Divorcing pressure from viscosity in incompressible Navier-Stokes dynamics

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Abstract

We show that in bounded domains with no-slip boundary conditions, the Navier-Stokes pressure can be determined in a such way that it is strictly dominated by viscosity. As a consequence, in a general domain we can treat the Navier-Stokes equations as a perturbed vector diffusion equation, instead of as a perturbed Stokes system. We illustrate the advantages of this view in a number of ways. In particular, we provide simple proofs of (i) local-in-time existence and uniqueness of strong solutions for an unconstrained formulation of the Navier-Stokes equations, and (ii) the unconditional stability and convergence of difference schemes that are implicit only in viscosity and explicit in both pressure and convection terms, requiring no solution of stationary Stokes systems or inf-sup conditions.

1 Introduction

The pressure term has always created problems for understanding the Navier-Stokes equations of incompressible flow. Pressure plays a role like a Lagrange multiplier to enforce the incompressibility constraint, and this has been a main source of difficulties. Our general aim in this paper is to show that the pressure can be obtained in a way that leads to considerable simplifications in both computation and analysis.

From the computational point of view, typical difficulties are related to the lack of an evolution equation for updating the pressure dynamically and the lack of useful boundary conditions for determining the pressure by solving boundary-value problems. Existing methods able to handle these difficulties are sophisticated and lack the robustness and flexibility that would be useful to address more complex problems. For example, finite element methods have required

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carefully arranging approximation spaces for velocity and pressure to satisfy an inf-sup compatibility condition [GR]. Projection methods too have typically encountered problems related to low-accuracy approximation of the pressure near boundaries [Ch, Te2, OID]. Yet much of the scientific and technological significance of the Navier-Stokes equations derives from their role in the modeling of physical phenomena such as lift, drag, boundary-layer separation and vortex shedding, for which the behavior of the pressure near boundaries is of great importance.

Our main results in this article indicate that in bounded domains with no-slip boundary conditions, the Navier-Stokes pressure can be determined in a such way that it is *strictly dominated by viscosity.* To explain, let us take Ω to be a bounded, connected domain in $\mathbb{R}^N$ $(N \geq 2)$ with C^3 boundary $\Gamma = \partial\Omega$. The Navier-Stokes equations for incompressible fluid flow in Ω with no-slip boundary conditions on Γ take the form

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \nu\Delta\vec{u} + \vec{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\vec{u} = 0 \quad \text{on } \Gamma. \tag{3}$$

Here $\vec{u}$ is the fluid velocity, p the pressure, and ν is the kinematic viscosity coefficient, assumed to be a fixed positive constant.

A standard way to determine p is via the Helmholtz-Hodge decomposition. We let $\mathcal{P}$ denote the Helmholtz projection operator onto divergence-free fields, and recall that it is defined as follows. Given any $\vec{a} \in L^2(\Omega, \mathbb{R}^N)$, there is a unique $q \in H^1(\Omega)$ with $\int_\Omega q = 0$ such that $\mathcal{P}\vec{a} := \vec{a} + \nabla q$ satisfies

$$0 = \int_\Omega (\mathcal{P}\vec{a}) \cdot \nabla\phi = \int_\Omega (\vec{a} + \nabla q) \cdot \nabla\phi \quad \text{for all } \phi \in H^1(\Omega). \tag{4}$$

The pressure p in (1) is determined by taking $\vec{a} = \vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}$. Then (1) is rewritten as

$$\partial_t \vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = 0. \tag{5}$$

In this formulation, solutions formally satisfy $\partial_t(\nabla \cdot \vec{u}) = 0$. Consequently the zero-divergence condition (2) needs to be imposed only on initial data. Nevertheless, the pressure is determined from (5) in principle even for velocity fields that do not respect the incompressibility constraint. However, the dissipation in (5) appears degenerate due to the fact that $\mathcal{P}$ annihilates gradients, so the analysis of (5) is usually restricted to spaces of divergence-free fields.

Alternatives are possible in which the pressure is determined differently when the velocity field has non-zero divergence. Instead of (5), we propose to consider

$$\partial_t \vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = \nu\nabla(\nabla \cdot \vec{u}). \tag{6}$$

Of course there is no difference as long as $\nabla \cdot \vec{u} = 0$. But we argue that (6) enjoys superior stability properties, for two reasons. The first is heuristic. The incompressibility constraint is enforced in a more robust way, because the divergence of velocity satisfies a weak form of the diffusion equation with no-flux (Neumann) boundary conditions — Due to (4), for all appropriate test functions ϕ we have

$$\int_\Omega \partial_t \vec{u} \cdot \nabla\phi = \nu \int_\Omega \nabla(\nabla\cdot\vec{u})\cdot\nabla\phi. \tag{7}$$

Taking $\phi = \nabla\cdot\vec{u}$ we get the dissipation identity

$$\frac{d}{dt}\frac{1}{2}\int_\Omega (\nabla\cdot\vec{u})^2 + \nu\int_\Omega |\nabla(\nabla\cdot\vec{u})|^2 = 0. \tag{8}$$

Due to the Poincaré inequality and the fact that $\int_\Omega \nabla\cdot\vec{u} = 0$, the divergence of velocity is smoothed and decays exponentially in L^2 norm. Naturally, if $\nabla\cdot\vec{u} = 0$ initially, this remains true for all later time, and one has a solution of the standard Navier-Stokes equations (1)–(3).

The second reason is much deeper. To explain, we recast (6) in the form (1) while explicitly identifying the separate contributions to the pressure term made by the convection and viscosity terms. Using the Helmholtz projection operator $\mathcal{P}$, we introduce the *Euler pressure* $p_{\rm E}$ and *Stokes pressure* $p_{\rm S}$ via the relations

$$\mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) = \vec{u}\cdot\nabla\vec{u} - \vec{f} + \nabla p_{\rm E}, \tag{9}$$

$$\mathcal{P}(-\Delta\vec{u}) = -\Delta\vec{u} + \nabla(\nabla\cdot\vec{u}) + \nabla p_{\rm S}. \tag{10}$$

This puts (6) into the form (1) with $p = p_{\rm E} + \nu p_{\rm S}$:

$$\partial_t\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p_{\rm E} + \nu\nabla p_{\rm S} = \nu\Delta\vec{u} + \vec{f}. \tag{11}$$

Identifying the Euler and Stokes pressure terms in this way allows one to focus separately on the difficulties peculiar to each. The Euler pressure is nonlinear, but of lower order. Since the Helmholtz projection is orthogonal, naturally the Stokes pressure satisfies

$$\int_\Omega |\nabla p_{\rm S}|^2 \le \int_\Omega |\Delta\vec{u}|^2 \qquad \text{if } \nabla\cdot\vec{u} = 0. \tag{12}$$

The key observation is that the Stokes pressure term is actually *strictly* dominated by the viscosity term, regardless of the divergence constraint. We regard the following theorem as the main achievement of this paper.

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be a connected bounded domain with C^3 boundary. Then for any $\varepsilon > 0$, there exists $C \ge 0$ such that for all vector fields $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, the Stokes pressure p_s determined by (10) satisfies*

$$\int_\Omega |\nabla p_s|^2 \le \beta\int_\Omega |\Delta\vec{u}|^2 + C\int_\Omega |\nabla\vec{u}|^2, \tag{13}$$

where $\beta = \frac{2}{3} + \varepsilon$.

This theorem allows one to see that (6) is *fully dissipative*. To begin to see why, recall that the Laplace operator $\Delta\colon H^2(\Omega)\cap H^1_0(\Omega)\to L^2(\Omega)$ is an isomorphism, and note that $\nabla p_{\rm S}$ is determined by $\Delta\vec u$ via

$$\nabla p_{\rm S} = (I-\mathcal{P}-\mathcal{Q})\Delta\vec u, \qquad \mathcal{Q}:=\nabla\nabla\cdot\Delta^{-1}. \tag{14}$$

Equation (6) can then be written

$$\begin{aligned}\partial_t\vec u+\mathcal{P}(\vec u\cdot\nabla\vec u-\vec f) &= \nu(\mathcal{P}+\mathcal{Q})\Delta\vec u\\ &= \nu\Delta\vec u-\nu(I-\mathcal{P}-\mathcal{Q})\Delta\vec u.\end{aligned} \tag{15}$$

Theorem 1 will allow us to regard the last term as a controlled perturbation.

We can take $\Delta\vec u=\vec g$ arbitrary in $L^2(\Omega,\mathbb{R}^N)$ and reinterpret Theorem 1 as follows. The last term in (13) can be interpreted as the squared norm of $\vec u$ in $H^1_0(\Omega,\mathbb{R}^N)$, giving the norm of $\vec g$ in the dual space $H^{-1}(\Omega,\mathbb{R}^N)$. Thus the conclusion of Theorem 1 is equivalent to the following estimate, which says that $I-\mathcal{P}$ is approximated by the bounded operator $\mathcal{Q}: L^2(\Omega,\mathbb{R}^N)\to\nabla H^1(\Omega)$:

Corollary 1 *For all vector fields $\vec g\in L^2(\Omega,\mathbb{R}^N)$ we have*

$$\|(I-\mathcal{P}-\mathcal{Q})\vec g\|^2_{L^2}\le\beta\|\vec g\|^2_{L^2}+C\|\vec g\|^2_{H^{-1}}. \tag{16}$$

There are several different ways to interpret the Stokes pressure as we have defined it. In this vein we make a few further observations. First, note that $\mathcal{P}\nabla(\nabla\cdot\vec u)=0$ for all $\vec u$ in $H^2\cap H^1_0(\Omega,\mathbb{R}^N)$, since $\nabla\cdot\vec u$ lies in $H^1(\Omega)$. Then

$$\nabla p_{\rm S}=(I-\mathcal{P})(\Delta\vec u-\nabla(\nabla\cdot u)). \tag{17}$$

Now $\mathcal{A}\vec u:=\Delta\vec u-\nabla(\nabla\cdot\vec u)$ has zero divergence in the sense of distributions and is in $L^2(\Omega,\mathbb{R}^N)$, so $\mathcal{A}\vec u$ lies in the space $H(\mathrm{div};\Omega)$ consisting of vector fields in $L^2(\Omega,\mathbb{R}^N)$ with divergence in $L^2(\Omega)$. By consequence, $\Delta p_{\rm S}=0$ in the sense of distributions and so $\nabla p_{\rm S}$ is in $H(\mathrm{div};\Omega)$ also. By a well-known trace theorem (see [GR], theorem 2.5), the normal components of $\mathcal{A}\vec u$ and $\nabla p_{\rm S}$ belong to the Sobolev space $H^{-1/2}(\Gamma)$, and from the definition of $\mathcal{P}$ we have

$$0=\int_\Omega(\nabla p_{\rm S}-\mathcal{A}\vec u)\cdot\nabla\phi=\int_\Gamma\phi\vec n\cdot(\nabla p_{\rm S}-\mathcal{A}\vec u) \tag{18}$$

for all $\phi\in H^1(\Omega)$. So $p_{\rm S}$ is determined as the zero-mean solution of the Neumann boundary-value problem

$$\Delta p_{\rm S}=0 \quad\text{in }\Omega,\qquad \vec n\cdot\nabla p_{\rm S}=\vec n\cdot(\Delta-\nabla\nabla\cdot)\vec u\quad\text{on }\Gamma. \tag{19}$$

Furthermore, in two and three dimensions, we have

$$\nabla p_{\rm S}=-(I-\mathcal{P})(\nabla\times\nabla\times\vec u) \tag{20}$$

due to the identity $\nabla\times\nabla\times\vec u=-\Delta\vec u+\nabla(\nabla\cdot u)$. Green's formula yields

$$\int_\Gamma\vec n\cdot(\nabla\times\nabla\times\vec u)\phi=\int_\Omega(\nabla\times\nabla\times\vec u)\cdot\nabla\phi=-\int_\Gamma(\nabla\times\vec u)\cdot(\vec n\times\nabla\phi) \tag{21}$$

and so $p_{\rm S}$ (with zero average) is determined through the weak formulation [JL]

$$\int_\Omega \nabla p_{\rm S}\cdot\nabla\phi = \int_\Gamma (\nabla\times\vec{u})\cdot(\vec{n}\times\nabla\phi) \quad \text{for all } \phi\in H^1(\Omega). \tag{22}$$

(Note that $\nabla\times\vec{u}\in H^{1/2}(\Gamma,\mathbb{R}^N)$, and $\vec{n}\times\nabla\phi\in H^{-1/2}(\Gamma,\mathbb{R}^N)$ by a standard trace theorem [GR, Theorem 2.11], since $\nabla\phi$ lies in $H(\mathrm{curl};\Omega)$, the space of vector fields in $L^2(\Omega,\mathbb{R}^N)$ with curl in L^2.)

As indicated by (19) or (22), the Stokes pressure is generated by the *tangential part of vorticity at the boundary.* In the whole space $\mathbb{R}^N$ or in the case of a periodic box without boundary, the Helmholtz projection is exactly given via Fourier transform by $\mathcal{P} = I - \mathcal{Q}$ and the Stokes pressure vanishes. Essentially, the Stokes pressure supplies the correction to this formula induced by the no-slip boundary conditions. By consequence, the results of the present paper should have nothing to do with the global regularity question for the three-dimensional Navier-Stokes equations. But as we have mentioned, many important physical phenomena modeled by the Navier-Stokes equations involve boundaries and boundary-layer effects, and it is exactly here where the Stokes pressure should play a key role.

The unconstrained formulation (6) is not without antecedents in the literature. Orszag et al. [OID] used the boundary condition in (19) as a way of enforcing consistency for a Neumann problem in the context of the projection method. After the results of this paper were completed, we found that the formulation (6) is exactly equivalent to one studied by Grubb and Solonnikov [GS1, GS2]. These authors also study several other types of boundary conditions, and argue that this formulation is parabolic in a nondegenerate sense. They perform an analysis based on a general theory of parabolic pseudo-differential initial-boundary value problems, and also show that for strong solutions, the divergence satisfies a diffusion equation with Neumann boundary conditions.

Due to our Theorem 1, we can treat the Navier-Stokes equations in bounded domains simply as a perturbation of the vector diffusion equation $\partial_t\vec{u} = \nu\Delta\vec{u}$, regarding both the pressure and convection terms as dominated by the viscosity term. This stands in contrast to the usual approach that regards the Navier-Stokes equations as a perturbation of the Stokes system $\partial_t\vec{u} = \nu\Delta\vec{u} - \nabla p$, $\nabla\cdot\vec{u} = 0$. Discussing this usual approach to analysis, Tartar [Ta2, p. 68] comments

> "The difficulty comes from the fact that one does not have adequate boundary conditions for p. …[S]ending the nonlinear term to play with f, one considers the Navier-Stokes equations as a perturbation of Stokes equation, and this is obviously not a good idea, but no one has really found how to do better yet."

By way of seeking to do better, in this paper we exploit Theorem 1 in a number of ways. In particular, we develop a simple proof of local-in-time existence and uniqueness for strong solutions of the unconstrained formulation (11) and consequently for the original Navier-Stokes equations, based upon demonstrating the *unconditional stability* of a simple time-discretization scheme with explicit time-stepping for the pressure and nonlinear convection terms and that is implicit only in the viscosity term.

The discretization that we use is related to a class of extremely efficient numerical methods for incompressible flow [Ti, Pe, JL, GuS]. Thanks to the explicit treatment of the convection and pressure terms, the computation of the momentum equation is completely decoupled from the computation of the kinematic pressure Poisson equation used to enforce incompressibility. No stationary Stokes solver is necessary to handle implicitly differenced pressure terms. For three-dimensional flow in a general domain, the computation of incompressible Navier-Stokes dynamics is basically reduced to solving a heat equation and a Poisson equation at each time step. This class of methods is very flexible and can be used with all kinds of spatial discretization methods [JL], including finite difference, spectral, and finite element methods. The stability properties we establish here should be helpful in analyzing these methods.

Indeed, we will show below that our stability analysis easily adapts to proving unconditional stability and convergence for corresponding fully discrete finite-element methods with C^1 elements for velocity and C^0 elements for pressure. It is important to note that we impose *no* inf-sup compatibility condition between the finite-element spaces for velocity and pressure. The inf-sup condition (also known as the Ladyzhenskaya-Babuška-Brezzi condition) has long been a central foundation for finite-element methods for all saddle-point problems including the stationary Stokes equation. Its beautiful theory is a masterpiece documented in many finite-element books. In the usual approach, the inf-sup condition serves to force the approximate solution to stay close to the divergence-free space where the Stokes operator $\mathcal{P}\Delta$ is dissipative. However, due to the fully dissipative nature of the unconstrained formulation (11) which follows as a consequence of Theorem 1, as far as our stability analysis in section 6 is concerned, the finite-element spaces for velocity and pressure can be completely unrelated.

The proof of Theorem 1 will be carried out in section 3. Important ingredients in the proof are: (i) an estimate near the boundary that is related to boundedness of the Neumann-to-Dirichlet map for boundary values of harmonic functions — this estimate is proved in section 2, see Theorem 2; and (ii) a representation formula for the Stokes pressure in terms of a part of velocity near and parallel to the boundary. In section 2 we also describe the space $\nabla\mathcal{S}_p$ of all possible Stokes pressure gradients (i.e., the range of $I - \mathcal{P} - \mathcal{Q}$). In $\mathbb{R}^3$ it turns out that this is the space of square-integrable vector fields that are *simultaneously gradients and curls* (see Theorem 4 in section 3.5 below).

In section 4 we establish the unconditional stability of the time-discretization scheme, and in section 5 we use this to study existence and uniqueness for strong solutions with no-slip boundary conditions. In section 6 we adapt the stability analysis to prove the unconditional stability and convergence of corresponding C^1/C^0 finite-element methods.

In section 7 we show that Theorem 1 also allows one to treat the linearized equations (an unconstrained version of the Stokes system) easily by analytic semigroup theory. We deal with non-homogeneous boundary conditions in section 8. From these results, in section 9 we deduce an apparently new result for the linear Stokes system, namely an isomorphism theorem between the solution space and a space of data for non-homogeneous side conditions in which only the average flux through the boundary vanishes.

2 Integrated Neumann-to-Dirichlet estimates in tubes

2.1 Notation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^3 boundary Γ. For any $\vec{x} \in \Omega$ we let $\Phi(\vec{x}) = \text{dist}(x, \Gamma)$ denote the distance from x to Γ. For any $s > 0$ we denote the set of points in Ω within distance s from Γ by

$$\Omega_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) < s\}, \tag{23}$$

and set $\Omega_s^c = \Omega \backslash \Omega_s$ and $\Gamma_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) = s\}$. Since Γ is C^3 and compact, there exists $s_0 > 0$ such that Φ is C^3 in Ω_{s_0} and its gradient is a unit vector, with $|\nabla\Phi(\vec{x})| = 1$ for every $\vec{x} \in \Omega_{s_0}$. We let

$$\vec{n}(\vec{x}) = -\nabla\Phi(\vec{x}), \tag{24}$$

then $\vec{n}(\vec{x})$ is the outward unit normal to $\Gamma_s = \partial\Omega_s^c$ for $s = \Phi(\vec{x})$, and $\vec{n} \in C^2(\bar{\Omega}_{s_0}, \mathbb{R}^N)$.

We let $\langle f, g\rangle_\Omega = \int_\Omega fg$ denote the L^2 inner product of functions f and g in Ω, and let $\|\cdot\|_\Omega$ denote the corresponding norm in $L^2(\Omega)$. We drop the subscript on the inner product and norm when the domain of integration is understood in context.

