg\{ & h^{mr} \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \|\boldsymbol{\phi}\|^r_{m,r,\Omega} \\
& + h^{2m}\Bigg(\|\boldsymbol{\phi}\|_{m,r,\Omega} + \|\mathbf{u}\|_{m,r,\Omega} + \|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\,\boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega} \Bigg) \Bigg\}\,, \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P \leq\ & C\, h^m \left(\|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\,\boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega} \right) \\
& + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'}\,, \qquad (4.66)
\end{aligned}
$$

$$
\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \leq C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T\,. \qquad (4.67)
$$

Proof: The result follows directly from Theorem 4.2, Lemma 4.6, and properties (4.60)–(4.64). ■

Remark 4.3 *The extension of Remark 4.1 to these approximation spaces is given by: If* $1/(|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|) \leq C$ *for some constant* $C > 0$ *and* $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_\infty \sim \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T$, *the estimates (4.65)–(4.67) may be written as*

$$
\begin{aligned}
& \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P + \|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \\
& \qquad \leq C\, h^m \Bigg\{ \|\boldsymbol{\phi}\|_{m,r,\Omega} + \|\mathbf{u}\|_{m,r,\Omega} + \|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\,\boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega} \Bigg\}\,. \qquad (4.68)
\end{aligned}
$$

5 Numerical Experiments

In this section we describe numerical experiments that support the theoretical results outlined in Sections 3 and 4. The first example illustrates the theoretical rate of convergence of the solution method and the second example illustrates the computed approximation for a benchmark physical problem. Computations are performed using the `FreeFEM++` finite element software package [19]. All computations below are performed in the lowest-order case ($k = 0$).

5.1 Example 1

For this example (similar to one in [16]) approximations are computed for a Ladyzhenskaya law fluid with $\nu_0 = 0$ and $\nu_1 = 1.0$. The computational domain is $\Omega = [0,2] \times [0,2]$, with $\mathbf{f}$ and $\mathbf{u}_\Gamma$ chosen so that the exact solution of (2.10)-(2.12) is given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad p = x_1 + x_2 \,,$$

with

$$u_1 = -(4.0 - x_1 - x_2)^\alpha \quad \text{and} \quad u_2 = -u_1$$

for α just large enough to ensure $\mathbf{f} = -\,div\,\boldsymbol{\psi} \in W^{\mu-\varepsilon,r'}(\Omega)$. It should be noted that $\alpha = -\frac{2}{r} + r' + \frac{\mu}{r-1} + \varepsilon$ ensures $\mathbf{f} \in W^{\mu,r'}(\Omega)$ for $\varepsilon > 0$.

Computations are performed on uniform meshes of decreasing size h and for selected values of r, α, and μ. For $1 < r < 2$, the resulting system of equations is nonlinear, and a fixed-point iteration is used to compute approximations. The fixed-point iteration is terminated when the pointwise maximum absolute difference in successive approximations falls below 10^{-5}. Results for the velocity, $\mathbf{u}$, the gradient of the velocity, $\boldsymbol{\phi}\,(= \nabla\mathbf{u})$, and the total stress, $\boldsymbol{\psi}$, are shown in Table 5.1.

For this example, $div\,\boldsymbol{\psi} \in W^{\mu-\varepsilon,r'}(\Omega)$ is the most singular of the quantities to be approximated. The observed experimental convergence rate for $\|div\,\boldsymbol{\psi} - div\,\boldsymbol{\psi}_h\|_{0,r'}$ of Ch^μ is in agreement with that predicted by (4.35). The experimental convergence rates observed for $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T$ and $\|\mathbf{u} - \mathbf{u}_h\|_U$ are both better than that given by (4.35).

5.2 Example 2

This example is the benchmark driven cavity problem. Driven cavity flows of power law fluids were computed using a mixed method by Manouzi and Farhloul in [22]. (In [22] the authors explicitly inverted the constitutive equation to obtain $\Phi_\alpha(\boldsymbol{\sigma}) = \nabla\mathbf{u}$, which was used in their formulation.)

For $\Omega = [0,1] \times [0,1]$, we have that $\mathbf{f} = \mathbf{0}$ in Ω, $\mathbf{u}_\Gamma = \mathbf{0}$ on $\Gamma \setminus \Gamma_{\text{top}}$ and $\mathbf{u}_\Gamma = [1 \quad 0]^{\mathrm{T}}$ on Γ_{top}, where Γ_{top} is the portion of the boundary satisfying $0 \le x_1 \le 1$ and $x_2 = 1$. Computations were performed for a power law fluid with $\nu_0 = 1.0$ and selected values of r. Figures 5.1, 5.2, and 5.3 show plots of the streamlines computed for $h = 1/32$ for $r = 2$, $r = 1.5$, and $r = 1.1$, respectively. As the power r in the constitutive law is decreased, we see a movement of the central vortex toward the top of the cavity, corresponding to an increase in viscosity.

