Directional Derivatives of Oblique Reflection Maps

Avishai Mandelbaum
School of Industrial Engineering and Management
Technion, Haifa, Israel

and

Kavita Ramanan
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213
kramanan@andrew.cmu.edu

Abstract. Given an oblique reflection map Γ and functions $\psi, \chi \in \mathcal{D}_{\lim}$ (the space of $\mathbb{R}^K$-valued functions that have left and right limits at every point), the directional derivative $\nabla_\chi \Gamma(\psi)$ of Γ along χ, evaluated at ψ, is defined to be the pointwise limit (as $\varepsilon \downarrow 0$) of the family of functions $\nabla_\chi^\varepsilon \Gamma(\psi) \doteq \varepsilon^{-1} [\Gamma(\psi + \varepsilon\chi) - \Gamma(\psi)]$. Directional derivatives are shown to exist and lie in $\mathcal{D}_{\lim}$ for oblique reflection maps associated with reflection matrices of the so-called Harrison-Reiman class. When ψ and χ are continuous, the convergence of $\nabla_\chi^\epsilon \Gamma(\psi)$ to $\nabla_\chi \Gamma(\psi)$ is shown to be uniform on compact subsets of continuity points of the limit $\nabla_\chi \Gamma(\psi)$ and the derivative $\nabla_\chi \Gamma(\psi)$ is shown to have an autonomous characterization as the unique fixed point of an associated map. Motivation for the study of directional derivatives stems from the fact that they arise as functional central limit approximations to time-inhomogeneous queueing networks as well as transient time-homogeneous queueing networks. This work also shows how the various types of discontinuities of the derivative $\nabla_\chi Gamma(\psi)$ are related to the reflection matrix and properties of the function $\Gamma(\psi)$. In the queueing network context, this describes the influence of the topology of the network and the states (of underloading, overloading or criticality) of the various queues in the network on the discontinuities of $\nabla_\chi \Gamma(\psi)$.