2.2 Statement of results

Our strategy for proving Theorem 1 crucially involves an integrated Neumann-to-Dirichlet–type estimate for harmonic functions in the tubular domains Ω_s for small $s > 0$.

The theorem below contains two estimates of this type. The first, (26) in part (i), can be obtained from a standard Neumann-to-Dirichlet estimate for functions harmonic in Ω, of the form

$$\beta_0 \int_{\Gamma_r} |(I - \vec{n}\vec{n}^t)\nabla p|^2 \le \int_{\Gamma_r} |\vec{n} \cdot \nabla p|^2, \tag{25}$$

by integrating over $r \in (0, s)$, provided one shows that $\beta_0 > 0$ can be chosen independent of r for small $r > 0$. On the first reading, the reader is encouraged to take (26) for granted and proceed directly to section 3.2 at this point; it is only necessary to replace (111) in the proof of Theorem 1 by the corresponding result from (26) to establish that the estimate in Theorem 1 is valid for *some* $\beta < 1$ depending upon Ω.

The second estimate, in part (ii), will be used with β_1 close to 1 to establish the full result in Theorem 1 for any number β greater than $\frac{2}{3}$, independent of the domain.

Theorem 2 *Let Ω be a bounded domain with C^3 boundary. (i) There exists $\beta_0 > 0$ such that for sufficiently small $s > 0$, whenever p is a harmonic function in Ω_s we have*

$$\beta_0 \int_{\Omega_s} |(I - \vec{n}\vec{n}^t)\nabla p|^2 \le \int_{\Omega_s} |\vec{n} \cdot \nabla p|^2. \tag{26}$$

(ii) Let $\beta_1 < 1$. Then for any sufficiently small $s > 0$, whenever p is a harmonic function in Ω_s and p_0 is constant on each component of Ω_s, we have

$$\beta_1 \int_{\Omega_s} |(I - \vec{n}\vec{n}^t)\nabla p|^2 \le \int_{\Omega_s} |\vec{n} \cdot \nabla p|^2 + \frac{24}{s^2} \int_{\Omega_s} |p - p_0|^2. \tag{27}$$

Our proof is motivated by the case of slab domains with periodic boundary conditions in the transverse directions. In this case the analysis reduces to estimates for Fourier series expansions in the transverse variables. For general domains, the idea is to approximate $-\Delta$ in thin tubular domains Ω_s by the Laplace-Beltrami operator on $\Gamma \times (0, s)$. This operator has a direct-sum structure, and we obtain the integrated Neumann-to-Dirichlet–type estimate by separating variables and expanding in series of eigenfunctions of the Laplace-Beltrami operator on Γ. For basic background in Riemannian geometry and the Laplace-Beltrami operator we refer to [Au] and [Ta].

2.3 Harmonic functions on $\Gamma \times (0, s)$

Geometric preliminaries. We consider the manifold $\mathcal{G} = \Gamma \times \mathcal{I}$ with $\mathcal{I} = (0, s)$ as a Riemannian submanifold of $\mathbb{R}^N \times \mathbb{R}$ with boundary $\partial\mathcal{G} = \Gamma \times \{0, s\}$. We let γ denote the metric on Γ induced from $\mathbb{R}^N$, let ι denote the standard Euclidean metric on $\mathcal{I}$, and let g denote the metric on the product space $\mathcal{G}$. Any vector $\vec{a}$ tangent to $\mathcal{G}$ at $z = (y, r)$ has components $\vec{a}_\Gamma$ tangent to Γ at y and $\vec{a}_\mathcal{I}$ tangent to $\mathcal{I}$ at r. For any two such vectors $\vec{a}$ and $\vec{b}$, we have

$$g(\vec{a}, \vec{b}) = \gamma(\vec{a}_\Gamma, \vec{b}_\Gamma) + \iota(\vec{a}_\mathcal{I}, \vec{b}_\mathcal{I}). \tag{28}$$

Given a C^1 function $z = (y, r) \mapsto f(y, r)$ on $\mathcal{G}$, its gradient $\nabla_\mathcal{G} f$ at z is a tangent vector to $\mathcal{G}$ determined from the differential via the metric, through requiring

$$g(\nabla_\mathcal{G} f, \vec{a}) = df \cdot \vec{a} \quad \text{for all } \vec{a} \in T_z\mathcal{G}. \tag{29}$$

By keeping r fixed, the function $y \mapsto f(y, r)$ determines the gradient vector $\nabla_\Gamma f$ tangent to Γ in similar fashion, and by keeping y fixed, the function $r \mapsto f(y, r)$ determines the gradient vector $\nabla_\mathcal{I} f$ tangent to $\mathcal{I}$. These gradients are also the components of $\nabla_\mathcal{G} f$:

$$(\nabla_\mathcal{G} f)_\Gamma = \nabla_\Gamma f, \qquad (\nabla_\mathcal{G} f)_\mathcal{I} = \nabla_\mathcal{I} f.$$

If $u = (u^1, \ldots, u^{N-1}) \mapsto y = (y^1, \ldots, y^N)$ is a local coordinate chart for Γ, the metric is given by $\gamma_{ij}\, du^i\, du^j$ (summation over repeated indices implied) with matrix elements

$$\gamma_{ij} = \frac{\partial y^k}{\partial u^i}\frac{\partial y^k}{\partial u^j}.$$

For $\mathcal{I} \subset \mathbb{R}$ the identity map serves as coordinate chart. In these coordinates the tangent vectors are written (in a form that aids in tracking coordinate changes) as

$$\nabla_\Gamma f = \gamma^{ij}\frac{\partial f}{\partial u^i}\frac{\partial}{\partial u^j}, \qquad \nabla_{\mathcal{I}} f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r}. \tag{30}$$

As usual, the matrix $(\gamma^{ij}) = (\gamma_{ij})^{-1}$. Given two C^1 functions $f, \tilde f$ on $\mathcal{G}$,

$$\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde f) = \gamma^{ij}\frac{\partial f}{\partial u^i}\frac{\partial \tilde f}{\partial u^j}, \qquad \iota(\nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde f) = \frac{\partial f}{\partial r}\frac{\partial \tilde f}{\partial r}. \tag{31}$$

In these coordinates, the (positive) Laplace-Beltrami operators on Γ and $\mathcal{I}$ respectively take the form

$$\Delta_\Gamma f = -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial u_i}\left(\sqrt{\gamma}\gamma^{ij}\frac{\partial}{\partial u_j} f\right), \qquad \Delta_{\mathcal{I}} f = -\frac{\partial^2}{\partial r^2} f, \tag{32}$$

where $\sqrt{\gamma} = \sqrt{\det(\gamma_{ij})}$ is the change-of-variables factor for integration on Γ — if a function f on Γ is supported in the range of the local coordinate chart then

$$\int_\Gamma f(y)\, dS(y) = \int_{\mathbb{R}^{N-1}} f(y(u))\, \sqrt{\gamma}\, du. \tag{33}$$

(Since orthogonal changes of coordinates in $\mathbb{R}^N$ and $\mathbb{R}^{N-1}$ leave the integral invariant, one can understand $\sqrt{\gamma}$ as the product of the singular values of the matrix $\partial y/\partial u$.)

Whenever $f \in H^1(\Gamma)$ and $\tilde f \in H^2(\Gamma)$, one has the integration-by-parts formula

$$\int_\Gamma f \Delta_\Gamma \tilde f = \int_\Gamma \gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde f). \tag{34}$$

One may extend Δ_Γ to be a map from $H^1(\Gamma) \to H^{-1}(\Gamma)$ by using this equation as a definition of $\Delta_\Gamma \tilde f$ as a functional on $H^1(\Gamma)$. In standard fashion [Ta], one finds that $I + \Delta_\Gamma : H^1(\Gamma) \to H^{-1}(\Gamma)$ is an isomorphism, and that $(I + \Delta_\Gamma)^{-1}$ is a compact self-adjoint operator on $L^2(\Gamma)$, hence $L^2(\Gamma)$ admits an orthonormal basis of eigenfunctions of Δ_Γ. Since the coefficient functions in (32) are C^1, standard interior elliptic regularity results ([GT, Theorem 8.8], [Ta, p. 306, Proposition 1.6]) imply that the eigenfunctions belong to $H^2(\Gamma)$. We denote the eigenvalues of Δ_Γ by ν_k^2, $k = 1, 2, \ldots$, with $0 = \nu_1 \le \nu_2 \le \ldots$ where $\nu_k \to \infty$ as $k \to \infty$, and let ψ_k be corresponding eigenfunctions forming an orthonormal basis of $L^2(\Gamma)$. If $\Delta_\Gamma \psi = 0$ then ψ is constant on each component of Γ, so if m is the number of components of Γ, then $0 = \nu_m < \nu_{m+1}$.

In the coordinates $\hat u = (u, r) \mapsto z = (y, r)$ for $\mathcal{G}$, the metric g takes the form $\gamma_{ij}\, du^i\, du^j + dr^2$, and the Laplace-Beltrami operator $\Delta_{\mathcal{G}} = \Delta_\Gamma + \Delta_{\mathcal{I}}$.

Similar considerations as above apply to $\Delta_{\mathcal{G}}$, except $\mathcal{G}$ has boundary. Whenever $f \in H^1_0(\mathcal{G})$ and $\tilde{f} \in H^2(\mathcal{G})$ we have

$$\int_{\mathcal{G}} f \Delta_{\mathcal{G}} \tilde{f} = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde{f}). \tag{35}$$

One extends $\Delta_{\mathcal{G}}$ to map $H^1(\mathcal{G})$ to $H^{-1}(\mathcal{G})$ by using this equation as a definition of $\Delta_{\mathcal{G}} \tilde{f}$ as a functional on $H^1_0(\mathcal{G})$.

We introduce notation for L^2 inner products and norms on $\mathcal{G}$ as follows:

$$\langle f, \tilde{f} \rangle_{\mathcal{G}} = \int_{\mathcal{G}} f \tilde{f} \qquad \|f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} |f|^2, \tag{36}$$

$$\langle \nabla_\Gamma f, \nabla_\Gamma \tilde{f} \rangle_{\mathcal{G}} = \int_{\mathcal{G}} \gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde{f}), \qquad \|\nabla_\Gamma f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} \gamma(\nabla_\Gamma f, \nabla_\Gamma f), \tag{37}$$

$$\langle \nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde{f} \rangle_{\mathcal{G}} = \int_{\mathcal{G}} (\partial_r f)(\partial_r \tilde{f}), \qquad \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} (\partial_r f)^2, \tag{38}$$

$$\langle \nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde{f} \rangle_{\mathcal{G}} = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde{f}) = \langle \nabla_\Gamma f, \nabla_\Gamma \tilde{f} \rangle_{\mathcal{G}} + \langle \nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde{f} \rangle_{\mathcal{G}}, \tag{39}$$

$$\|\nabla_{\mathcal{G}} f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} f) = \|\nabla_\Gamma f\|^2_{\mathcal{G}} + \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}}. \tag{40}$$

Lemma 1 *Suppose* $f \in H^1(\mathcal{G})$ *and* $\Delta_{\mathcal{G}} f = 0$ *on* $\mathcal{G} = \Gamma \times \mathcal{I}$ *where* $\mathcal{I} = (0, s)$. *Then, (i) there exists* $\hat{\beta}_0 \in (0, 1)$ *independent of* f *such that*

$$\hat{\beta}_0 \|\nabla_\Gamma f\|^2_{\mathcal{G}} \le \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}}, \tag{41}$$

and (ii)

$$\|\nabla_\Gamma f\|^2_{\mathcal{G}} \le \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}} + \frac{12}{s^2} \|f - f_0\|^2_{\mathcal{G}} \tag{42}$$

whenever f_0 *is constant on* $\Gamma_i \times (0, s)$ *for every component* Γ_i *of* Γ.

Proof: Suppose $\Delta_{\mathcal{G}} f = 0$ on $\mathcal{G}$. Since the coefficient functions in (32) are C^1, the aforementioned interior elliptic regularity results imply that that $f \in H^2_{\text{loc}}(\mathcal{G})$. For any $r \in (0, s)$, fixing r yields a trace of f in $H^1(\Gamma)$, and as a function of r, we can regard $f = f(y, r)$ as in the space $L^2([a, b], H^2(\Gamma)) \cap H^2([a, b], L^2(\Gamma))$ for any closed interval $[a, b] \subset (0, s)$. Now, for each r we have the $L^2(\Gamma)$-convergent expansion

$$f(y, r) = \sum_k \hat{f}(k, r) \psi_k(y) \tag{43}$$

where

$$\hat{f}(k, r) = \int_\Gamma f(y, r) \psi_k(y)\, dS(y). \tag{44}$$

For each $k \in \mathbb{N}$, the map $r \mapsto \hat{f}(k, r)$ is in $H^2_{\text{loc}}(0, s)$ and

$$\partial_r \hat{f}(k, r) = \int_\Gamma \partial_r f(y, r) \psi_k(y)\, dS(y). \tag{45}$$

For any smooth $\xi \in C_0^\infty(0,s)$, taking $\tilde f(y,r) = \psi_k(y)\xi(r)$ we compute that

$$\nabla_\Gamma \tilde f = \xi(r)\nabla_\Gamma \psi_k, \qquad \partial_r \tilde f = \psi_k \partial_r \xi, \tag{46}$$

and so by (35), (28), and (34), we have

$$\begin{aligned} 0 = \int_{\mathcal G} (\Delta_{\mathcal G} f)\tilde f &= \int_{\mathcal I}\int_\Gamma \Big(\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde f) + (\partial_r f)(\partial_r \tilde f)\Big) \\ &= \int_{\mathcal I} \xi(r) \int_\Gamma \gamma(\nabla_\Gamma f, \nabla_\Gamma \psi_k) + \int_{\mathcal I} (\partial_r \xi) \int_\Gamma (\partial_r f)\psi_k \\ &= \int_{\mathcal I} \xi(r) \int_\Gamma f \Delta_\Gamma \psi_k + \int_{\mathcal I} (\partial_r \xi)\partial_r \hat f(k,r) \\ &= \int_0^s \Big(\xi(r)\nu_k^2 \hat f(k,r) + (\partial_r \xi)\partial_r \hat f(k,r)\Big)\, dr. \end{aligned} \tag{47}$$

Therefore $\hat f(k,\cdot)$ is a weak solution of $\partial_r^2 \hat f = \nu_k^2 \hat f$ in $H^2_{\mathrm{loc}}(0,s)$ and hence is C^2 and it follows that whenever $\nu_k \neq 0$, there exist a_k, b_k such that

$$\hat f(k,r) = a_k \sinh \nu_k \tau + b_k \cosh \nu_k \tau, \qquad \tau = r - s/2. \tag{48}$$

Now

$$\|f\|^2_{\mathcal G} = \sum_k \int_0^s |\hat f(k,r)|^2\, dr, \tag{49}$$

$$\|\nabla_\Gamma f\|^2_{\mathcal G} = \sum_k \int_0^s |\nu_k \hat f(k,r)|^2\, dr, \tag{50}$$

$$\|\nabla_{\mathcal I} f\|^2_{\mathcal G} = \sum_k \int_0^s |\partial_r \hat f(k,r)|^2\, dr. \tag{51}$$

Let $\gamma_k = \int_{-s/2}^{s/2} \sinh^2 \nu_k \tau\, d\tau$. Then γ_k increases with k, and

$$\gamma_k + s = \int_{-s/2}^{s/2} \cosh^2 \nu_k \tau\, d\tau \ge \int_{-s/2}^{s/2} (1 + \nu_k^2 \tau^2)\, d\tau \ge \frac{\nu_k^2 s^3}{12}. \tag{52}$$

Whenever $\nu_k \neq 0$ we get

$$\int_0^s |\hat f(k,r)|^2\, dr = |a_k|^2 \gamma_k + |b_k|^2(\gamma_k + s), \tag{53}$$

$$\int_0^s |\partial_r \hat f(k,r)|^2\, dr = \nu_k^2(|a_k|^2(\gamma_k + s) + |b_k|^2 \gamma_k), \tag{54}$$

and since $\hat\beta_0(\gamma_k + s) \le \gamma_k$ where $\hat\beta_0 = \gamma_{m+1}/(\gamma_{m+1} + s)$, it follows

$$\hat\beta_0 \int_0^s |\nu_k \hat f(k,r)|^2\, dr \le \int_0^s |\partial_r \hat f(k,r)|^2\, dr, \tag{55}$$

$$\int_0^s |\nu_k \hat f(k,r)|^2\, dr \le \int_0^s |\partial_r \hat f(k,r)|^2\, dr + \frac{12}{s^2} \int_0^s |\hat f(k,r)|^2\, dr. \tag{56}$$

The results in (i) and (ii) follow by summing over k. $\square$

2.4 Global coordinates on $\Gamma \times (0, s)$

It will be important for comparison with the Laplacian on Ω_s to coordinatize $\mathcal{G}$ for small $s > 0$ *globally* via the coordinate chart $\Omega_s \to \mathcal{G}$ given by

$$x \mapsto z = (y, r) = (x + \Phi(x)\vec{n}(x), \Phi(x)). \tag{57}$$

In these coordinates, the metric on $\mathcal{G}$ that is inherited from $\mathbb{R}^{N+1}$ has the representation $g_{ij}\, dx^i\, dx^j$ with matrix elements given by

$$g_{ij} = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} + \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}. \tag{58}$$

Let us write $\partial_i = \partial/\partial x^i$ and let $\nabla f = (\partial_1 f, \dots, \partial_N f)$ denote the usual gradient vector in $\mathbb{R}^N$. The components of $\vec{n}$ are $n_i = -\partial_i \Phi$ and so $\partial_i n_j = \partial_j n_i$, meaning the matrix $\nabla\vec{n}$ is symmetric. Since $|\vec{n}|^2 = 1$ we have $n_i \partial_j n_i = 0 = n_i \partial_i n_j$. Then the $N \times N$ matrix

$$\frac{\partial y}{\partial x} = I - \vec{n}\vec{n}^t + \Phi\nabla\vec{n} = (I - \vec{n}\vec{n}^t)(I + \Phi\nabla\vec{n})(I - \vec{n}\vec{n}^t), \tag{59}$$

and the matrix

$$G = (g_{ij}) = (I - \vec{n}\vec{n}^t)(I + \Phi\nabla\vec{n})^2(I - \vec{n}\vec{n}^t) + \vec{n}\vec{n}^t = (I + \Phi\nabla\vec{n})^2. \tag{60}$$

With $\sqrt{g} = \sqrt{\det G}$, the integral of a function f on $\mathcal{G}$ in terms of these coordinates is given by

$$\int_{\mathcal{G}} f = \int_{\Omega_s} f \sqrt{g}\, dx. \tag{61}$$

Given two C^1 functions f, $\tilde{f}$ on $\mathcal{G}$, we claim that the following formulae are valid in the coordinates from (57):

$$g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde{f}) = (\nabla f)^t G^{-1}(\nabla\tilde{f}) = g^{ij}\partial_i f \partial_j \tilde{f}, \tag{62}$$

$$\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde{f}) = (\nabla f)^t (I - \vec{n}\vec{n}^t) G^{-1} (I - \vec{n}\vec{n}^t)(\nabla\tilde{f}), \tag{63}$$

$$\iota(\nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde{f}) = (\vec{n}\cdot\nabla f)(\vec{n}\cdot\nabla\tilde{f}) = (\nabla f)^t \vec{n}\vec{n}^t(\nabla\tilde{f}). \tag{64}$$