	h	$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,r}$	rate	$\|div\,\boldsymbol{\psi} - div\,\boldsymbol{\psi}_h\|_{0,r'}$	rate	$\|\mathbf{u} - \mathbf{u}_h\|_{0,r}$	rate
	1	2.5481		0.8014		37.3797	
$r = 3/2$	1/2	1.2633	1.01	0.4459	0.85	19.6284	0.93
$\mu = 1$	1/4	0.6218	1.02	0.2426	0.88	9.8677	0.99
$\alpha = 11/3$	1/8	0.3080	1.01	0.1299	0.90	4.9294	1.00
	1/16	0.1534	1.01	0.0687	0.92	2.4623	1.00
	1	1.3341		0.2556		10.5023	
$r = 3/2$	1/2	0.6899	0.95	0.1824	0.49	5.3111	0.98
$\mu = 1/2$	1/4	0.3405	1.02	0.1294	0.49	2.6503	1.00
$\alpha = 8/3$	1/8	0.1677	1.02	0.0917	0.50	1.3223	1.00
	1/16	0.0832	1.01	0.0648	0.50	0.6605	1.00
	1	2.6967		1.3410		4721.1800	
$r = 5/4$	1/2	1.3109	1.04	0.7234	0.89	2553.9800	0.89
$\mu = 1$	1/4	0.6325	1.05	0.3833	0.92	1285.9000	0.99
$\alpha = 37/5$	1/8	0.3094	1.03	0.2007	0.93	635.6200	1.02
	1/16	0.1533	1.01	0.1042	0.95	315.0940	1.01
	1	1.4671		0.1661		363.2130	
$r = 5/4$	1/2	0.7461	0.98	0.1176	0.50	191.1110	0.93
$\mu = 1/2$	1/4	0.3604	1.05	0.0832	0.50	94.7585	1.01
$\alpha = 27/5$	1/8	0.1746	1.05	0.0588	0.50	46.8215	1.02
	1/16	0.0860	1.02	0.0416	0.50	23.2479	1.01

Table 5.1: Approximation errors and rates of convergence for Example 1.

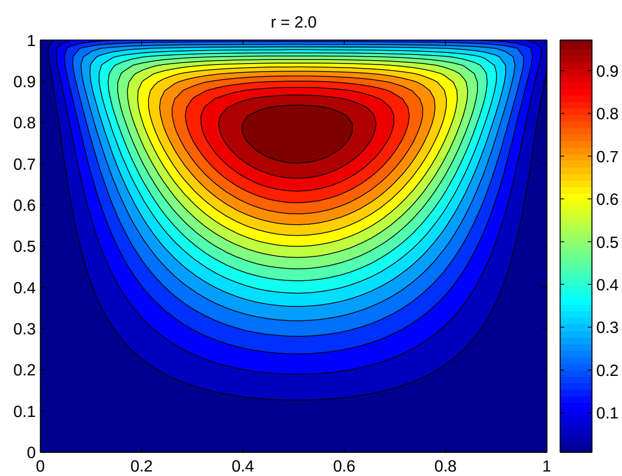

Figure 5.1: Streamlines for $r = 2.0$, driven cavity

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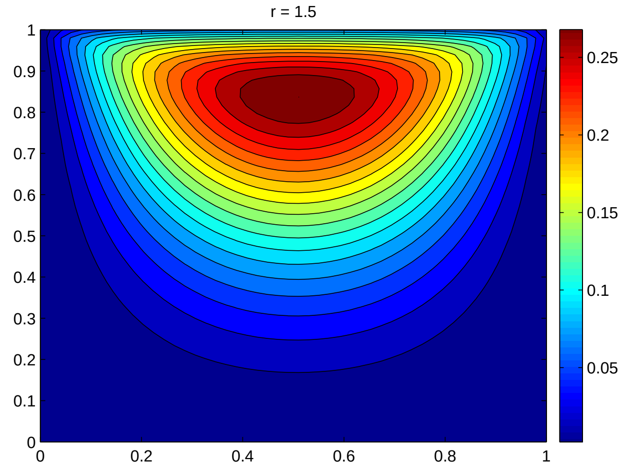

Figure 5.2: Streamlines for $r = 1.5$, driven cavity

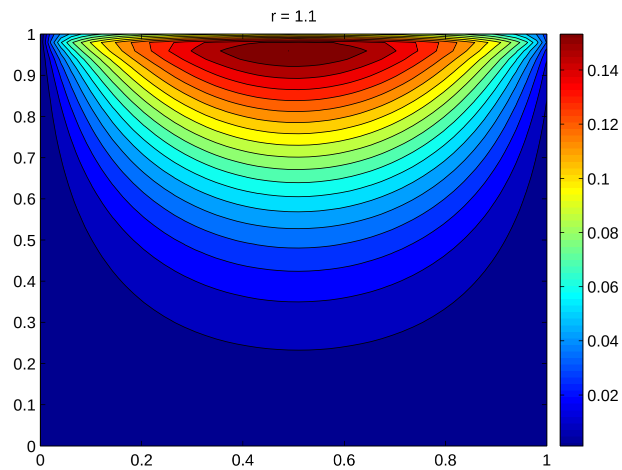

Figure 5.3: Streamlines for $r = 1.1$, driven cavity

Lemma 4.1.

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