On optimal regularity of free boundary problems and a conjecture of De Giorgi

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Abstract

We present a general theory to study optimal regularity for a large class of nonlinear elliptic systems satisfying general boundary conditions and in the presence of a geometric transmission condition on the free-boundary. As an application we give a full positive answer to a conjecture of De Giorgi on the analyticity of local minimizers of the Mumford-Shah functional.

1 Introduction

In the seminal papers [23] and [24], Kinderlehrer, Nirenberg and Spruck started a systematic study of the higher regularity of solutions of elliptic systems on one or both sides of a free hypersurface Γ subject to overdetermined boundary conditions on Γ.

The key ingredient in their proof is the partial hodograph transformation (see [18], [22], [8, 9]) which is defined in terms of a function $u(x)$ in $\Omega \cup \Gamma \subset \mathbb{R}^N$, satisfying on Γ the condition $u = 0$, by the mapping

$$x \mapsto y = (x', u),$$

where $x' = (x_1, \ldots, x_{N-1})$. If u is a solution of the elliptic free boundary problem this change of variable transforms the problem into a new one where the boundary now is contained in a hyperplane.

It is clear that this method is strongly hinged to the presence of Dirichlet boundary conditions and thus the important problem of the regularity of solutions in the case of Neumann boundary conditions has remained open until now, with the exception of the two–dimensional case, where duality arguments can be applied to change Neumann conditions into Dirichlet (see e.g. [26]).

Moreover, although in [23] and [24] several model problems involving either equations or systems of decoupled equations were studied, as already remarked by the authors in [24], corresponding results for general elliptic systems are still missing.

The main purpose of this paper is to present a general theory to study optimal regularity for a large class of second order nonlinear elliptic systems satisfying general boundary conditions and in the presence of a geometric condition on the free-boundary, which in several applications of interest accounts for the interaction either between the physical system and the outside environment (one-phase free boundary problems), or between two different systems (two-phase free boundary problems). We will refer to this condition as *transmission condition*.

More precisely, in the case of one-phase free boundary problems, we consider systems of the form

$$F_k\left(x,\mathbf{u},D\mathbf{u},D^2\mathbf{u}\right)=0 \quad \text{in } \Omega, \quad 1\le k\le n, \tag{1.1}$$

$$G_h\left(x,\mathbf{u},D\mathbf{u},\nu\right)=0 \quad \text{on } \Gamma, \quad 1\le h\le \mu, \tag{1.2}$$

$$H\left(x,\mathbf{u},D\mathbf{u},\nu,D_\tau\nu\right)=0 \quad \text{on } \Gamma, \tag{1.3}$$

where $\Gamma\subset\partial\Omega$ is the free boundary, ν is the normal to the free boundary, and $D_\tau\nu$ denotes the tangential gradient. Here the system (1.1) is assumed to be elliptic and the boundary conditions (1.2) complementing for (1.1), in the sense of Agmon, Douglis, Nirenberg (see [2], [3], [30]), while the derivative of the function H with respect to the variable $D_\tau\nu$ is positive definite.

Assuming some initial degree of regularity on $\mathbf{u}$ and Γ we prove optimal regularity and analyticity (depending on the regularity of F_k, G_h, and H) of $\mathbf{u}$ and Γ .

To illustrate our results we consider the important special case

$$\begin{array}{ll} \Delta u=g\left(x,u,\nabla u\right) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu}=b\left(x,u\right) & \text{on } \Gamma, \\ \mathcal{K}=h\left(x,u,\nabla u\right) & \text{on } \Gamma, \end{array} \tag{1.4}$$

where $\mathcal{K}$ denotes the mean curvature of Γ. If g, b, and h are analytic and u and Γ of class $C^{1,\alpha}$ then u and Γ are analytic. To the best of our knowledge this is one of the first results in the literature on the analyticity for free boundary problems without Dirichlet condition for $N>2$.

Note that if we replace the transmission condition with a condition of the form

$$f\left(x,u,\nabla u\right)=0 \text{ on } \Gamma,$$

then the result is false in general, as illustrated by the example

$$\begin{array}{ll} \Delta u=0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \Gamma, \\ |\nabla u|=1 & \text{on } \Gamma. \end{array}$$

If $N>2$ then the function $u(x)=x_1$ solves the problem with Γ any C^1 hypersurface of the form $x_N=\gamma\left(x_2,\ldots,x_{N-1}\right)$ (see [18]).