Of course (62) simply expresses the metric in the x-coordinates from (57). To prove (64), first note that along any curve $\tau \mapsto x(\tau)$ satisfying $\partial_\tau x = \vec{n}(x)$ we find $\partial_\tau \vec{n}(x) = n_j \partial_j n_i = 0$, so $\vec{n}(x)$ is constant and the curve is a straight line segment. Hence in the chart from (57), $\vec{n}(x) = \vec{n}(y)$ and we have $x = y - r\vec{n}(y)$. Given a C^1 function f then, we find that in these Ω_s-coordinates,

$$\partial_r f(y, r) = (\partial_r x_j)(\partial_j f) = n_j \partial_j f = \vec{n}\cdot\nabla f, \tag{65}$$

and (64) follows from (31). Finally, (63) follows directly from (62) and (64) using (28) — since $\vec{n}\vec{n}^t\nabla\vec{n} = 0$ we have $\vec{n}\vec{n}^t G = \vec{n}\vec{n}^t$ so $\vec{n}\vec{n}^t = \vec{n}\vec{n}^t G^{-1}$ and hence

$$(I - \vec{n}\vec{n}^t)G^{-1}(I - \vec{n}\vec{n}^t) = G^{-1} - \vec{n}\vec{n}^t. \tag{66}$$

2.5 Proof of Theorem 2

Let $\beta_1 < 1$. Suppose $\Delta p = 0$ in Ω_s. We may assume $p \in H^1(\Omega_s)$ without loss of generality by establishing the result in subdomains where $\Phi(x) \in (a,b)$ with $[a,b] \subset (0,s)$ and taking $a \to 0$, $b \to s$. We write

$$p = p_1 + p_2,$$

where $p_1 \in H_0^1(\Omega_s)$ is found by solving a weak form of $\Delta_{\mathcal{G}} p_1 = \Delta_{\mathcal{G}} p$:

$$\langle \nabla_{\mathcal{G}} p_1, \nabla_{\mathcal{G}} \phi \rangle_{\mathcal{G}} = \langle \nabla_{\mathcal{G}} p, \nabla_{\mathcal{G}} \phi \rangle_{\mathcal{G}} \quad \text{for all } \phi \in H_0^1(\Omega_s). \tag{67}$$

For small $s > 0$, $G = (g_{ij}) = I + O(s)$ and $\sqrt{g} = 1 + O(s)$. Since $\langle \nabla p, \nabla p_1 \rangle = 0$, taking $\phi = p_1$ we have

$$\|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}^2 = \int_{\Omega_s} (\nabla p)^t (G^{-1}\sqrt{g} - I) \nabla p_1 \, dx \le Cs \|\nabla p\|_{\Omega_s} \|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}, \tag{68}$$

where C is a constant independent of s. By Poincaré's inequality we also have

$$\|p_1\|_{\mathcal{G}}^2 \le \frac{s^2}{\pi^2} \|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}^2 \tag{69}$$

since the eigenvalues of $\Delta_{\mathcal{G}}$ on the product space $\Gamma \times [0,s]$ with Dirichlet boundary conditions all have the form $\mu = \nu_k^2 + j^2\pi^2/s^2$ for $j, k \in \mathbb{N}$, so that $\mu \ge \pi^2/s^2$.

Let us first prove part (ii). For $0 < \varepsilon < 1$, using (63), (61) and (37) we deduce

$$\begin{aligned}
\|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 &\le (1 + Cs) \|\nabla_\Gamma p\|_{\mathcal{G}}^2 \\
&\le (1 + Cs) \left((1+\varepsilon) \|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 + (1 + \varepsilon^{-1}) \|\nabla_\Gamma p_1\|_{\mathcal{G}}^2 \right) \\
&\le (1 + Cs)(1+\varepsilon) \left(\|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 + \varepsilon^{-1} C^2 s^2 \|\nabla p\|_{\Omega_s}^2 \right). \quad (70)
\end{aligned}$$

Now $p_2 = p - p_1$ satisfies $\Delta_{\mathcal{G}} p_2 = 0$ in Ω_s and $p_2 \in H^1(\mathcal{G})$, hence for any p_0 constant on each component of Ω_s we have

$$\begin{aligned}
\|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 &\le \|\nabla_{\mathcal{I}} p_2\|_{\mathcal{G}}^2 + \frac{12}{s^2} \|p_2 - p_0\|_{\mathcal{G}}^2, \quad (71) \\
\|\nabla_{\mathcal{I}} p_2\|_{\mathcal{G}}^2 &\le (1+\varepsilon) \|\nabla_{\mathcal{I}} p\|_{\mathcal{G}}^2 + (1 + \varepsilon^{-1}) \|\nabla_{\mathcal{I}} p_1\|_{\mathcal{G}}^2 \\
&\le (1+\varepsilon)(1+Cs) \left(\|\vec{n} \cdot \nabla p\|_{\Omega_s}^2 + \varepsilon^{-1} C^2 s^2 \|\nabla p\|_{\Omega_s}^2 \right), \quad (72) \\
\frac{12}{s^2} \|p_2 - p_0\|_{\mathcal{G}}^2 &\le \frac{24}{s^2} \left(\|p - p_0\|_{\mathcal{G}}^2 + \|p_1\|_{\mathcal{G}}^2 \right) \\
&\le \frac{24}{s^2} \|p - p_0\|_{\mathcal{G}}^2 + \frac{24}{\pi^2} \|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}^2 \\
&\le \frac{24}{s^2} (1 + Cs) \|p - p_0\|_{\Omega_s}^2 + C^2 s^2 \|\nabla p\|_{\Omega_s}^2 \quad (73)
\end{aligned}$$

Presuming $Cs < \frac{1}{3}\varepsilon$, assembling these estimates yields

$$\begin{aligned}
\|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 &\le (1+\varepsilon)^4 \left(\|\vec{n} \cdot \nabla p\|_{\Omega_s}^2 + \frac{24}{s^2} \|p - p_0\|_{\Omega_s}^2 + \varepsilon \|\nabla p\|_{\Omega_s}^2 \right) \\
&\le (1+\varepsilon)^5 \left(\|\vec{n} \cdot \nabla p\|_{\Omega_s}^2 + \frac{24}{s^2} \|p - p_0\|_{\Omega_s}^2 + \varepsilon \|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 \right), \quad (74)
\end{aligned}$$

since $|\nabla p|^2 = |\vec{n}\cdot\nabla p|^2 + |(I-\vec{n}\vec{n}^t)\nabla p|^2$. Fixing $\varepsilon > 0$ small so that $(1+\varepsilon)^{-5} - \varepsilon > \beta_1$ proves part (ii).

To prove part (i), instead of (71) we use

$$\hat{\beta}_0 \|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 \le \|\nabla_{\mathcal{I}} p_2\|_{\mathcal{G}}^2 \tag{75}$$

(from part (i) of Lemma 1) together with (70) and (72) and obtain

$$\begin{aligned} \hat{\beta}_0 \|(I-\vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 &\le (1+\varepsilon)^4 \left(\|\vec{n}\cdot\nabla p\|_{\Omega_s}^2 + \frac{2\varepsilon}{9}\|\nabla p\|_{\Omega_s}^2 \right) \\ &\le (1+\varepsilon)^5 \left(\|\vec{n}\cdot\nabla p\|_{\Omega_s}^2 + \varepsilon \|(I-\vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 \right). \end{aligned} \tag{76}$$

Now taking $\varepsilon > 0$ so small that $\varepsilon(1+\varepsilon)^5 < \hat{\beta}_0$ finishes the proof. □

3 Analysis of the Stokes pressure

The main purpose of this section is to prove Theorem 1. We also describe the range of the map $\vec{u} \mapsto \nabla p_{\rm S}$ from velocity fields to Stokes pressure gradients. For motivation for the proof of Theorem 1, the reader can proceed directly to section 3.3 at this point. Here, we first establish some key preliminary results.

3.1 An L^2 estimate

The following L^2 estimate on the Stokes pressure will be used to obtain the full result of Theorem 1 for arbitrary $\beta > \frac{2}{3}$. It is not needed to prove the weaker statement that (13) holds for some $\beta < 1$.

Lemma 2 *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be any bounded connected domain with $C^{1,1}$ boundary. For any $\varepsilon > 0$, there is a constant $C \ge 0$ so that for any $\vec{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$, the associated Stokes pressure p_S defined by* (10) *with zero average satisfies*

$$\|p_s\| \le \varepsilon \|\Delta \vec{u}\| + C\|\vec{u}\|. \tag{77}$$

Proof: For any $\phi \in L^2(\Omega)$, define ψ by

$$\Delta\psi = \phi - \bar{\phi}, \qquad \vec{n}\cdot\nabla\psi\big|_\Gamma = 0, \tag{78}$$

where $\bar{\phi}$ is the average value of ϕ over Ω. Recall $\bar{p}_{\rm S} = 0$. Then,

$$\langle p_{\rm S}, \phi\rangle = \langle p_{\rm S}, \phi - \bar{\phi}\rangle = \langle p_{\rm S}, \Delta\psi\rangle = -\langle \nabla p_{\rm S}, \nabla\psi\rangle. \tag{79}$$

From (22), we know $\langle \nabla p_{\rm S}, \nabla\psi\rangle = \langle \nabla\times\vec{u}, \vec{n}\times\nabla\psi\rangle_\Gamma$ when $N = 2$ or 3. For general N, using the notation $\partial_i := \partial/\partial x_i$ and automatic summation upon repeated indices, from (19) we derive

$$\langle \nabla p_{\rm S}, \nabla\psi\rangle = \frac{1}{2}\int_\Gamma (\partial_j u_i - \partial_i u_j)(n_j\partial_i\psi - n_i\partial_j\psi). \tag{80}$$

Plug (80) into (79), take the absolute value and use the trace theorem to get

$$|\langle p_{\rm S}, \phi\rangle| \le c_0 \|\nabla\vec u\|_{L^2(\Gamma)} \|\nabla\psi\|_{L^2(\Gamma)} \le c_1 \|\vec u\|_{H^{3/2}(\Omega)} \|\psi\|_{H^{3/2}(\Omega)}. \tag{81}$$

By the regularity theory for Poisson's equation (78),

$$\|\psi\|_{H^{3/2}(\Omega)} \le c_2 \|\phi - \bar\phi\| \le c_2 \|\phi\|. \tag{82}$$

By a standard interpolation theorem, for any $\delta > 0$, there is a constant c, so

$$\|\vec u\|_{H^{3/2}(\Omega)} \le \delta \|\Delta \vec u\| + c\|\vec u\|. \tag{83}$$

Plugging (82) and (83) into (81), we get

$$|\langle p_{\rm S}, \phi\rangle| \le \big(\delta\|\Delta\vec u\| + c\|\vec u\|\big) c_1 c_2 \|\phi\|. \tag{84}$$

Thus,

$$\|p_{\rm S}\| = \sup_{\phi\in L^2} \frac{|\langle p_{\rm S}, \phi\rangle|}{\|\phi\|} \le \varepsilon \|\Delta\vec u\| + c c_1 c_2 \|\vec u\|. \qquad \square \tag{85}$$

3.2 Identities at the boundary

A key part of the proof of Theorem 1 involves boundary values of two quantities that involve the decomposition of $\vec u = (I - \vec n\vec n^t)\vec u + \vec n\vec n^t\vec u$ into parts parallel and normal to the boundary. Our goal in this subsection is to prove the following.

Lemma 3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^3. Then for any $\vec u \in H^2(\Omega, \mathbb{R}^N)$ with $\vec u|_\Gamma = 0$, the following is valid on Γ:*

(i) $\nabla\cdot\big((I - \vec n\vec n^t)\vec u\big) = 0$ *in* $H^{1/2}(\Gamma)$.

(ii) $\vec n \cdot (\Delta - \nabla\nabla\cdot)\big(\vec n\vec n^t\vec u\big) = 0$ *in* $H^{-1/2}(\Gamma)$.

The proof will reduce to the case $\vec u \in C^2(\bar\Omega, \mathbb{R}^N)$, due to the following density result.

Lemma 4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{2,\alpha}$ where $0 < \alpha < 1$. Then for any $\vec u \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, there exists a sequence $\vec u_k \in C^{2,\alpha}(\bar\Omega)$ such that $\vec u_k|_\Gamma = 0$ and $\|\vec u_k - \vec u\|_{H^2(\Omega)} \to 0$.*

Proof: Define $\vec f = \Delta\vec u$. Since $\vec f \in L^2$, we can find a sequence $\vec f_k \in C^1(\bar\Omega)$ so that $\|\vec f_k - \vec f\|_{L^2} \to 0$. Construct $\vec u_k$ by solving

$$\Delta \vec u_k = \vec f_k, \qquad \vec u_k\big|_{\partial\Omega} = 0.$$

Classical elliptic regularity theory in Hölder spaces (see [GT], theorem 15.13) says that a unique $\vec u_k$ exists and is in $C^{2,\alpha}(\bar\Omega)$. By standard regularity theory in Sobolev spaces,

$$\|\vec u_k - \vec u\|_{H^2} \le C\|\vec f_k - \vec f\|_{L^2} \to 0. \qquad \square$$

Proof of Lemma 3: To begin, recall $\vec{n} = -\nabla\Phi$. Equality of mixed partial derivatives yields $\partial_j n_i = \partial_i n_j$ for all $i, j = 1, \dots, N$. Together with the fact $n_i n_i = 1$, we infer that for small $s > 0$, throughout Ω_s we have

$$n_i \partial_j n_i = 0 \quad \text{and} \quad n_i \partial_i n_j = 0. \tag{86}$$

(i) First, for any $f \in C^1(\bar{\Omega})$, if $f = 0$ on Γ then $\nabla f \parallel \vec{n}$ on Γ, which means

$$(I - \vec{n}\vec{n}^t)\nabla f = 0, \quad \text{or} \quad (\partial_k - n_k n_j \partial_j) f = 0 \quad \text{for } k = 1, \dots, N. \tag{87}$$

Now suppose $\vec{u} \in C^2(\bar{\Omega}, \mathbb{R}^N)$ with $\vec{u} = 0$ on Γ. Then, after taking derivatives in Ω_s for some $s > 0$ and then taking the trace on Γ, using (87) we get

$$\nabla \cdot \big((I - \vec{n}\vec{n}^t)\vec{u}\big) = \partial_j \big(u_j - n_j n_k u_k\big) = \partial_j u_j - n_j n_k \partial_j u_k = \partial_j u_j - \partial_k u_k = 0.$$

For general $\vec{u} \in H^2(\Omega, \mathbb{R}^N)$ with $\vec{u}|_\Gamma = 0$, the expression $\nabla \cdot \big((I - \vec{n}\vec{n}^t)\vec{u}\big)$ is in $H^1(\Omega_s)$ for small $s > 0$ and hence is in $H^{1/2}(\Gamma)$ by a trace theorem. After approximating $\vec{u}$ using Lemma 4 we obtain the result in (i).

(ii) At first we suppose $\vec{u} \in C^2(\bar{\Omega}, \mathbb{R}^N)$ with $\vec{u} = 0$ on Γ. We claim in fact that for any $f \in C^2(\bar{\Omega})$ with $f\big|_\Gamma = 0$,

$$\vec{n} \cdot (\Delta - \nabla\nabla\cdot)(\vec{n} f) = 0 \quad \text{on } \Gamma. \tag{88}$$

This yields (ii) by taking $f = \vec{n} \cdot \vec{u}$. We prove (88) in two steps.

1. The formula in (i) holds in $C(\Gamma)$ if $\vec{u}$ is C^1. Since $I - \vec{n}\vec{n}^t = (I - \vec{n}\vec{n}^t)^2$, we can use $\vec{u} = (I - \vec{n}\vec{n}^t)\nabla f$ in (i) to find that

$$\nabla \cdot \big((I - \vec{n}\vec{n}^t)\nabla f\big) = 0 \qquad \text{on } \Gamma. \tag{89}$$

2. Using (86) it is easy to verify the following identities in Ω_s:

$$\vec{n} \cdot \Delta(\vec{n} f) = \Delta f + f\, \vec{n} \cdot \Delta \vec{n}, \tag{90}$$

$$\vec{n} \cdot \nabla\nabla \cdot (\vec{n} f) = (\vec{n}\vec{n}^t) : \nabla^2 f + (\nabla \cdot \vec{n})\vec{n} \cdot \nabla f + f\vec{n} \cdot \nabla\nabla \cdot \vec{n}, \tag{91}$$

$$\nabla \cdot (\vec{n}\vec{n}^t \nabla f) = (\vec{n}\vec{n}^t) : \nabla^2 f + (\nabla \cdot \vec{n})\vec{n} \cdot \nabla f. \tag{92}$$

Here $(\vec{n}\vec{n}^t) : \nabla^2 f := n_i n_j \partial_i \partial_j f$. It directly follows that

$$\vec{n} \cdot (\Delta - \nabla\nabla\cdot)(\vec{n} f) = \nabla \cdot (I - \vec{n}\vec{n}^t)\nabla f + f\vec{n} \cdot (\Delta - \nabla\nabla\cdot)\vec{n}. \tag{93}$$

Using this with (89) proves (88), and establishes (ii) when $\vec{u} \in C^2(\bar{\Omega})$ with $\vec{u} = 0$ on Γ.

To establish (ii) for arbitrary $\vec{u} \in H^2(\Omega, \mathbb{R}^N)$, we restrict to Ω_s for small s and let $\vec{a} = (\Delta - \nabla\nabla\cdot)(\vec{n}\vec{n}^t\vec{u})$. Then $\vec{a} \in L^2(\Omega_s, \mathbb{R}^N)$ and $\nabla \cdot \vec{a} = 0$ in the sense of distributions, so $\vec{a} \in H(\text{div}; \Omega_s)$ and a well-known trace theorem (see [GR], theorem 2.5) yields that the map $H^2(\Omega_s, \mathbb{R}^N) \to H(\text{div}; \Omega_s) \to H^{-1/2}(\Gamma)$ given by $\vec{u} \mapsto \vec{a} \mapsto \vec{n} \cdot \vec{a}$ is continuous. To conclude the proof, simply apply the approximation lemma above to infer $\vec{n} \cdot \vec{a}|_\Gamma = 0$. $\square$

3.3 Identities for the Stokes pressure

Given $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, recall that $\mathcal{P}(\nabla\nabla \cdot \vec{u}) = 0$, so that the Stokes pressure defined in (10) satisfies

$$\nabla p_{\mathrm{S}} = \Delta\vec{u} - \nabla\nabla \cdot \vec{u} - \mathcal{P}\Delta\vec{u} = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}. \tag{94}$$

Also recall that whenever $\vec{a} \in L^2(\Omega, \mathbb{R}^N)$ and $\nabla \cdot \vec{a} \in L^2(\Omega)$, $\vec{n} \cdot \vec{a} \in H^{-1/2}(\Gamma)$ by the trace theorem for $H(\mathrm{div}; \Omega)$. If $\nabla \cdot \vec{a} = 0$ and $\vec{n} \cdot \vec{a}|_\Gamma = 0$, then we have $\langle \vec{a}, \nabla\phi \rangle = 0$ for all $\phi \in H^1(\Omega)$ and this means $(I - \mathcal{P})\vec{a} = 0$. Thus, the Stokes pressure is not affected by any part of the velocity field that contributes nothing to $\vec{n} \cdot \vec{a}|_\Gamma$ where $\vec{a} = (\Delta - \nabla\nabla\cdot)\vec{u}$. Indeed, this means that the Stokes pressure is not affected by the part of the velocity field in the interior of Ω away from the boundary, nor is it affected by the normal component of velocity near the boundary, since $\vec{n} \cdot (\Delta - \nabla\nabla\cdot)(\vec{n}\vec{n}^t\vec{u})|_\Gamma = 0$ by Lemma 3.

This motivates us to focus on the part of velocity near and parallel to the boundary. We make the following decomposition. Let $\rho : [0, \infty) \to [0, 1]$ be a smooth decreasing function with $\rho(t) = 1$ for $t < \frac{1}{2}$ and $\rho(t) = 0$ for $t \geq 1$. For small $s > 0$, the cutoff function given by $\xi(x) = \rho(\Phi(x)/s)$ is C^3, with $\xi = 1$ when $\Phi(x) < \frac{1}{2}s$ and $\xi = 0$ when $\Phi(x) \geq s$. Then we can write

$$\vec{u} = \vec{u}_\perp + \vec{u}_\parallel \tag{95}$$

where

$$\vec{u}_\perp = (1 - \xi)\vec{u} + \xi\vec{n}\vec{n}^t\vec{u}, \qquad \vec{u}_\parallel = \xi(I - \vec{n}\vec{n}^t)\vec{u}. \tag{96}$$

Since $\vec{u}_\perp = (\vec{n}\vec{n}^t)\vec{u}$ in $\Omega_{s/2}$, with $\vec{a}_\perp = (\Delta - \nabla\nabla\cdot)\vec{u}_\perp$ we have

$$\vec{a}_\perp \in L^2(\Omega, \mathbb{R}^N), \qquad \nabla \cdot \vec{a}_\perp = 0 \quad \text{and} \quad \vec{n} \cdot \vec{a}_\perp|_\Gamma = 0 \tag{97}$$

by Lemma 3(ii). Hence $\langle \vec{a}_\perp, \nabla\phi \rangle = 0$ for all $\phi \in H^1(\Omega)$, that is,

$$(I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}_\perp = 0. \tag{98}$$

Combining this with (94) and (95) proves part (i) of the following.

Lemma 5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^3 boundary, and let $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$. Let p_s and $\vec{u}_\parallel$ be defined as in (94) and (96) respectively. Then*

(i) The Stokes pressure is determined by $\vec{u}_\parallel$ according to the formula

$$\nabla p_s = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}_\parallel. \tag{99}$$

(ii) For any $q \in H^1(\Omega)$ that satisfies $\Delta q = 0$ in the sense of distributions,

$$\langle \Delta\vec{u}_\parallel - \nabla p_s, \nabla q \rangle = 0. \tag{100}$$

(iii) In particular we can let $q = p_s$ in (ii), so $\langle \Delta\vec{u}_\parallel - \nabla p_s, \nabla p_s \rangle = 0$ and

$$\|\Delta\vec{u}_\parallel\|^2 = \|\Delta\vec{u}_\parallel - \nabla p_s\|^2 + \|\nabla p_s\|^2. \tag{101}$$

Proof: We already proved (i). For (ii), note by Lemma 3(i) we have

$$\nabla\cdot\vec u_\parallel|_\Gamma = 0, \tag{102}$$

so $\nabla\cdot\vec u_\parallel \in H^1_0(\Omega)$, thus $\langle\nabla\nabla\cdot\vec u_\parallel,\nabla q\rangle = -\langle\nabla\cdot\vec u_\parallel,\Delta q\rangle = 0$. Now (i) entails

$$\langle\nabla p_{\rm s},\nabla q\rangle = \langle\Delta\vec u_\parallel,\nabla q\rangle. \tag{103}$$

This proves (ii), and then (iii) follows by the L^2 orthogonality. □

3.4 Proof of Theorem 1

Let $\varepsilon > 0$ and $\beta = \frac{2}{3}+\varepsilon$. We fix $\beta_1 < 1$ such that $1+\varepsilon_0 := \beta(1+\frac12\beta_1^2) > 1$, and fix $s > 0$ small so Theorem 2 (ii) applies in Ω_s with this β_1. Let $\vec u \in H^2\cap H^1_0(\Omega,\mathbb{R}^N)$ and define the Stokes pressure $\nabla p_{\rm s}$ by (10) and the decomposition $\vec u = \vec u_\perp + \vec u_\parallel$ as in the previous subsection. Then by part (iii) of Lemma 5 we have

$$\|\Delta\vec u\|^2 = \|\Delta\vec u_\perp\|^2 + 2\langle\Delta\vec u_\perp,\Delta\vec u_\parallel\rangle + \|\Delta\vec u_\parallel-\nabla p_{\rm s}\|^2 + \|\nabla p_{\rm s}\|^2. \tag{104}$$

We will establish the Theorem with the help of two further estimates.
Claim 1: For any $\varepsilon_1 > 0$, there exists a constant $C_1 > 0$ independent of $\vec u$ such that

$$\langle\Delta\vec u_\perp,\Delta\vec u_\parallel\rangle \ge -\varepsilon_1\|\Delta\vec u\|^2 - C_1\|\nabla\vec u\|^2. \tag{105}$$

Claim 2: For any $\varepsilon_1 > 0$ there exists a constant C_2 independent of $\vec u$ such that

$$\|\Delta\vec u_\parallel-\nabla p_{\rm s}\|^2 \ge \frac{\beta_1^2}{2}\|\nabla p_{\rm s}\|^2 - \varepsilon_1\|\Delta\vec u\|^2 - C_2\|\nabla\vec u\|^2. \tag{106}$$

Proof of claim 1: From the definitions in (96), we have

$$\Delta\vec u_\perp = \xi\vec n\vec n^t\Delta\vec u + (1-\xi)\Delta\vec u + R_1, \qquad \Delta\vec u_\parallel = \xi(I-\vec n\vec n^t)\Delta\vec u + R_2, \tag{107}$$

where $\|R_1\|+\|R_2\| \le C\|\nabla\vec u\|$ with C independent of $\vec u$. Since $I-\vec n\vec n^t = (I-\vec n\vec n^t)^2$,

$$\big(\xi\vec n\vec n^t\Delta\vec u + (1-\xi)\Delta\vec u\big)\cdot\big(\xi(I-\vec n\vec n^t)\Delta\vec u\big) = 0 + \xi(1-\xi)|(I-\vec n\vec n^t)\Delta\vec u|^2 \ge 0.$$

This means the leading term of $\langle\Delta\vec u_\perp,\Delta\vec u_\parallel\rangle$ is non-negative. Using the inequality $|\langle a,b\rangle| \le (\varepsilon_1/C)\|a\|^2 + (4C/\varepsilon_1)\|b\|^2$ and the bounds on R_1 and R_2 to estimate the remaining terms, it is easy to obtain (105).
Proof of claim 2: Recall that $\vec u_\parallel$ is supported in Ω_s, and note

$$\Delta\vec u_\parallel = \xi(I-\vec n\vec n^t)\Delta\vec u + R_3 \tag{108}$$

where $\|R_3\| \le C\|\nabla\vec u\|$. Since $\vec n\cdot(I-\vec n\vec n^t)\Delta\vec u = 0$ we find

$$\|\vec n\cdot\Delta\vec u_\parallel\|_{\Omega_s} \le C_2\|\nabla\vec u\| \tag{109}$$

with $C_2 > 0$ independent of $\vec{u}$. We use $|a+b|^2 \geq (1-\varepsilon_2)|b|^2 - |a|^2/\varepsilon_2$ to get

$$\begin{aligned}\|\Delta\vec{u}_\parallel - \nabla p_\mathrm{s}\|^2_\Omega &\geq \int_{\Omega_s^c} |\nabla p_\mathrm{s}|^2 + \int_{\Omega_s} |\vec{n}\cdot(\Delta\vec{u}_\parallel - \nabla p_\mathrm{s})|^2 \\ &\geq \int_{\Omega_s^c} |\nabla p_\mathrm{s}|^2 + (1-\varepsilon_2)\int_{\Omega_s} |\vec{n}\cdot\nabla p_\mathrm{s}|^2 - \frac{1}{\varepsilon_2}\int_{\Omega_s} |\vec{n}\cdot\Delta\vec{u}_\parallel|^2. \qquad (110)\end{aligned}$$

Next we use part (ii) of Theorem 2 with $p_0 = 0$ and with $\beta_1 \int_{\Omega_s} |\vec{n}\cdot\nabla p|^2$ added to both sides, together with Lemma 2 and Poincaré's inequality, to deduce that

$$\frac{\beta_1}{2}\int_{\Omega_s} |\nabla p_\mathrm{s}|^2 \leq \int_{\Omega_s} |\vec{n}\cdot\nabla p_\mathrm{s}|^2 + \varepsilon_1 \int_{\Omega_s} |\Delta\vec{u}|^2 + C\int_{\Omega_s} |\nabla\vec{u}|^2. \qquad (111)$$

Taking $1-\varepsilon_2 = \beta_1$ and combining (109), (110) and (111) establishes Claim 2.

Now we conclude the proof of the theorem. Combining the two claims with (104), we get

$$(1+3\varepsilon_1)\|\Delta\vec{u}\|^2 \geq \left(1 + \frac{\beta_1^2}{2}\right)\|\nabla p_\mathrm{s}\|^2 - (C_2 + 2C_1)\|\nabla\vec{u}\|^2. \qquad (112)$$

Multiplying by β and taking $\varepsilon_1 > 0$ so that $3\varepsilon_1 < \varepsilon_0$ concludes the proof. □

3.5 The space of Stokes pressures

According to (21)–(22), the space of Stokes pressures, obtainable via (10) from velocity fields $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, can be characterized as the space

$$\mathcal{S}_p := \{p \in H^1(\Omega)/\mathbb{R} \mid \Delta p = 0 \text{ in } \Omega \text{ and } \vec{n}\cdot\nabla p|_\Gamma \in \mathcal{S}_\Gamma\}, \qquad (113)$$

where $\mathcal{S}_\Gamma$ is the subspace of $H^{-1/2}(\Gamma)$ given by

$$\mathcal{S}_\Gamma := \{f = \vec{n}\cdot(\Delta - \nabla\nabla\cdot)\vec{u}|_\Gamma \mid \ \vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)\}. \qquad (114)$$

The Stokes pressure p with zero average is determined uniquely by $f = \vec{n}\cdot\nabla p|_\Gamma \in \mathcal{S}_\Gamma$, with $\|p\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1/2}(\Gamma)}$ by the Lax-Milgram lemma.

The space $\mathcal{S}_\Gamma$ may be characterized as follows.

Theorem 3 *Assume $\Omega \subset \mathbb{R}^N$ is a bounded, connected domain and its boundary Γ is of class C^3. Denote the connected components of Γ by Γ_i, $i = 1, \ldots, m$. Then*

$$\mathcal{S}_\Gamma = \{f \in H^{-1/2}(\Gamma) \mid \int_{\Gamma_i} f = 0 \ \ \textit{for } i = 1, \ldots, m\},$$

and moreover, the map $\vec{u} \mapsto \vec{n}\cdot(\Delta - \nabla\nabla\cdot)\vec{u}|_\Gamma$ from $H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$ to $\mathcal{S}_\Gamma$ admits a bounded right inverse.

Proof. First we check the necessity of the integral conditions. Let $u \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$ and let $f = \vec{n}\cdot(\Delta - \nabla\nabla\cdot)\vec{u}|_\Gamma$. For each connected component Γ_i of Γ, there is an $s_i > 0$ small enough and a smooth cut-off function ρ_i

defined in Ω which satisfies $\rho_i(x) = 1$ when $\mathrm{dist}(x, \Gamma_i) < s_i$ and $\rho_i(x) = 0$ when $\mathrm{dist}(x, \Gamma_j) < s_i$ for all $j \neq i$. Let $\vec{a} = (\Delta - \nabla\nabla\cdot)(\rho_i \vec{u})$. Then $\vec{a} \in L^2(\Omega, \mathbb{R}^N)$ and $\nabla \cdot \vec{a} = 0$, so

$$\int_{\Gamma_i} f = \int_{\Gamma} \vec{n} \cdot \vec{a} = \int_{\Omega} \nabla \cdot \vec{a} = 0. \tag{115}$$

Next, let $f \in H^{-1/2}(\Gamma)$ with $\int_{\Gamma_i} f = 0$ for all i. Treating each boundary component separately, we can then solve the problem

$$\Delta_\Gamma \psi = -f \quad \text{on } \Gamma, \qquad \int_{\Gamma_i} \psi = 0 \quad \text{for } i = 1, \ldots, m, \tag{116}$$

where Δ_Γ is the (positive) Laplace-Beltrami operator on Γ. Denote the mapping $f \mapsto \psi$ by T. Then $T\colon H^{-1}(\Gamma) \to H^1(\Gamma)$ is bounded ([Au, theorem 1.71, theorem 4.7], [Ta, p. 306, Proposition 1.6]). Also $T\colon L^2(\Gamma) \to H^2(\Gamma)$ is bounded, by elliptic regularity theory [Ta, p. 306, Proposition 1.6]. So, interpolation implies (see [LM, vol I, p. 37, Remark 7.6])

$$\|\psi\|_{H^{3/2}(\Gamma)} \leq C \|f\|_{H^{-1/2}(\Gamma)}. \tag{117}$$

Now by an inverse trace theorem [RR, Theorem 6.109], there exists a map $\psi \mapsto q \in H^3(\Omega)$ with

$$q = 0 \ \text{ and } \ \vec{n} \cdot \nabla q = \psi \quad \text{on } \Gamma, \qquad \|q\|_{H^3(\Omega)} \leq C \|\psi\|_{H^{3/2}(\Gamma)}. \tag{118}$$

We may assume q is supported in a small neighborhood of Γ. Define

$$\vec{u} = (I - \vec{n}\vec{n}^t) \nabla q. \tag{119}$$

Then $f \mapsto \vec{u}$ is bounded from $\mathcal{S}_\Gamma$ to $H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$. We claim

$$\vec{n} \cdot (\Delta - \nabla\nabla\cdot) \vec{u} = f \quad \text{on } \Gamma. \tag{120}$$

The proof of this claim amounts to showing, by calculations similar to those in the proof of Lemma 3, that the normal derivative $\vec{n} \cdot \nabla$ and normal projection $\vec{n}\vec{n}^t$ commute on the boundary with the tangential gradient and divergence operators $(I - \vec{n}\vec{n}^t)\nabla$ and $\nabla \cdot (I - \vec{n}\vec{n}^t)$ for the functions involved.

First, since $\vec{n} \cdot \vec{u} = 0$, by expanding $\Delta(\vec{n} \cdot \vec{u})$ we get

$$\vec{n} \cdot \Delta \vec{u} = -(\Delta \vec{n}) \cdot \vec{u} - 2 \nabla \vec{n} : \nabla \vec{u} = 0 \quad \text{on } \Gamma, \tag{121}$$

since for each i, ∇n_i is tangential and ∇u_i is normal to Γ — indeed, using $\partial_j n_i = \partial_i n_j$ and (86) and (87), we have that

$$\nabla \vec{n} : \nabla \vec{u} = (\partial_j n_i)(\partial_j u_i) = (\partial_i n_j)(n_j n_k \partial_k u_i) = 0 \quad \text{on } \Gamma. \tag{122}$$

Next we calculate in Ω that

$$\vec{n} \cdot \nabla\nabla \cdot \vec{u} = \nabla \cdot (\vec{n} \cdot \nabla \vec{u}) - \nabla \vec{n} : \nabla \vec{u}. \tag{123}$$

Note that $\vec{n}\cdot\nabla(\vec{n}\vec{n}^t) = 0$ by (86), so $\vec{n}\cdot\nabla$ commutes with $I - \vec{n}\vec{n}^t$ in Ω. Then since $\vec{u} = (I - \vec{n}\vec{n}^t)\vec{u}$ from (119) we get

$$\vec{n}\cdot\nabla\vec{u} = (I - \vec{n}\vec{n}^t)(\vec{n}\cdot\nabla)\vec{u} = (I - \vec{n}\vec{n}^t)(\vec{n}\cdot\nabla)\nabla q. \tag{124}$$

Now

$$(\vec{n}\cdot\nabla)\nabla q = \nabla(\vec{n}\cdot\nabla q) - \vec{a} \tag{125}$$

where

$$a_i = (\partial_i n_j)(\partial_j q) = (\partial_j n_i)(\partial_j q) \tag{126}$$

This quantity lies in $H^2(\Omega)$ and vanishes on Γ since $\nabla q = (\vec{n}\vec{n}^t)\nabla q$ on Γ. (This can be proved by approximation using Lemma 4.) Using part (i) of Lemma 3, we have that $\nabla\cdot(I - \vec{n}\vec{n}^t)\vec{a} = 0$ on Γ. Combining (121)–(125) we conclude that

$$\vec{n}\cdot(\Delta - \nabla\nabla\cdot)\vec{u} = -\nabla\cdot(I - \vec{n}\vec{n}^t)\nabla(\vec{n}\cdot\nabla q) \quad \text{on } \Gamma. \tag{127}$$

But it is well known that at any point x where $\Phi(x) = r \in (0, s)$, for any smooth function ϕ on Ω_s,

$$\nabla\cdot(I - \vec{n}\vec{n}^t)\nabla\phi = \Delta\phi - (\nabla\cdot\vec{n})(\vec{n}\cdot\nabla\phi) - (\vec{n}\cdot\nabla)^2\phi = -\Delta_{\Gamma_r}(\phi|_{\Gamma_r}). \tag{128}$$

where Δ_{Γ_r} is the Laplace-Beltrami operator on Γ_r. So taking $r \to 0$ we see that the right hand side of (127) is exactly $-\Delta_\Gamma(\vec{n}\cdot\nabla q|_\Gamma)$. So by (116) and (118) we have established the claim in (120). This finishes the proof. □

Remark 1. Given a velocity field $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^3)$, the associated Stokes pressure is determined by the normal component at the boundary of the curl of the vorticity $\omega = \nabla\times\vec{u}$, which is a vector field in $H^1(\Omega, \mathbb{R}^3)$. A question related to Theorem 3 is whether the space $\mathcal{S}_\Gamma$ of such boundary values $\vec{n}\cdot\nabla\times\omega$ is constrained in any way, as compared to the space of boundary values $\vec{n}\cdot\nabla\times\vec{v}$ where $\vec{v} \in H^1(\Omega, \mathbb{R}^3)$ is arbitrary.

The answer is no. In [Te1, Appendix I, Proposition 1.3], Temam proves

$$\nabla\times H^1(\Omega, \mathbb{R}^3) = \{\vec{g} \in L^2(\Omega, \mathbb{R}^3) \mid \nabla\cdot\vec{g} = 0, \int_{\Gamma_i} \vec{n}\cdot\vec{g} = 0 \ \forall i\}. \tag{129}$$

Clearly $\mathcal{S}_\Gamma \subset \vec{n}\cdot\nabla\times H^1(\Omega, \mathbb{R}^3)$ by (114). For the other direction, let $\vec{v} \in H^1(\Omega, \mathbb{R}^3)$ be arbitrary, and let $f = \vec{n}\cdot\nabla\times\vec{v}|_\Gamma$. By (129) or otherwise, $f \in H^{-1/2}(\Gamma)$ and $\int_{\Gamma_i} f = 0$ for all i, hence $f \in \mathcal{S}_\Gamma$. This shows that for $N = 3$,

$$\mathcal{S}_\Gamma = \vec{n}\cdot\nabla\times H^1(\Omega, \mathbb{R}^3). \tag{130}$$

A related point is that for $N = 3$, the space of Stokes pressure gradients $\nabla\mathcal{S}_p$ can be characterized as *the space of simultaneous gradients and curls.*

Theorem 4 *Assume $\Omega \subset \mathbb{R}^3$ is a bounded, connected domain and its boundary Γ is of class C^3. Then*

$$\nabla\mathcal{S}_p = \nabla H^1(\Omega) \cap \nabla\times H^1(\Omega, \mathbb{R}^3). \tag{131}$$

Proof. Indeed, $\nabla \mathcal{S}_p \subset \nabla \times H^1$ by (129) and Theorem 3. On the other hand, if $\vec{g} = \nabla \times \vec{v} = \nabla p$ then $\Delta p = \nabla \cdot \vec{g} = 0$ and $\vec{n} \cdot \nabla p|_\Gamma \in \mathcal{S}_\Gamma$ by (129) and Theorem 3, so $\nabla p \in \nabla \mathcal{S}_p$. □

Remark 2. In the book [Te1] (see Theorem 1.5) Temam establishes the orthogonal decomposition $L^2(\Omega, \mathbb{R}^N) = H \oplus H_1 \oplus H_2$, which means that for any $g \in L^2(\Omega, \mathbb{R}^N)$,

$$\vec{g} = \mathcal{P}\vec{g} + \nabla q + \nabla \Delta^{-1} \nabla \cdot \vec{g}, \tag{132}$$

where q satisfies $\Delta q = 0$ and $\vec{n} \cdot \nabla q|_\Gamma = \vec{n} \cdot (\vec{g} - \nabla \Delta^{-1} \nabla \cdot \vec{g})$. By contrast, we have shown

$$\vec{g} = \mathcal{P}\vec{g} + \nabla p + \nabla \nabla \cdot \Delta^{-1} \vec{g} \tag{133}$$

where p satisfies $\Delta p = 0$ and $\vec{n} \cdot \nabla p|_\Gamma = \vec{n} \cdot (\vec{g} - \nabla \nabla \cdot \Delta^{-1} \vec{g})$, i.e., p is the Stokes pressure associated with $\Delta^{-1}\vec{g}$. Thus the map $\vec{g} \mapsto \nabla p - \nabla q$ is the commutator $\nabla \Delta^{-1} \nabla \cdot - \nabla \nabla \cdot \Delta^{-1}$. The decomposition (132) is orthogonal, and q satisfies $\left\langle \vec{n} \cdot \nabla q, 1 \right\rangle_\Gamma = 0$. In our decomposition (133), the gradient terms are not orthogonal, but the Stokes pressure term enjoys the bounds stated in Corollary 1, and if Γ is not connected, it has the extra property that $\left\langle \vec{n} \cdot \nabla p, 1 \right\rangle_{\Gamma_i} = 0$ for every i.

4 Unconditional stability of time discretization with pressure explicit

In this section we exploit Theorem 1 to establish the unconditional stability of a simple time discretization scheme for the initial-boundary-value problem for (6), our unconstrained formulation of the Navier-Stokes equations. We focus here on the case of two and three dimensions. In subsequent sections we shall proceed to prove an existence and uniqueness theorem based on this stability result.

Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary Γ of class C^3. We consider the initial-boundary-value problem

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p_{\mathrm{E}} + \nu \nabla p_{\mathrm{S}} = \nu \Delta \vec{u} + \vec{f} \qquad (t > 0,\ x \in \Omega), \tag{134}$$

$$\vec{u} = 0 \qquad (t \geq 0,\ x \in \Gamma), \tag{135}$$

$$\vec{u} = \vec{u}_{\mathrm{in}} \qquad (t = 0,\ x \in \Gamma). \tag{136}$$

We assume $\vec{u}_{\mathrm{in}} \in H^1_0(\Omega, \mathbb{R}^N)$ and $\vec{f} \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$ for some given $T > 0$. As before, the Euler and Stokes pressures p_{E} and p_{S} are defined by the relations

$$\mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) = \vec{u} \cdot \nabla \vec{u} - \vec{f} + \nabla p_{\mathrm{E}}, \tag{137}$$

$$\mathcal{P}(-\Delta \vec{u}) = -\Delta \vec{u} + \nabla(\nabla \cdot \vec{u}) + \nabla p_{\mathrm{S}}. \tag{138}$$

Theorem 1 tells us that the Stokes pressure can be strictly controlled by the viscosity term. This allows us to treat the pressure term explicitly, so that the update of pressure is decoupled from that of velocity. This can make corresponding fully discrete numerical schemes very efficient (see [JL]). Here, through Theorem 1, we will prove that the following spatially continuous time discretization scheme has surprisingly good stability properties:

$$\frac{\vec{u}^{n+1}-\vec{u}^n}{\Delta t} - \nu\Delta\vec{u}^{n+1} = \vec{f}^n - \vec{u}^n\cdot\nabla\vec{u}^n - \nabla p_{\rm E}^n - \nu\nabla p_{\rm S}^n, \tag{139}$$

$$\nabla p_{\rm E}^n = (I-\mathcal{P})(\vec{f}^n - \vec{u}^n\cdot\nabla\vec{u}^n), \tag{140}$$

$$\nabla p_{\rm S}^n = (I-\mathcal{P})\Delta\vec{u}^n - \nabla(\nabla\cdot\vec{u}^n), \tag{141}$$

$$\vec{u}^n\big|_\Gamma = 0. \tag{142}$$

We set

$$\vec{f}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \vec{f}(t)\,dt, \tag{143}$$

and take $\vec{u}^0 \in H^2\cap H_0^1(\Omega,\mathbb{R}^N)$ to approximate $\vec{u}_{\rm in}$ in $H_0^1(\Omega,\mathbb{R}^N)$. It is evident that for all $n = 0,1,2,\ldots$, given $\vec{u}^n \in H^2\cap H_0^1$ one can determine $\nabla p_{\rm E}^n \in L^2$ and $\nabla p_{\rm S}^n \in L^2$ from (140) and (141) and advance to time step $n+1$ by solving (139) as an elliptic boundary-value problem with Dirchlet boundary values to obtain $\vec{u}^{n+1}$.

This simple scheme is related to one studied by Timmermans et al. [Ti]. In the time-differencing scheme described in [Ti] for the linear Stokes equation, the pressure $p^n = p_{\rm E}^n + \nu p_{\rm S}^n$ is updated in nearly equivalent fashion, if one omits the velocity correction step that imposes zero divergence, and uses first-order time differences in (15) and (18) of [Ti]. Also see [Pe, GuS, JL].

Let us begin making estimates — our main result is stated as Theorem 5 below. Dot (139) with $-\Delta u^{n+1}$ and use (140) and $\|I-\mathcal{P}\| \le 1$ to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2 + \|\nabla\vec{u}^{n+1}-\nabla\vec{u}^n\|^2\Big) + \nu\|\Delta\vec{u}^{n+1}\|^2 \\
&\quad\le \|\Delta\vec{u}^{n+1}\|\Big(2\|\vec{f}^n - \vec{u}^n\cdot\nabla\vec{u}^n\| + \nu\|\nabla p_{\rm S}^n\|\Big) \\
&\quad\le \frac{\varepsilon_1}{2}\|\Delta\vec{u}^{n+1}\|^2 + \frac{2}{\varepsilon_1}\|\vec{f}^n - \vec{u}^n\cdot\nabla\vec{u}^n\|^2 + \frac{\nu}{2}\big(\|\Delta\vec{u}^{n+1}\|^2 + \|\nabla p_{\rm S}^n\|^2\big)
\end{aligned} \tag{144}$$

for any $\varepsilon_1 > 0$. (This is not optimal for $\nabla p_{\rm E}^n$ but is convenient.) This gives

$$\begin{aligned}
&\frac{1}{\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2\Big) + (\nu-\varepsilon_1)\|\Delta\vec{u}^{n+1}\|^2 \\
&\quad\le \frac{8}{\varepsilon_1}\Big(\|\vec{f}^n\|^2 + \|\vec{u}^n\cdot\nabla\vec{u}^n\|^2\Big) + \nu\|\nabla p_{\rm S}^n\|^2.
\end{aligned} \tag{145}$$

Fix any β with $\frac{2}{3} < \beta < 1$. By Theorem 1 one has

$$\nu\|\nabla p_{\rm S}^n\|^2 \le \nu\beta\|\Delta\vec{u}^n\|^2 + \nu C_\beta\|\nabla\vec{u}^n\|^2. \tag{146}$$

Using this in (145), one obtains

$$\frac{1}{\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2-\|\nabla\vec{u}^n\|^2\Big)+(\nu-\varepsilon_1)\big(\|\Delta\vec{u}^{n+1}\|^2-\|\Delta\vec{u}^n\|^2\big)$$
$$+(\nu-\varepsilon_1-\nu\beta)\|\Delta\vec{u}^n\|^2$$
$$\le\frac{8}{\varepsilon_1}\big(\|\vec{f}^n\|^2+\|\vec{u}^n\cdot\nabla\vec{u}^n\|^2\big)+\nu C_\beta\|\nabla\vec{u}^n\|^2. \tag{147}$$

At this point there are no remaining difficulties with controlling the pressure. It remains only to use the viscosity to control the nonlinear term. We focus on the physically most interesting cases $N=2$ and 3. We make use of Ladyzhenskaya's inequalities [La]

$$\int_{\mathbb{R}^N} g^4 \le 2\left(\int_{\mathbb{R}^N} g^2\right)\left(\int_{\mathbb{R}^N}|\nabla g|^2\right) \qquad (N=2), \tag{148}$$
$$\int_{\mathbb{R}^N} g^4 \le 4\left(\int_{\mathbb{R}^N} g^2\right)^{1/2}\left(\int_{\mathbb{R}^N}|\nabla g|^2\right)^{3/2} \qquad (N=3), \tag{149}$$

valid for $g\in H^1(\mathbb{R}^N)$ with $N=2$ and 3 respectively, together with the fact that the standard bounded extension operator $H^1(\Omega)\to H^1(\mathbb{R}^N)$ is also bounded in L^2 norm, to infer that for all $g\in H^1(\Omega)$,

$$\|g\|_{L^4}^2 \le C\|g\|_{L^2}\|g\|_{H^1} \qquad (N=2), \tag{150}$$
$$\|g\|_{L^3}^2 \le \|g\|_{L^2}^{2/3}\|g\|_{L^4}^{4/3} \le C\|g\|_{L^2}\|g\|_{H^1} \qquad (N=3). \tag{151}$$

Using that $H^1(\Omega)$ embeds into L^4 and L^6, these inequalities lead to the estimates

$$\int_\Omega |\vec{u}^n\cdot\nabla\vec{u}^n|^2 \le \begin{cases}\|\vec{u}^n\|_{L^4}^2\|\nabla\vec{u}^n\|_{L^4}^2 \le C\|u\|_{L^2}\|\nabla\vec{u}^n\|_{L^2}^2\|\nabla\vec{u}^n\|_{H^1} & (N=2),\\ \|\vec{u}^n\|_{L^6}^2\|\nabla\vec{u}^n\|_{L^3}^2 \le C\|\nabla\vec{u}^n\|_{L^2}^3\|\nabla\vec{u}^n\|_{H^1} & (N=3).\end{cases} \tag{152}$$

By the elliptic regularity estimate $\|\nabla\vec{u}\|_{H^1}\le\|\vec{u}\|_{H^2}\le C\|\Delta\vec{u}\|$, we conclude

$$\|\vec{u}^n\cdot\nabla\vec{u}^n\|^2 \le \begin{cases}\varepsilon_2\|\Delta\vec{u}^n\|^2+4C\varepsilon_2^{-1}\|\vec{u}^n\|^2\|\nabla\vec{u}^n\|^4 & (N=2),\\ \varepsilon_2\|\Delta\vec{u}^n\|^2+4C\varepsilon_2^{-1}\|\nabla\vec{u}^n\|^6 & (N=2 \text{ or } 3).\end{cases} \tag{153}$$

for any $\varepsilon_2>0$. Plug this into (147) and take ε_1, $\varepsilon_2>0$ satisfying $\nu-\varepsilon_1>0$ and $\varepsilon:=\nu-\varepsilon_1-\nu\beta-8\varepsilon_2/\varepsilon_1>0$. We get

$$\frac{1}{\Delta t}\big(\|\nabla\vec{u}^{n+1}\|^2-\|\nabla\vec{u}^n\|^2\big)+(\nu-\varepsilon_1)\big(\|\Delta\vec{u}^{n+1}\|^2-\|\Delta\vec{u}^n\|^2\big)+\varepsilon\|\Delta\vec{u}^n\|^2$$
$$\le\frac{8}{\varepsilon_1}\|\vec{f}^n\|^2+\frac{32C}{\varepsilon_1\varepsilon_2}\|\nabla\vec{u}^n\|^6+\nu C_\beta\|\nabla\vec{u}^n\|^2. \tag{154}$$

A simple discrete Gronwall-type argument leads to our main stability result:

Theorem 5 *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with C^3 boundary, and assume $\vec{f} \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ for some given $T > 0$ and $\vec{u}^0 \in H_0^1(\Omega,\mathbb{R}^N)\cap H^2(\Omega,\mathbb{R}^N)$. Consider the time-discrete scheme (139)-(143). Then there exist positive constants T^* and C_3, such that whenever $n\Delta t \le T^*$, we have*

$$\sup_{0\le k\le n} \|\nabla \vec{u}^k\|^2 + \sum_{k=0}^{n} \|\Delta \vec{u}^k\|^2 \Delta t \le C_3, \tag{155}$$

$$\sum_{k=0}^{n-1} \left(\left\| \frac{\vec{u}^{k+1} - \vec{u}^k}{\Delta t} \right\|^2 + \|\vec{u}^k \cdot \nabla \vec{u}^k\|^2 \right) \Delta t \le C_3. \tag{156}$$

The constants T^* and C_3 depend only upon Ω, ν and

$$M_0 := \|\nabla \vec{u}^0\|^2 + \nu\Delta t \|\Delta \vec{u}^0\|^2 + \int_0^T \|\vec{f}\|^2.$$

Proof: Put

$$z_n = \|\nabla \vec{u}^n\|^2 + (\nu - \varepsilon_1)\Delta t \|\Delta \vec{u}^n\|^2, \quad w_n = \varepsilon \|\Delta \vec{u}^n\|^2, \quad b_n = \|\vec{f}^n\|^2, \tag{157}$$

and note that from (143) we have that as long as $n\Delta t \le T$,

$$\sum_{k=0}^{n-1} \|\vec{f}_k\|^2 \Delta t \le \int_0^T |\vec{f}(t)|^2 \, dt. \tag{158}$$

Then by (154),

$$z_{n+1} + w_n \Delta t \le z_n + C\Delta t (b_n + z_n + z_n^3), \tag{159}$$

where we have replaced $\max\{8/\varepsilon_1, 32C/(\varepsilon_1\varepsilon_2), \nu C_\beta\}$ by C. Summing from 0 to $n-1$ and using (158) yields

$$z_n + \sum_{k=0}^{n-1} w_k \Delta t \le CM_0 + C\Delta t \sum_{k=0}^{n-1} (z_k + z_k^3) =: y_n. \tag{160}$$

The quantities y_n so defined increase with n and satisfy

$$y_{n+1} - y_n = C\Delta t (z_n + z_n^3) \le C\Delta t (y_n + y_n^3). \tag{161}$$

Now set $F(y) = \ln(\sqrt{1+y^2}/y)$ so that $F'(y) = -(y+y^3)^{-1}$. Then on $(0,\infty)$, F is positive, decreasing and convex, and we have

$$F(y_{n+1}) - F(y_n) = F'(\xi_n)(y_{n+1} - y_n) \ge -\frac{y_{n+1} - y_n}{y_n + y_n^3} \ge -C\Delta t, \tag{162}$$

whence

$$F(y_n) \ge F(y_0) - Cn\Delta t = F(CM_0) - Cn\Delta t. \tag{163}$$

Choosing any $T^* > 0$ so that $C_* := F(CM_0) - CT^* > 0$, we infer that as long as $n\Delta t \le T^*$ we have $y_n \le F^{-1}(C_*)$, and this together with (160) yields the stability estimate (155).

Now, using (153) and elliptic regularity, we get from (155) that

$$\sum_{k=0}^{n} \|\vec{u}^k \cdot \nabla \vec{u}^k\|^2 \Delta t \le C \sum_{k=0}^{n} \|\nabla \vec{u}^k\|_{L^2}^2 \|\nabla \vec{u}^k\|_{H^1}^2 \Delta t \le C \sum_{k=0}^{n} \|\Delta \vec{u}^k\|^2 \Delta t \le C. \tag{164}$$

Then the difference equation (139) yields

$$\sum_{k=0}^{n-1} \left\| \frac{\vec{u}^{k+1} - \vec{u}^k}{\Delta t} \right\|^2 \Delta t \le C. \tag{165}$$

This yields (156) and finishes the proof of the Theorem. □

5 Existence and uniqueness of strong solutions

The stability estimates in Theorem 5 lead directly to the following existence and uniqueness theorem for strong solutions of the unconstrained formulation (6) of the Navier-Stokes equations. Regarding the constrained Navier-Stokes equations there are of course many previous works; see [Am] for a recent comprehensive treatment. For unconstrained formulations of the Navier-Stokes equations with a variety of boundary conditions including the one considered in the present paper, Grubb and Solonnikov [GS1, GS2] lay out a general existence theory in anisotropic Sobolev spaces using a theory of pseudodifferential initial-boundary-value problems developed by Grubb.

Theorem 6 *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ of class C^3, and let $\vec{f} \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$, $\vec{u}_{\rm in} \in H_0^1(\Omega,\mathbb{R}^N)$. Then, there exists $T^* > 0$ depending only upon Ω, ν and $M_1 := \|\nabla \vec{u}_{\rm in}\|^2 + \int_0^T \|\vec{f}\|^2$, so that a unique strong solution of (134)-(136) exists on $[0,T^*]$, with*

$$\begin{aligned} &\vec{u} \in L^2(0,T^*;H^2(\Omega,\mathbb{R}^N)) \cap H^1(0,T^*;L^2(\Omega,\mathbb{R}^N)), \\ &\nabla p = \nabla p_E + \nu \nabla p_S \in L^2(0,T^*;L^2(\Omega,\mathbb{R}^N)), \end{aligned}$$

where p_E and p_S are as in (9) and (10). Moreover, $\vec{u} \in C([0,T^*],H^1(\Omega,\mathbb{R}^N))$, and $\nabla \cdot \vec{u} \in C^\infty((0,T^*],C^\infty(\Omega))$ is a classical solution of the heat equation with no-flux boundary conditions. The map $t \mapsto \|\nabla \cdot \vec{u}\|^2$ is smooth for $t > 0$ and we have the dissipation identity

$$\frac{d}{dt}\frac{1}{2}\|\nabla \cdot \vec{u}\|^2 + \nu \|\nabla(\nabla \cdot \vec{u})\|^2 = 0. \tag{166}$$

Proof of existence: We shall give a simple proof of existence based on the finite difference scheme considered in section 4, using a classical compactness argument [Ta1, Te1, LM]. However, in contrast to similar arguments in other sources, for example by Temam [Te1] for a time-discrete scheme with implicit differencing of pressure terms, we do not make any use of regularity theory for stationary Stokes systems.

First we smooth the initial data. Given $\vec{u}_{\rm in} \in H^1_0(\Omega,\mathbb{R}^N)$ and $\Delta t > 0$, determine $\vec{u}^0$ in $H^1_0 \cap H^2(\Omega,\mathbb{R}^N)$ by solving $(I - \Delta t\,\Delta)\vec{u}^0 = \vec{u}_{\rm in}$. An energy estimate yields

$$\|\nabla\vec{u}^0\|^2 + \Delta t\|\Delta\vec{u}^0\|^2 \le \|\nabla\vec{u}_{\rm in}\|\|\nabla\vec{u}^0\| \le \|\nabla\vec{u}_{\rm in}\|^2.$$

Then $\|\Delta t\,\Delta\vec{u}^0\|^2 = O(\Delta t)$ as $\Delta t \to 0$, so $\vec{u}^0 \to \vec{u}_{\rm in}$ strongly in L^2 and weakly in H^1. The stability constant C_3 in Theorem 5 is then uniformly bounded independent of Δt.

We define the discretized solution $\vec{u}^n$ by (139)-(142) of section 4, and note

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} + \mathcal{P}(\vec{u}^n\cdot\nabla\vec{u}^n - \vec{f}^n - \nu\Delta\vec{u}^n) = \nu\Delta(\vec{u}^{n+1} - \vec{u}^n) + \nu\nabla\nabla\cdot\vec{u}^n. \quad (167)$$

With $t_n = n\Delta t$, we put $\vec{u}_{\Delta t}(t_n) = \vec{U}_{\Delta t}(t_n) = \vec{u}_n$ for $n = 0,1,2,\ldots$, and define $\vec{u}_{\Delta t}(t)$ and $\vec{U}_{\Delta t}(t)$ on each subinterval $[t_n, t_n + \Delta t)$ through linear interpolation and as piecewise constant respectively:

$$\vec{u}_{\Delta t}(t_n + s) = \vec{u}^n + s\left(\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t}\right), \qquad s \in [0,\Delta t), \quad (168)$$

$$\vec{U}_{\Delta t}(t_n + s) = \vec{u}^n, \qquad s \in [0,\Delta t). \quad (169)$$

Then (167) means that whenever $t > 0$ with $t \ne t_n$,

$$\partial_t\vec{u}_{\Delta t} + \mathcal{P}(\vec{U}_{\Delta t}\cdot\nabla\vec{U}_{\Delta t} - \vec{f}_{\Delta t} - \nu\Delta\vec{U}_{\Delta t}) = \nu\Delta(\vec{U}_{\Delta t}(\cdot + \Delta t) - \vec{U}_{\Delta t}) + \nu\nabla\nabla\cdot\vec{U}_{\Delta t}, \quad (170)$$

where $\vec{f}_{\Delta t}(t) = \vec{f}^n$ for $t \in [t_n, t_n + \Delta t)$.

We will use the simplified notation $X(Y)$ to denote a function space of the form $X([0,T^*], Y(\Omega,\mathbb{R}^N))$, and we let $Q = \Omega \times [0,T^*]$ where T^* is given by Theorem 5. The estimates in Theorem 5 say that $\vec{u}_{\Delta t}$ is bounded in the Hilbert space

$$V_0 := L^2(H^2 \cap H^1_0) \cap H^1(L^2), \quad (171)$$

and also that $\vec{U}_{\Delta t}$ is bounded in $L^2(H^2)$, uniformly for $\Delta t > 0$. Moreover, estimate (155) says $\vec{u}_{\Delta t}$ is bounded in $C(H^1)$. This is also a consequence of the embedding $V_0 \hookrightarrow C(H^1)$, see [Ta1, p. 42] or [Ev, p. 288].

Along some subsequence $\Delta t_j \to 0$, then, we have that $\vec{u}_{\Delta t}$ converges weakly in V_0 to some $\vec{u} \in V_0$, and $\vec{U}_{\Delta t}$ and $\vec{U}_{\Delta t}(\cdot + \Delta t)$ converge weakly in $L^2(H^2)$ to some $\vec{U}_1$ and $\vec{U}_2$ respectively. Since clearly $V_0 \hookrightarrow H^1(Q)$, and since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, we have that $\vec{u}_{\Delta t} \to \vec{u}$ strongly in $L^2(Q)$. Note that by estimate (156),

$$\|\vec{u}_{\Delta t} - \vec{U}_{\Delta t}\|^2_{L^2(Q)} \le \|\vec{U}_{\Delta t}(\cdot + \Delta t) - \vec{U}_{\Delta t}\|^2_{L^2(Q)} = \sum_{k=0}^{n-1}\|\vec{u}^{n+1} - \vec{u}^n\|^2\Delta t \le C\Delta t^2. \quad (172)$$

Therefore $\vec{U}_{\Delta t}(\cdot + \Delta t)$ and $\vec{U}_{\Delta t}$ converge to $\vec{u}$ strongly in $L^2(Q)$ also, so $\vec{U}_1 = \vec{U}_2 = \vec{u}$.

We want to show $\vec{u}$ is a strong solution of (134) by passing to the limit in (170). From the definition of $\vec{f}^n$ in (143), it is a standard result which can be proved by using a density argument that

$$\|\vec{f} - \vec{f}_{\Delta t}\|^2_{L^2(Q)} \to 0 \quad \text{as } \Delta t \to 0.$$

We are now justified in passing to the limit weakly in $L^2(Q)$ in all terms in (170) except the nonlinear term, which (therefore) converges weakly to some $\vec{w} \in L^2(Q)$. But since $\nabla\vec{U}_{\Delta t}$ converges to $\nabla\vec{u}$ weakly and $\vec{U}_{\Delta t}$ to $\vec{u}$ strongly in $L^2(Q)$, we can conclude $\vec{U}_{\Delta t} \cdot \nabla\vec{U}_{\Delta t}$ converges to $\vec{u} \cdot \nabla\vec{u}$ in the sense of distributions on Q. So $\vec{w} = \vec{u} \cdot \nabla\vec{u}$, and upon taking limits in (170) it follows that

$$\partial_t\vec{u} + \mathcal{P}\big(\vec{u} \cdot \nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}\big) = \nu\nabla\nabla \cdot \vec{u}. \tag{173}$$

That is, $\vec{u}$ is indeed a strong solution of (134). That $\vec{u}(0) = \vec{u}_{\text{in}}$ is a consequence of the continuity of the map $\vec{u} \to \vec{u}(0)$ from V_0 through $C(H^1)$ to $H^1(\Omega, \mathbb{R}^N)$.

It remains to study $\nabla \cdot \vec{u}$. Dot (173) with $\nabla\phi$, $\phi \in H^1(\Omega)$. We get

$$\int_\Omega \partial_t\vec{u} \cdot \nabla\phi = \nu \int_\Omega \nabla(\nabla \cdot \vec{u}) \cdot \nabla\phi. \tag{174}$$

This says that $w = \nabla \cdot \vec{u}$ is a weak solution of the heat equation with Neumann boundary conditions:

$$\partial_t w = \nu\Delta w \quad \text{in } \Omega, \qquad \vec{n} \cdot \nabla w = 0 \quad \text{on } \Gamma. \tag{175}$$

Indeed, the operator $A := \nu\Delta$ defined on $L^2(\Omega)$ with domain

$$D(A) = \{w \in H^2(\Omega) \mid \vec{n} \cdot \nabla w = 0 \text{ on } \Gamma\} \tag{176}$$

is self-adjoint and non-positive, so generates an analytic semigroup. For any $\phi \in D(A)$ we have that $t \mapsto \langle w(t), \phi\rangle = -\langle u(t), \nabla\phi\rangle$ is absolutely continuous, and using (174) we get $(d/dt)\langle w(t), \phi\rangle = \langle w(t), A\phi\rangle$ for a. e. t. By Ball's characterization of weak solutions of abstract evolution equations [Ba], $w(t) = e^{At}w(0)$ for all $t \in [0, T^*]$. It follows $w \in C([0, T^*], L^2(\Omega))$, and $w(t) \in D(A^m)$ for every $m > 0$ [Pa, theorem 6.13]. Since $A^m w(t) = e^{A(t-\tau)}A^m w(\tau)$ if $0 < \tau < t$ we infer that for $0 < t \le T^*$, $w(t)$ is analytic in t with values in $D(A^m)$. Using interior estimates for elliptic equations, we find $w \in C^\infty((0, T^*], C^\infty(\Omega))$ as desired. The dissipation identity follows by dotting with w.

This finishes the proof of existence. $\square$

Proof of uniqueness: Suppose $\vec{u}_1$ and $\vec{u}_2$ are both solutions of (134)–(136) belonging to V_0. Put $\vec{u} = \vec{u}_1 - \vec{u}_2$ and $\nabla p_{\rm S} = (I-\mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}$. Then $\vec{u}(0) = 0$ and

$$\partial_t \vec{u} + \mathcal{P}\big(\vec{u}_1 \cdot \nabla\vec{u} + \vec{u}\cdot\nabla\vec{u}_2\big) = \nu\Delta\vec{u} - \nu\nabla p_{\rm S}. \tag{177}$$

Dot with $-\Delta\vec{u}$ and use Theorem 1 to get

$$\big\langle \nu\Delta\vec{u} - \nu\nabla p_{\rm S}, -\Delta\vec{u}\big\rangle \le -\frac{\nu}{2}\|\Delta\vec{u}\|^2 + \frac{\nu}{2}\|\nabla p_{\rm S}\|^2 \le -\frac{\nu\beta}{2}\|\Delta\vec{u}\|^2 + C\|\nabla\vec{u}\|^2. \tag{178}$$

Next, use the Cauchy-Schwarz inequality for the nonlinear terms, estimating them as follows in a manner similar to (150)-(152), using that $\vec{u}_1$ and $\vec{u}_2$ are a priori bounded in H^1 norm:

$$\|\vec{u}_1\cdot\nabla\vec{u}\|\,\|\Delta\vec{u}\| \le C\|\nabla\vec{u}_1\|\|\nabla\vec{u}\|^{1/2}\|\Delta\vec{u}\|^{3/2} \le \varepsilon\|\Delta\vec{u}\|^2 + C\|\nabla\vec{u}\|^2, \tag{179}$$

$$\|\vec{u}\cdot\nabla\vec{u}_2\|\,\|\Delta\vec{u}\| \le C\|\nabla\vec{u}\|\|\nabla\vec{u}_2\|_{H^1}\|\Delta\vec{u}\| \le \varepsilon\|\Delta\vec{u}\|^2 + C\|\Delta\vec{u}_2\|^2\|\nabla\vec{u}\|^2. \tag{180}$$

Lastly, since $\vec{u} \in V_0$ we infer that $\big\langle \partial_t\vec{u}, -\Delta\vec{u}\big\rangle \in L^1(0,T)$ and $t \mapsto \|\nabla\vec{u}\|^2$ is absolutely continuous with

$$\big\langle \partial_t\vec{u}, -\Delta\vec{u}\big\rangle = \frac{1}{2}\frac{d}{dt}\|\nabla\vec{u}\|^2. \tag{181}$$

This can be shown by using the density of smooth functions in V_0; see [Ev, p. 287] for a detailed proof of a similar result.

Through this quite standard-style approach, we get

$$\frac{d}{dt}\|\nabla\vec{u}\|^2 + \alpha\|\Delta\vec{u}\|^2 \le C(1 + \|\Delta\vec{u}_2\|^2)\|\nabla\vec{u}\|^2 \tag{182}$$

for some positive constants α and C. Because $\|\Delta\vec{u}_2\|^2 \in L^1(0,T)$, by Gronwall's inequality we get $\|\nabla\vec{u}\| \equiv 0$. This proves the uniqueness. $\square$

Since the interval of existence $[0, T_*]$ depends only upon M_1, in standard fashion we may extend the unique strong solution to a maximal interval of time, and infer that the approximations considered above converge to this solution up to the maximal time.

Corollary 2 *Given the assumptions of Theorem 6, system (134)-(136) admits a unique strong solution $\vec{u}$ on a maximal interval $[0, T_{\max})$ with the property that if $T_{\max} < T$ then*

$$\|\vec{u}(t)\|_{H^1} \to \infty \quad \text{as } t \to T_{\max}. \tag{183}$$

For every $\hat{T} \in [0, T_{\max})$, the approximations $\vec{u}_{\Delta t}$ constructed in (168) converge to $\vec{u}$ weakly in

$$L^2([0,\hat{T}], H^2 \cap H^1_0(\Omega, \mathbb{R}^N)) \cap H^1([0,\hat{T}], L^2(\Omega, \mathbb{R}^N))$$

and strongly in $L^2([0,\hat{T}] \times \Omega, \mathbb{R}^N)$.

6 Unconditional stability and convergence for C^1/C^0 finite element methods without inf-sup conditions

The simplicity of the stability proof for the time-discrete scheme in section 4 allows us to easily establish the unconditional stability and convergence (up to the maximal time of existence for the strong solution) of corresponding fully discrete finite-element methods that use C^1 elements for the velocity field and C^0 elements for pressure. To motivate the discretization, we write the unconstrained Navier-Stokes formulation (11) in weak form as follows, in terms of total pressure $p = p_{\rm E} + \nu p_{\rm S}$:

$$\left\langle \vec{u}_t + \nabla p - \nu\Delta\vec{u} + \vec{u}\cdot\nabla\vec{u} - \vec{f}, \Delta\vec{v}\right\rangle = 0 \quad \forall v \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N), \tag{184}$$

$$\left\langle \nabla p + \nu\nabla\nabla\cdot\vec{u} - \nu\Delta\vec{u} + \vec{u}\cdot\nabla\vec{u} - \vec{f}, \nabla\phi\right\rangle = 0 \quad \forall \phi \in H^1(\Omega). \tag{185}$$

We suppose that for some sequence of positive values of h approaching zero, $X_h \subset H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$ is a finite-dimensional space containing the approximate velocity field, and suppose $Y_h \subset H^1(\Omega)/\mathbb{R}$ is a finite-dimensional space containing approximate pressures. We assume these spaces have the approximation property that

$$\forall\vec{v} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)\ \forall h\ \exists\vec{v}_h \in X_h, \quad \|\Delta(\vec{v} - \vec{v}_h)\| \to 0 \quad \text{as } h \to 0, \tag{186}$$

$$\forall\phi \in H^1(\Omega)/\mathbb{R}\ \forall h\ \exists\phi_h \in Y_h, \quad \|\nabla(\phi - \phi_h)\| \to 0 \quad \text{as } h \to 0. \tag{187}$$

As we have emphasized in the introduction to this paper, we impose *no* inf-sup condition between the spaces X_h and Y_h. (We remark that in general, practical finite element methods usually use spaces defined on domains that approximate the given Ω. For simplicity here we suppose Ω can be kept fixed, such that finite-element spaces X_h and Y_h can be found as described with C^1 elements for velocity and C^0 elements for pressure. Though generally impractical, in principle this should be possible whenever Ω has a piecewise polynomial C^3 boundary.)

We discretize (184)-(185) in a straightforward way, implicitly only in the viscosity term and explicitly in the pressure and nonlinear terms. The resulting scheme was also derived in [JL] and is equivalent to a space discretization of the scheme in (139)–(143). Given the approximate velocity $\vec{u}^h_n$ at the n-th time step, we determine $p^n_h \in Y_h$ and $\vec{u}^{n+1}_h \in X_h$ by requiring

$$\left\langle \nabla p^n_h + \nu\nabla\nabla\cdot\vec{u}^n_h - \nu\Delta\vec{u}^n_h + \vec{u}^n_h\cdot\nabla\vec{u}^n_h - \vec{f}^n, \nabla\phi_h\right\rangle = 0 \quad \forall\phi_h \in Y_h, \tag{188}$$

$$\left\langle \frac{\nabla\vec{u}^{n+1}_h - \nabla\vec{u}^n_h}{\Delta t}, \nabla\vec{v}_h\right\rangle + \left\langle \nu\Delta\vec{u}^{n+1}_h, \Delta\vec{v}_h\right\rangle = \left\langle \nabla p^n_h + \vec{u}^n_h\cdot\nabla\vec{u}^n_h - \vec{f}^n, \Delta\vec{v}_h\right\rangle \quad \forall\vec{v}_h \in X_h. \tag{189}$$

Stability. We are to show the scheme above is unconditionally stable. First, we take $\phi_h = p_h$ in (188). Due to the fact that

$$\left\langle \mathcal{P}(\Delta - \nabla\nabla\cdot)\vec{u}^n_h, \nabla p^n_h\right\rangle = 0,$$

we directly deduce from the Cauchy-Schwarz inequality that

$$\|\nabla p_h^n\| \le \|\nu \nabla p_{\rm S}(u_h^n)\| + \|\vec u_h^n \cdot \nabla \vec u_h^n - \vec f^n\| \tag{190}$$

where

$$\nabla p_{\rm S}(u_h^n) = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec u_h^n \tag{191}$$

is the Stokes pressure associated with $\vec u_h^n$. (Note $\nabla p_{\rm S}(u_h^n)$ need not lie in the space Y_h). Now, taking $\vec v_h = \vec u_h^{n+1}$ in (189) and arguing just as in (144), we obtain an exact analog of (145), namely

$$\begin{aligned} \frac{1}{\Delta t}\Big(\|\nabla \vec u_h^{n+1}\|^2 - \|\nabla \vec u_h^n\|^2\Big) + (\nu - \varepsilon_1)\|\Delta \vec u_h^{n+1}\|^2 \\ \le \frac{8}{\varepsilon_1}\Big(\|\vec f^n\|^2 + \|\vec u_h^n \cdot \nabla \vec u_h^n\|^2\Big) + \nu\|\nabla p_{\rm S}(\vec u_h^n)\|^2. \end{aligned} \tag{192}$$

Proceeding now exactly as in section 4 leads to the following unconditional stability result.

Theorem 7 *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with C^3 boundary, and suppose spaces $X_h \subset H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$, $Y_h \subset H^1(\Omega)/\mathbb{R}$ satisfy (186)–(187). Assume $\vec f \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$ for some given $T > 0$ and $\vec u_h^0 \in X_h$. Consider the finite-element scheme (188)-(189) with (143). Then there exist positive constants T^* and C_4, such that whenever $n\Delta t \le T^*$, we have*

$$\sup_{0\le k\le n} \|\nabla \vec u_h^k\|^2 + \sum_{k=0}^{n} \|\Delta \vec u_h^k\|^2 \Delta t \le C_4, \tag{193}$$

$$\sum_{k=0}^{n-1}\left(\left\|\frac{\vec u_h^{k+1} - \vec u_h^k}{\Delta t}\right\|^2 + \|\vec u_h^k \cdot \nabla \vec u_h^k\|^2\right)\Delta t \le C_4. \tag{194}$$

The constants T^* and C_4 depend only upon Ω, ν and

$$M_{0h} := \|\nabla \vec u_h^0\|^2 + \nu \Delta t \|\Delta \vec u_h^0\|^2 + \int_0^T \|\vec f\|^2.$$

Convergence. We prove the convergence of the finite-element scheme described above by taking $h \to 0$ to obtain the solution of the time-discrete scheme studied in section, then $\Delta t \to 0$ as before. Because of the uniqueness of the solution of the time-discrete scheme and of the strong solution of the PDE, it suffices to prove convergence for some subsequence of any given sequence of values of h tending toward 0. The bounds obtained in Theorem 7 make this rather straightforward.

Fix $\Delta t > 0$. The bounds in Theorem 7 and in (190) imply that for all positive integers $n < T_*/\Delta t$, the $\vec u_h^n$ are bounded in $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and the ∇p_h^n are bounded in $L^2(\Omega, \mathbb{R}^N)$ uniformly in h. So from any sequence of h approaching zero, we may extract a subsequence along which we have weak limits

$$\vec u_h^n \rightharpoonup \vec u^n \text{ in } H^2(\Omega, \mathbb{R}^N), \quad \nabla p_h^n \rightharpoonup \nabla p^n, \ \vec u_h^n \cdot \nabla \vec u_h^n \rightharpoonup \vec w^n \text{ in } L^2(\Omega, \mathbb{R}^N) \tag{195}$$

for all n. Then $\vec{u}_h^n \to \vec{u}^n$ strongly in $H_0^1(\Omega,\mathbb{R}^N)$ and so $\vec{w}^n = \vec{u}^n\cdot\nabla\vec{u}^n$ since the nonlinear term converges strongly in L^1.

Now, for any $\vec{v}\in H^2\cap H_0^1(\Omega,\mathbb{R}^N)$ and $\phi\in H^1(\Omega)$, by assumption there exist $\vec{v}_h\in X_h$, $\phi_h\in H^1(\Omega)$ such that $\vec{v}_h\to\vec{v}$ strongly in $H^2(\Omega,\mathbb{R}^N)$ and $\nabla\phi_h\to\nabla\phi$ strongly in $L^2(\Omega,\mathbb{R}^N)$. Applying these convergence properties in (188)–(189) yields that the weak limits in (195) satisfy

$$\left\langle \nabla p^n + \nu\nabla\nabla\cdot\vec{u}^n - \nu\Delta\vec{u}^n + \vec{u}^n\cdot\nabla\vec{u}^n - \vec{f}^n, \nabla\phi\right\rangle = 0, \tag{196}$$

$$\left\langle \frac{\vec{u}^{n+1}-\vec{u}^n}{\Delta t} - \nu\Delta\vec{u}^{n+1} + \nabla p^n + \vec{u}^n\cdot\nabla\vec{u}^n - \vec{f}^n, \Delta\vec{v}\right\rangle = 0. \tag{197}$$

But this means exactly that $\vec{u}^n$ satisfies (139) with $p^n = p_{\mathrm{E}}^n + \nu p_{\mathrm{S}}^n$, where p_{E}^n and p_{S}^n are given by (140)–(141). So in the limit $h\to 0$ we obtain the solution of the time-discrete scheme studied in section (4). Then the limit $\Delta t\to 0$ yields the unique strong solution on a maximal time interval as established in section 5.

7 Semigroup approach for the homogeneous linear case

There are many other approaches to existence theory for the Navier-Stokes equations, of course — Galerkin's method, mollification, semigroup theory, etc. We will not discuss any of them here, except to note that the linearization of the unconstrained system (11) can be treated easily by analytic semigroup theory using Theorem 1. Take $\nu = 1$ without loss of generality, and consider (11) without the nonlinear and forcing terms, i.e., consider the unconstrained Stokes equation

$$\vec{u}_t - \Delta\vec{u} + \nabla p_{\mathrm{S}} = 0 \qquad (t>0,\ x\in\Omega), \tag{198}$$

with the no-slip boundary condition (135) and initial condition (136), where ∇p_{S} is given by (10) as before. In the space $X = L^2(\Omega,\mathbb{R}^N)$ define operators B_0 and B_1 by

$$B_0\vec{u} = -\Delta\vec{u}, \qquad B_1\vec{u} = \nabla p_{\mathrm{S}} = (I-\mathcal{P})\Delta\vec{u} - \nabla\nabla\cdot\vec{u}, \tag{199}$$

with domain $D(B_0) = D(B_1) = H^2\cap H_0^1(\Omega,\mathbb{R}^N)$. Then B_0 is a positive self-adjoint operator in X with compact resolvent, and by using Theorem 1 together with the interpolation estimate

$$\|\nabla\vec{u}\| \le \varepsilon\|\Delta\vec{u}\| + C_\varepsilon\|u\|$$

valid for any $\varepsilon > 0$ for all $\vec{u}\in D(B_0)$, we deduce that

$$\|B_1\vec{u}\| \le a\|B_0\vec{u}\| + K\|\vec{u}\| \tag{200}$$

for all $\vec{u}\in D(B_0)$, where a and K are positive constants, with $a<1$.

Theorem 8 *The unconstrained Stokes operator $B = B_0 + B_1$ in the space $X = L^2(\Omega, \mathbb{R}^N)$ is sectorial and generates an analytic semigroup. The resolvent of B is compact and the spectrum of B consists entirely of isolated eigenvalues of finite multiplicity, all of which are positive. Moreover, for any $\alpha \geq 0$, given $\vec{u}_{\text{in}} \in D(B^\alpha)$ equation (198) has the solution*

$$\vec{u} = e^{-Bt}\vec{u}_{\text{in}} \in C([0,T], D(B^\alpha)) \cap C^\infty((0,T], D(B^m))$$

for any $T > 0$ and all $m > 0$, and this is the unique weak solution of $\partial_t\vec{u} + B\vec{u} = 0$, $\vec{u}(0) = \vec{u}_{\text{in}}$ in the sense of Ball [Ba].

Proof. That B is sectorial is a consequence of (200) and the self-adjointness of B_0. Indeed, by a theorem on the perturbation of sectorial operators [He, p. 19, theorem 1.3.2], it suffices to show that for some $\phi_0 < \pi/2$,

$$a \sup_{\lambda \in S_0} \|B_0(\lambda - B_0)^{-1}\| < 1 \tag{201}$$

where $S_0 \subset \mathbb{C}$ is the sector where $\phi_0 < |\arg \lambda| \leq \pi$. By expanding any element of (complexified) X with respect to an orthonormal basis of eigenfunctions of B_0, for any $\lambda \notin \sigma(B_0)$ we get

$$\|B_0(\lambda - B_0)^{-1}\| = \sup_{\mu \in \sigma(B_0)} \left| \frac{\mu}{\lambda - \mu} \right|.$$

Fix $\tilde{a} \in (a, 1)$. For any $\mu > 0$, we have $|\mu| \leq |\lambda - \mu|$ whenever $\Re\lambda \leq 0$, and it is straightforward to check that whenever $\Re\lambda > 0$ and $|\Im\lambda| > \tilde{a}|\lambda|$, then $\tilde{a}|\mu| \leq |\lambda - \mu|$. Then (201) follows, proving that B is sectorial.

That $(\lambda - B)^{-1}$ is compact for $\lambda \notin \sigma(B) \cup \sigma(B_0)$ follows from the compactness of $(\lambda - B_0)^{-1}$ together with the identity

$$(\lambda - B)^{-1} = (\lambda - B_0)^{-1} + (\lambda - B_0)^{-1} B_1 (\lambda - B)^{-1}.$$

It follows that the spectrum of B is discrete, consisting only of isolated eigenvalues of finite multiplicity [Ka, III.6.29].

Suppose now that $(\lambda - B)\vec{u} = 0$ for some non-zero $\vec{u} \in D(B)$, so $\lambda\vec{u} = -\mathcal{P}\Delta\vec{u} - \nabla\nabla\cdot\vec{u}$. Then the function $w = \nabla \cdot \vec{u}$ satisfies $\lambda w = -\Delta w$ in Ω, $\vec{n} \cdot \nabla w = 0$ on Γ, i.e., $(\lambda + A)w = 0$ (see (176)). So if $\lambda \notin \sigma(-A) \subset \mathbb{R}_+$, then $\nabla \cdot \vec{u} = 0$, and since $\vec{n} \cdot \vec{u} = 0$ on Γ we have $\vec{u} = \mathcal{P}\vec{u}$. Then

$$\lambda\langle \vec{u}, \vec{u} \rangle = \langle -\mathcal{P}\Delta\vec{u}, \vec{u} \rangle = \langle -\Delta\vec{u}, \mathcal{P}\vec{u} \rangle = \|\nabla\vec{u}\|^2,$$

so $\lambda > 0$. If $\lambda = 0$, then $\nabla \cdot \vec{u}$ is constant, but $\int_\Omega \nabla \cdot \vec{u} = 0$ so $\nabla \cdot \vec{u} = 0$ and arguing as above we infer $\vec{u} = 0$. Hence 0 is not an eigenvalue, and so 0 is in the resolvent set of B.

Lastly, for any $\alpha \geq 0$, given $\vec{u}_{\text{in}} \in D(B^\alpha)$, the regularity results for $e^{-Bt}\vec{u}_{\text{in}}$ are standard consequences of the fact that B^α is an isomorphism between its domain and X and commutes with e^{-Bt} [Pa, p. 74, Theorem 6.13]. For uniqueness, see [Ba]. □

Remark 3. The equation $B\vec{u} = \vec{f}$ has an interesting interpretation in terms of a stationary Stokes system with prescribed divergence. Given any $\vec{f} \in L^2(\Omega, \mathbb{R}^N)$ there is a unique $\vec{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ such that $B\vec{u} = \vec{f}$, since 0 is in the resolvent set of B by the above theorem. We can write $\mathcal{P}\vec{f} = \vec{f} + \nabla q$ where $q \in H^1(\Omega)$ with $\int_\Omega q = 0$. Since $B\vec{u} = -\mathcal{P}\Delta\vec{u} - \nabla\nabla\cdot\vec{u}$, we have $\mathcal{P}\vec{f} = -\mathcal{P}\Delta\vec{u}$, so $\nabla(q + \nabla\cdot\vec{u}) = 0$. Let $p_{\rm S}$ be the Stokes pressure associated with $\vec{u}$. Then $(\vec{u}, p_{\rm S})$ form a solution to the Stokes system

$$-\Delta\vec{u} + \nabla p_{\rm S} = \vec{f} \quad \text{in } \Omega, \tag{202}$$

$$-\nabla\cdot\vec{u} = q \quad \text{in } \Omega, \tag{203}$$

$$\vec{u} = 0 \quad \text{on } \Gamma. \tag{204}$$

As a corollary, we can characterize the domains of positive integer powers of B by using the regularity theory for the stationary Stokes equation (see for example [Soh, p. 123, theorem 1.5.3] or [Te1, p. 23, proposition 2.2]).

Corollary 3 *Let Ω be a bounded domain with C^{2m} boundary Γ, where $m > 1$ is an integer. Then*

$$D(B^m) = \{\vec{u} \mid \vec{u} \in H^{2m}(\Omega, \mathbb{R}^N),\ \vec{u} = B\vec{u} = \ldots = B^{m-1}\vec{u} = 0 \text{ on } \Gamma\}.$$

Proof: When $m = 1$, the conclusion is true. Suppose it is true when $m = k-1$. When $m = k$, take any $\vec{u} \in D(B^k)$. By the definition of $D(B^k)$, we have $\vec{u} \in D(B^{k-1})$ and $B\vec{u} \in D(B^{k-1})$. By assumption, $\vec{f} := B\vec{u} \in H^{2k-2}$ and $B^{k-1}\vec{u} = B^{k-2}(B\vec{u}) = 0$ on Γ. Since $\mathcal{P}$ is bounded on H^{2k-2} [Te1, I, Remark 1.6] we find that $q \in H^{2k-1}(\Omega)$. Now (202)-(204) hold, and we can use the regularity theory of the stationary Stokes equation cited above to conclude $\vec{u} \in H^{2k}(\Omega, \mathbb{R}^N)$. This finishes the proof. □

Remark 4. We note that B and B_0 have the same domain and that $D(B_0^{1/2})$ is the closure of $D(B_0) = H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ in norm equivalent to

$$\|\vec{u}\|^2_{X^{1/2}} = \|B_0^{1/2}\vec{u}\|^2 = \big\langle -\Delta\vec{u}, \vec{u}\big\rangle = \|\nabla\vec{u}\|^2,$$

the ordinary H^1 norm. So $D(B_0^{1/2}) = H_0^1(\Omega, \mathbb{R}^N)$. It is known that if B has *bounded imaginary powers* then for $0 < \alpha < 1$, $D(B^\alpha)$ can be obtained by interpolation between X and $D(B) = D(B_0)$ and so $D(B^\alpha) = D(B_0^\alpha)$. The result that indeed $B + cI$ has bounded imaginary powers for some $c > 0$ apparently follows from a recent analysis of Abels [Ab] related to the formulation of Grubb and Solonnikov (although the final result in [Ab] is stated in terms of the constrained Stokes operator in divergence-free spaces).

8 Non-homogeneous side conditions

Looking back at the Stokes pressure $p_{\rm S}$ associated with $\vec{u}$, one recognizes that the no-slip boundary condition for $\vec{u}$ was essential for getting the crucial equalities

(99)-(101) using Lemma 3. So the important question arises, if general boundary conditions $\vec{u} = \vec{g}$ on Γ are imposed, do we still have an unconstrained formulation like (134)-(136)? Moreover, what can we say if the velocity field is not divergence free but is specified as $\nabla \cdot \vec{u} = h$? Such issues are likely to be relevant in the analysis of problems involving complex fluids and low Mach number flows, for example.

In this section we develop and study an unconstrained formulation for such non-homogeneous problems. In this new formulation, $\nabla \cdot \vec{u} - h$ satisfies the heat equation with no-flux boundary conditions. The main theorem of this section establishes existence and uniqueness for strong solutions.

8.1 An unconstrained formulation

Consider the Navier-Stokes equations with non-homogeneous boundary conditions and divergence constraint:

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} + \vec{f} \qquad (t > 0, x \in \Omega), \tag{205}$$

$$\nabla \cdot \vec{u} = h \qquad (t \geq 0, x \in \Omega), \tag{206}$$

$$\vec{u} = \vec{g} \qquad (t \geq 0, x \in \Gamma), \tag{207}$$

$$\vec{u} = \vec{u}_{\text{in}} \qquad (t = 0, x \in \Omega). \tag{208}$$

What we have done before can be viewed as replacing the divergence constraint (206) by decomposing the pressure via the formulae in (9) and (10) in such a way that the divergence constraint is enforced automatically. It turns out that in the non-homogeneous case a very similar procedure works. One can simply use the Helmholtz decomposition to identify Euler and Stokes pressure terms *exactly as before* via the formulae (9) and (10), but in addition another term is needed in the total pressure to deal with the inhomogeneities. Equation (6) is replaced by

$$\partial_t \vec{u} + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f} - \nu \Delta \vec{u}) + \nabla p_{gh} = \nu \nabla (\nabla \cdot \vec{u}). \tag{209}$$

The equation that determines the inhomogeneous pressure p_{gh} can be found by dotting with $\nabla \phi$ for $\phi \in H^1(\Omega)$, formally integrating by parts and plugging in the side conditions: We require

$$\langle \nabla p_{gh}, \nabla \phi \rangle = -\langle \partial_t (\vec{n} \cdot \vec{g}), \phi \rangle_\Gamma + \langle \partial_t h, \phi \rangle + \langle \nu \nabla h, \nabla \phi \rangle \tag{210}$$

for all $\phi \in H^1(\Omega)$. With this definition, we see from (209) that

$$\langle \partial_t \vec{u}, \nabla \phi \rangle - \langle \partial_t (\vec{n} \cdot \vec{g}), \phi \rangle_\Gamma + \langle \partial_t h, \phi \rangle = \langle \nu \nabla (\nabla \cdot \vec{u} - h), \nabla \phi \rangle \tag{211}$$

for every $\phi \in H^1(\Omega)$. This will mean $w := \nabla \cdot \vec{u} - h$ is a weak solution of

$$\partial_t w = \nu \Delta w \ \text{ in } \Omega, \qquad \vec{n} \cdot \nabla w = 0 \ \text{ on } \Gamma, \tag{212}$$

with initial condition $w = \nabla \cdot \vec{u}_{\text{in}} - h\big|_{t=0}$. So the divergence constraint will be enforced through exponential diffusive decay as before (see (232) below).

The total pressure in (205) now has the representation

$$p = p_{\mathrm{E}} + \nu p_{\mathrm{S}} + p_{gh}, \tag{213}$$

where the Euler pressure p_{E} and the Stokes pressure p_{S} are determined exactly by (9) and (10) as before, and p_{gh} is determined up to a constant by the forcing functions g and h through the weak-form pressure Poisson equation (210). (See Lemma 6 below.) Our unconstrained formulation of (205)-(208) then takes the form

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p_{\mathrm{E}} + \nu\nabla p_{\mathrm{S}} + \nabla p_{gh} = \nu\Delta\vec{u} + \vec{f} \qquad (t>0,\ x\in\Omega), \tag{214}$$

$$\vec{u} = \vec{g} \qquad (t\ge 0,\ x\in\Gamma), \tag{215}$$

$$\vec{u} = \vec{u}_{\mathrm{in}} \qquad (t=0,\ x\in\Omega). \tag{216}$$

Although the definition of Stokes pressure does not require a no-slip velocity field, clearly the analysis that we performed in section 2 does rely in crucial ways on no-slip boundary conditions. So in order to analyze the new unconstrained formulation, we will decompose the velocity field $\vec{u}$ in two parts. We introduce a fixed field $\tilde{u}$ in $\Omega\times[0,T]$ that satisfies $\tilde{u} = \vec{g}$ on Γ, and let

$$\vec{v} = \vec{u} - \tilde{u}. \tag{217}$$

Then $\vec{v} = 0$ on Γ. With this $\vec{v}$, similar to (9) and (10) we introduce

$$\nabla q_{\mathrm{E}} = (\mathcal{P} - I)(\vec{v}\cdot\nabla\vec{v} - \vec{f}), \qquad \nabla q_{\mathrm{S}} = (I-\mathcal{P})\Delta\vec{v} - \nabla\nabla\cdot\vec{v}. \tag{218}$$

Then we can rewrite (214) as an equation for $\vec{v}$:

$$\partial_t\vec{v} + \vec{v}\cdot\nabla\vec{v} + \nabla q_{\mathrm{E}} + \nu\nabla q_{\mathrm{S}} + \mathcal{P}(\tilde{u}\cdot\nabla\vec{v} + \vec{v}\cdot\nabla\tilde{u}) = \nu\Delta\vec{v} + \vec{f} - \tilde{f}, \tag{219}$$

where

$$\tilde{f} := \partial_t\tilde{u} + \mathcal{P}(\tilde{u}\cdot\nabla\tilde{u} - \nu\Delta\tilde{u}) - \nu\nabla\nabla\cdot\tilde{u} + \nabla p_{gh}. \tag{220}$$

8.2 Existence, uniqueness and dissipation identity

We will first answer questions concerning the existence and regularity of $\tilde{u}$ and p_{gh}, then state an existence and uniqueness result for strong solutions of the unconstrained formulation (214)–(216). Let Ω be a bounded, connected domain in $\mathbb{R}^N$ ($N = 2$ or 3) with boundary Γ of class C^3. We assume

$$\vec{u}_{\mathrm{in}} \in H_{uin} := H^1(\Omega,\mathbb{R}^N), \tag{221}$$

$$\vec{f} \in H_f := L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \tag{222}$$

$$\vec{g} \in H_g := H^{3/4}(0,T;L^2(\Gamma,\mathbb{R}^N)) \cap L^2(0,T;H^{3/2}(\Gamma,\mathbb{R}^N)) \cap \{\vec{g} \mid \partial_t(\vec{n}\cdot\vec{g}) \in L^2(0,T;H^{-1/2}(\Gamma))\}, \tag{223}$$

$$h \in H_h := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1)'(\Omega)). \tag{224}$$

Here $(H^1)'$ is the space dual to H^1. We also make the compatibility assumptions

$$\vec{g} = \vec{u}_{\rm in} \quad \text{when } t = 0,\ x \in \Gamma, \tag{225}$$

$$\big\langle \partial_t(\vec{n}\cdot\vec{g}), 1\big\rangle_\Gamma = \big\langle \partial_t h, 1\big\rangle_\Omega. \tag{226}$$

We remark that most of the literature on nonhomogeneous Navier-Stokes problems [La, Sol, Gr1, GS1, GS2] treats the constrained case with $h = 0$ in Ω and imposes the condition $\vec{n}\cdot\vec{g} = 0$ on Γ. Amann recently studied very weak solutions without imposing the latter condition, but only in spaces of very low regularity that exclude the case considered here [Am2].

We define

$$V := L^2(0,T; H^2(\Omega,\mathbb{R}^N)) \cap H^1(0,T; L^2(\Omega,\mathbb{R}^N)), \tag{227}$$

and note we have the embeddings ([Ta1, p. 42], [Ev, p. 288], [Te1, p. 176])

$$V \hookrightarrow C([0,T], H^1(\Omega,\mathbb{R}^N)), \quad H_h \hookrightarrow C([0,T], L^2(\Omega)). \tag{228}$$

Notice that we have always used an arrow or tilde to denote a vector. So, without confusion, we can use $Y(\Omega)$ to denote $Y(\Omega,\mathbb{R}^N)$ or $Y(\Omega)$ as appropriate, and further use $X(Y(\Omega))$ to denote $X(0,T;Y(\Omega))$.

Lemma 6 *Assume (221)-(226). Then, there exists some $\tilde{u} \in V$ that satisfies*

$$\tilde{u}(0) = \vec{u}_{\rm in}, \quad \tilde{u}\big|_\Gamma = \vec{g}, \tag{229}$$

and there exists $p_{gh} \in L^2(H^1(\Omega)/\mathbb{R})$ satisfying (210). Moreover,

$$\|\tilde{u}\|_V^2 \le C\big(\|\vec{g}\|^2_{H^{3/4}(L^2(\Gamma))\cap H^{3/2}(\Gamma))} + \|\vec{u}_{\rm in}\|^2_{H^1(\Omega)}\big), \tag{230}$$

$$\|p_{gh}\|_{L^2(H^1(\Omega)/\mathbb{R})} \le C\big(\|\partial_t(\vec{n}\cdot\vec{g})\|_{L^2(H^{-1/2}(\Gamma))} + \|h\|_{L^2(H^1)\cap H^1((H^1)')}\big). \tag{231}$$

Proof: (i) By a trace theorem of Lions and Magenes [LM, vol II, Theorem 2.3], the fact $\vec{g} \in H^{3/4}(L^2(\Gamma)) \cap L^2(H^{3/2}(\Gamma))$ together with (221) and the compatibility condition (225) implies the existence of $\tilde{u} \in V$ satisfying (229).

(ii) One applies the Lax-Milgram lemma for a.e. t to (210) in the space of functions in $H^1(\Omega)$ with zero average. We omit the standard details. □

Theorem 9 *Let Ω be a bounded, connected domain in $\mathbb{R}^N$ ($N = 2$ or 3) and assume (221)-(226). Then there exists $T^* > 0$ so that a unique strong solution of (214)-(216) exists on $[0,T^*]$, with*

$$\vec{u} \in L^2(0,T^*; H^2(\Omega,\mathbb{R}^N)) \cap H^1(0,T^*; L^2(\Omega,\mathbb{R}^N)),$$
$$p = \nu p_S + p_E + p_{gh} \in L^2(0,T^*; H^1(\Omega)/\mathbb{R}),$$

where p_E and p_S are defined in (9) and (10) after introducing the $\tilde{u}$ and p_{gh} from Lemma 6. Moreover, $\vec{u} \in C([0,T^*], H^1(\Omega,\mathbb{R}^N))$ and

$$\nabla\cdot\vec{u} - h \in L^2(0,T^*; H^1(\Omega)) \cap H^1(0,T^*; (H^1)'(\Omega))$$

is a smooth solution of the heat equation for $t > 0$ with no-flux boundary conditions. The map $t \mapsto \|\nabla\cdot\vec{u}-h\|^2$ is smooth for $t > 0$ and we have the dissipation identity

$$\frac{d}{dt}\frac{1}{2}\|\nabla\cdot\vec{u}-h\|^2+\nu\|\nabla(\nabla\cdot\vec{u}-h)\|^2=0. \tag{232}$$

If we further assume $h \in H_{h.s} := L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ and $\nabla\cdot\vec{u}_{\rm in} \in H^1(\Omega)$, then

$$\nabla\cdot\vec{u} \in L^2(0,T^*;H^2(\Omega))\cap H^1(0,T^*;L^2(\Omega)).$$

Proof: First rewrite (214) as (219). Then we note that there are only two differences between (219) and (134):

(i) There is an extra forcing term $\tilde{f}$ in (219). But by Lemma 6, all terms in $\tilde{f}$ are known to be in $L^2(L^2(\Omega))$ and thus they won't be a problem.

(ii) Equation (219) has some extra linear terms:

$$\mathcal{P}(\tilde{u}\cdot\nabla\vec{v}+\vec{v}\cdot\nabla\tilde{u}). \tag{233}$$

We know $\tilde{u} \in V \hookrightarrow C([0,T],H^1(\Omega,\mathbb{R}^N))$, so we can discretize these terms explicitly by setting $\tilde{u}^n = \tilde{u}(n\Delta t)$. Similar to (153), we get

$$\|\mathcal{P}(\tilde{u}\cdot\nabla\vec{v})\|^2 \le \varepsilon\|\Delta\vec{v}\|^2+\frac{C}{\varepsilon}\|\tilde{u}\|_{H^1}^4\|\nabla\vec{v}\|^2. \tag{234}$$

We estimate the other term in (233) by using Gagliardo-Nirenberg inequalities [Fr, Thm. 10.1] and the Sobolev embeddings of H^1 into L^3 and L^6:

$$\|\vec{v}\|_{L^\infty} \le \begin{cases} C\|\Delta\vec{v}\|_{L^{3/2}}^{1/2}\|\vec{v}\|_{L^3}^{1/2} \le C\|\Delta\vec{v}\|^{1/2}\|\nabla\vec{v}\|^{1/2} & (N=2),\\ C\|\Delta\vec{v}\|^{1/2}\|\vec{v}\|_{L^6}^{1/2} \le C\|\Delta\vec{v}\|^{1/2}\|\nabla\vec{v}\|^{1/2} & (N=3).\end{cases} \tag{235}$$

Then for $N = 2$ and 3 we have

$$\|\mathcal{P}(\vec{v}\cdot\nabla\tilde{u})\|^2 \le \|\vec{v}\|_{L^\infty}^2\|\nabla\tilde{u}\|^2 \le \varepsilon\|\Delta\vec{v}\|^2+\frac{C}{\varepsilon}\|\tilde{u}\|_{H^1}^4\|\nabla\vec{v}\|^2. \tag{236}$$

With these estimates, the rest of the proof of existence and uniqueness is essentially the same as that of Theorem 6, and therefore we omit the details.

To prove the regularity of $\nabla\cdot\vec{u}$, we argue in a manner similar to the proof of Theorem 6. We go from (214) to (209) by using (9) and (10). Then using (210) we get (211) for any $\phi \in H^1(\Omega)$. With $w = \nabla\cdot\vec{u}-h$, taking $\phi \in D(A)$ as in (176), we have

$$\langle w,\phi\rangle = \langle \vec{n}\cdot\vec{g},\phi\rangle_\Gamma - \langle\vec{u},\nabla\phi\rangle - \langle h,\phi\rangle, \tag{237}$$

therefore $t \mapsto \langle w,\phi\rangle$ is absolutely continuous, and (211) yields $(d/dt)\langle w,\phi\rangle = \langle w,A\phi\rangle$ for a.e. t. This means w is a weak solution in the sense of Ball [Ba], and the rest of the proof goes as before.

If we further assume $h \in H_{h.s}$ and $\nabla \cdot \vec{u}_{\rm in} \in H^1(\Omega)$, then $w(0) \in H^1(\Omega)$. We claim

$$H^1(\Omega) = D((-A)^{1/2}). \tag{238}$$

Then semigroup theory yields $w \in C([0,T^*], D((-A)^{1/2}))$, so since

$$0 = \big\langle -\Delta w, \partial_t w - \nu\Delta w\big\rangle = \frac{d}{dt}\frac{1}{2}\|\nabla w\|^2 + \nu\|\Delta w\|^2 \tag{239}$$

for $t > 0$, we deduce $w \in L^2(0,T^*;H^2(\Omega)) \cap H^1(0,T^*;L^2(\Omega))$, and $\nabla \cdot \vec{u}$ is in the same space.

To prove (238), note $X := D((-A)^{1/2})$ is the closure of $D(A)$ from (176) in the norm given by

$$\|w\|_X^2 = \|w\|^2 + \|(-A)^{1/2}w\|^2 = \big\langle (I-\nu\Delta)w, w\big\rangle = \int_\Omega |w|^2 + \nu|\nabla w|^2.$$

Clearly $X \subset H^1(\Omega)$. For the other direction, let $w \in H^1(\Omega)$ be arbitrary. We may suppose $w \in C^\infty(\bar\Omega)$ since this space is dense in $H^1(\Omega)$. Now we only need to construct a sequence of C^2 functions $w_n \to 0$ in H^1 norm with $\vec{n}\cdot\nabla w_n = \vec{n}\cdot\nabla w$ on Γ. This is easily accomplished using functions of the form $w_n(x) = \xi_n(\mathrm{dist}(x,\Gamma))\vec{n}\cdot\nabla w(x)$, where $\xi_n(s) = \xi(ns)/n$ with ξ smooth and satisfying $\xi(0)=0$, $\xi'(0)=1$ and $\xi(s)=0$ for $s>1$. This proves (238).

We can prove the uniqueness by the same method as in Theorem 6. □

9 Isomorphism theorems for non-homogeneous Stokes systems

Drop the nonlinear term and consider the non-homogeneous Stokes system:

$$\partial_t\vec{u} + \nabla p - \nu\Delta\vec{u} = \vec{f} \qquad (t>0,\ x\in\Omega), \tag{240}$$

$$\nabla\cdot\vec{u} = h \qquad (t\ge 0,\ x\in\Omega), \tag{241}$$

$$\vec{u} = \vec{g} \qquad (t\ge 0,\ x\in\Gamma), \tag{242}$$

$$\vec{u} = \vec{u}_{\rm in} \qquad (t=0,\ x\in\Omega). \tag{243}$$

The unconstrained formulation is

$$\partial_t\vec{u} + \nabla p - \nu\Delta\vec{u} = \vec{f} \qquad (t>0,\ x\in\Omega), \tag{244}$$

$$\vec{u} = \vec{g} \qquad (t\ge 0,\ x\in\Gamma), \tag{245}$$

$$\vec{u} = \vec{u}_{\rm in} \qquad (t=0,\ x\in\Omega), \tag{246}$$

with

$$\nabla p = (I-\mathcal{P})\vec{f} + \nu\nabla p_{\rm s} + \nabla p_{gh}, \tag{247}$$

where $p_{\rm s}$ and p_{gh} are defined as before via (10) and (210).

The aim of this section is to obtain an isomorphism between the space of solutions and the space of data $\{\vec{f},\vec{g},h,\vec{u}_{\rm in}\}$, for this unconstrained formulation and for the original Stokes system. In examining this question we are motivated by the classic works of Lions and Magenes [LM] which provide a satisfactory description of the correspondence between solutions and data for elliptic boundary value problems. In the spirit of these results, a satisfactory theory of a given system of partial differential equations should describe exactly how, in the space of all functions involved, the manifold of solutions can be parametrized. Yet we are not aware of any such complete treatment of the non-homogeneous Stokes system. (See further remarks on this issue below.)

First we consider the mapping from data to solution. Thanks to the absence of the nonlinear term, we can repeat much easier what we did in the proof of Theorems 6 and 9 and get the global existence and uniqueness of a strong solution of (244)-(247) under the same assumptions as Theorem 9. The data $\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\}$ lie inside the space

$$\Pi_F := H_f \times H_g \times H_h \times H_{uin} \tag{248}$$

from (221)–(224), and need to satisfy the compatibility conditions (225)–(226). Corresponding to such data, we get a unique solution $\vec{u}$ of (244)-(247) in the space

$$\begin{aligned} H_u := L^2(0,T; H^2(\Omega,\mathbb{R}^N)) \cap H^1(0,T; L^2(\Omega,\mathbb{R}^N)) \\ \cap \{\vec{u} \mid \partial_t(\vec{n}\cdot\vec{u})|_\Gamma \in L^2(0,T; H^{-1/2}(\Gamma))\}. \end{aligned} \tag{249}$$

The total pressure p lies in

$$H_p := L^2(0,T; H^1(\Omega)/\mathbb{R}), \tag{250}$$

and the pair $\{\vec{u}, p\}$ satisfies (240), (242) and (243). As in Theorem 9, we can show $w = \nabla\cdot\vec{u} - h$ satisfies a heat equation with no-flux boundary conditions. Equation (241) says that $w = 0$, and this will hold if and only if $w(0) = 0$, i.e., the following additional compatibility condition holds:

$$\nabla\cdot\vec{u}_{\rm in} = h(0). \tag{251}$$

For the non-homogeneous Stokes system (240)–(243), then, we define the data and solution spaces by

$$\Pi_{F.c} := \Big\{\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\} \in \Pi_F : \text{ (225), (226) and (251) hold } \Big\}, \tag{252}$$

$$\Pi_U := H_u \times H_p. \tag{253}$$

From what we have said so far, we get a map $\Pi_{F.c} \to \Pi_U$ by solving the unconstrained system (244)–(247). Due to the absence of nonlinear terms, the estimates in the proof ensure that this map is bounded. In the other direction, given $\{\vec{u}, p\} \in \Pi_U$, we simply define $\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\}$ using (240)–(243) and check that this lies in $\Pi_{F.c}$.

Note that in Theorem 9, one has more regularity on $\nabla\cdot\vec u$ if one assumes more on $\nabla\cdot\vec u_{\rm in}$ and h. Correspondingly, like $H_{h.s}$ defined in Theorem 9, we introduce spaces of stronger regularity by

$$H_{uin.s} := H^1(\Omega,\mathbb{R}^N)\cap\{\vec u_{\rm in}\;\big|\;\nabla\cdot\vec u_{\rm in}\in H^1(\Omega)\}, \tag{254}$$

$$\Pi_{F.s} := H_f\times H_g\times H_{h.s}\times H_{uin.s}. \tag{255}$$

The solution $\vec u$ then lies in

$$H_{u.s} := L^2(0,T;H^2(\Omega,\mathbb{R}^N))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^N)) \\ \cap\{\vec u\;\big|\;\nabla\cdot\vec u\in L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\}. \tag{256}$$

(Note, if $\vec u\in H_{u.s}$ then $\partial_t\vec u\in L^2(H(\mathrm{div};\Omega))$ so $\vec n\cdot\partial_t\vec u\in L^2(H^{-1/2}(\Gamma))$.) So as an alternative to the spaces in (252)–(253), we also obtain an isomorphism between the data and solution spaces with stronger regularity defined by

$$\Pi_{F.c.s} := \Big\{\{\vec f,\vec g,h,\vec u_{\rm in}\}\in\Pi_{F.s}:\ (225),\ (226)\text{ and }(251)\text{ hold}\Big\}, \tag{257}$$

$$\Pi_{U.s} := H_{u.s}\times H_p. \tag{258}$$

Summarizing, we have proved the following isomorphism theorem for the non-homogeneous Stokes system (240)–(243).

Theorem 10 *Let Ω be a bounded, connected domain in $\mathbb{R}^N$ with N any positive integer ≥ 2, and let $T>0$. The map $\{\vec f,\vec g,h,\vec u_{\rm in}\}\mapsto\{\vec u,p\}$, given by solving the unconstrained system (244)–(247), defines an isomorphism from $\Pi_{F.c}$ onto Π_U. The same solution procedure defines an isomorphism from $\Pi_{F.c.s}$ onto $\Pi_{U.s}$.*

Remark 5. For the standard Stokes system with zero-divergence constraints $\nabla\cdot\vec u_{\rm in}=0$ and $h=0$, existence and uniqueness results together with the estimates

$$\sup_{0\le t\le T}\|\vec u(t)\|_{H^1}+\|\vec u\|_{L^2(0,T;H^2)}+\|p\|_{L^2(0,T;H^1/\mathbb{R})} \\ \le C\big(\|\vec f\|_{L^2(0,T;L^2)}+\|\vec u_{\rm in}\|_{H^1}+\|\vec g\|_{H^{3/4}(L^2(\Gamma))}+\|\vec g\|_{L^2(H^{3/2}(\Gamma))}\big) \tag{259}$$

were obtained in the classic work of Solonnikov [Sol, Theorem 15], where more general L^p estimates were also proved. (Also see [GS1, GS2].) However, instead of the necessary compatibility condition

$$\int_\Gamma \vec n\cdot\vec g=0, \tag{260}$$

Solonnikov made the stronger constraining assumption that both the data $\vec g$ and solution $\vec u$ have zero normal component on Γ, and correspondingly his estimates do not contain a term $\|\partial_t(\vec n\cdot\vec g)\|_{L^2(H^{-1/2}(\Gamma))}$ on the right hand side of (259). (Note that when $\nabla\cdot\vec u_{\rm in}=0$ and $h=0$, we have $\int_\Gamma\vec n\cdot\vec g|_{t=0}=\int_\Omega\nabla\cdot\vec u_{\rm in}=0$ by (225), whence (260) is equivalent to (226).)

Remark 6. For the unconstrained Stokes system (244)-(246) there is an extra subtlety in determining an isomorphism from data to solution. We obtain a unique solution pair $\{\vec{u}, p\} \in \Pi_U$ given any data $\{\vec{f}, \vec{g}, h, \vec{u}_{\text{in}}\} \in \Pi_F$ that satisfy only the compatibility conditions (225) and (226) *without* (251). Consequently the map from data to $\{\vec{u}, p\}$ is not one-to-one. And, in the other direction, given $\{\vec{u}, p\}$, we can recover

$$\vec{f} = \partial_t \vec{u} + \nabla p - \nu \Delta \vec{u}, \qquad \vec{g} = \vec{u}\big|_\Gamma, \qquad \vec{u}_{\text{in}} = \vec{u}|_{t=0}. \tag{261}$$

But how are we to recover h? We need to use the fact, that follows from the definition of p_{gh} in (210), that $\nabla \cdot \vec{u} - h$ satisfies a heat equation with no-flux boundary conditions. In fact, to be able to recover h we need to know one more item, h_{in}, the initial value of h. We have

$$h = \nabla \cdot \vec{u} - w \tag{262}$$

where w is the solution of

$$\partial_t w = \nu \Delta w \ \text{ in } \Omega, \qquad \vec{n} \cdot \nabla w = 0 \ \text{ on } \Gamma, \qquad w(0) = \nabla \cdot \vec{u}|_{t=0} - h_{\text{in}}. \tag{263}$$

This procedure indicates that we should count the triple $\{\vec{u}, p, h_{\text{in}}\}$ as our solution in order to build an isomorphism with the data. Of course, the regularity of h_{in} must match that of h, recalling the embeddings in (228).

Consequently, we see that solving the unconstrained system (244)–(247) defines an isomorphism between the data spaces

$$\tilde{\Pi}_{F.c} := \Big\{ \{\vec{f}, \vec{g}, h, \vec{u}_{\text{in}}\} \in \Pi_F : \ (225) \text{ and } (226) \text{ hold} \Big\}, \tag{264}$$

$$\tilde{\Pi}_{F.c.s} := \Big\{ \{\vec{f}, \vec{g}, h, \vec{u}_{\text{in}}\} \in \Pi_{F.s} : \ (225) \text{ and } (226) \text{ hold} \Big\}, \tag{265}$$

and, respectively, the solution spaces for $\{\vec{u}, p, h_{\text{in}}\}$ given by

$$\Pi_{U.w} = H_u \times H_p \times H_{hin}, \qquad H_{hin} = L^2(\Omega), \tag{266}$$

$$\Pi_{U.s} = H_{u.s} \times H_p \times H_{hin.s}, \qquad H_{hin.s} = H^1(\Omega). \tag{267}$$

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