All the regularity results obtained for (1.1)-(1.3) continue to hold for general two-phase free boundary problems. In particular if we consider the analogue of (1.4)

$$\begin{array}{ll} \Delta u^\pm=g^\pm\left(x,u^\pm,\nabla u^\pm\right) & \text{in } \Omega^\pm, \\ \dfrac{\partial u^\pm}{\partial\nu}=b^\pm\left(x,u^+,u^-\right) & \text{on } \Gamma, \\ \mathcal{K}=h\left(x,u^+,u^-,\nabla u^+,\nabla u^-\right) & \text{on } \Gamma, \end{array}$$

where now $\Gamma:=\partial\Omega^+\cap\partial\Omega^-$ is the free boundary, we can prove optimal regularity and analyticity of the free boundary under proper smoothness assumptions on the data and on the initial regularity of $(u^{\pm}, \Gamma)$. This result gives a positive answer to a conjecture of De Giorgi on the analyticity of local minimizers of the Mumford-Shah functional (see [4], [14], [15])

$$F_g(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \alpha \int_{\Omega \setminus K} |u - g|^2 \, dx + \beta \mathcal{H}^{N-1}(K \cap \Omega),$$

defined for all pairs (u, K) where K is a closed set and u is of class C^1 in $\Omega \setminus K$, precisely

Conjecture (De Giorgi) *If* g is analytic and (u, K) *is a local minimizer of the functional* F_g *such that* $K \cap A$ *is a* $C^{1,\gamma}$ *manifold for some open set* A, *then* $K \cap A$ *is analytic.*

A partial answer to this conjecture in the case $N = 2$ and $\alpha = 0$ was given in [26] (see also [4], [13] and the references contained within for previous results on the C^∞ regularity).

Our approach allows us to prove the full conjecture for general free discontinuity problems of the general form.

The paper is organized as follows. In Section 2 we present some preliminary results on elliptic systems and complementing conditions. In Section 3 we state and prove the main theorem on the optimal regularity for one-phase systems of the form (1.1)-(1.3) and we give the corresponding result for two-phase systems. In Section 4 we present several applications to systems of physical interest including free discontinuity problems and other free boundary problems arising from the elasticity theory.

Finally, in order to show the flexibility of our approach, in Section 5 we present a further result for the stationary Navier-Stokes equations with a free-capillarity condition which, although not included in the general framework of Section 3, can be treated in a similar way.

2 Preliminaries

In what follows let $\Omega \subset \mathbb{R}^N$ be an open set and define

$$\mathcal{D} := (\mathcal{D}_1, \ldots, \mathcal{D}_N), \quad \mathcal{D}_j := \frac{1}{i} \frac{\partial}{\partial y_j} \quad 1 \le j \le N.$$

Let $L_{kj}(y, \mathcal{D})$, $1 \le j, k \le n$, be linear differential operators with continuous complex valued coefficients. Consider the system of partial differential equations in the dependent variables $u^1, \ldots, u^n$

$$\sum_{j=1}^{n} L_{kj}(y, \mathcal{D}) u^j(y) = f_k(y) \quad \text{in } \Omega, \quad 1 \le k \le n. \tag{2.1}$$

To each equation we assign an integer weight $s_k \le 0$ and to each dependent variable an integer weight $t_j \ge 0$ such that

$$\begin{aligned} \text{order } L_{kj}(y, \mathcal{D}) \le s_k + t_j \quad &\text{in } \Omega, \quad 1 \le k \le n, \\ \max_k s_k = 0,& \end{aligned}$$

where we use the convention that $L_{kj}(y, \mathcal{D}) \equiv 0$ if $s_k + t_j < 0$. If we write

$$L_{kj}(y, \mathcal{D}) = \sum_{|\alpha| \le s_k + t_j} a_{kj}^{\alpha}(y) \mathcal{D}^{\alpha},$$

then the *principal part* of $L_{kj}(y, \mathcal{D})$ is denoted by $L'_{kj}(y, \mathcal{D})$ where the polynomial

$$L'_{kj}(y, \xi) := \sum_{|\alpha| = s_k + t_j} a_{kj}^{\alpha}(y) \xi^{\alpha}, \quad \xi \in \mathbb{R}^N$$

is the *principal symbol* of $L_{kj}(y, \mathcal{D})$.

Definition 2.1 *We say that the system* (2.1) *is elliptic at* y_0 *if the matrix*

$$(L'_{kj}(y_0,\xi))_{kj} \tag{2.2}$$

is non-degenerate for each $\xi \in \mathbb{R}^N \setminus \{0\}$*, and for each pair of independent vectors* ξ*,* $\eta \in \mathbb{R}^N$ *the polynomial*

$$p(z) = \det L'_{kj}(y_0, \xi + z\eta) \tag{2.3}$$

has exactly $\mu = \frac{1}{2}\deg p$ *roots with positive imaginary part and* $\mu = \frac{1}{2}\deg p$ *roots with negative imaginary part.*

The condition about the roots is automatic if $N \geq 3$. The notion of ellipticity is invariant under smooth transformations of the independent variables. If $\Psi : \Omega_1 \to \Omega$ is a diffeomorphism then the principal symbol of the transformed operator is

$$\left(L'_{kj}\left(\Psi(x), D\Psi(x)^T\xi\right)\right)_{kj}.$$

A general system of equations

$$F_k\left(y, \mathbf{u}(y), \mathcal{D}\mathbf{u}(y), \ldots, \mathcal{D}^\ell \mathbf{u}(y)\right) = 0 \quad \text{in } \Omega, \quad 1 \leq k \leq n, \tag{2.4}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathcal{D}^m$ stands for the set of all partial derivatives of order m, is *elliptic for the solution* $\mathbf{u}$ at the point $y_0 \in \Omega$ if there exist weights $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ such that the linearized equations

$$\sum_{j=1}^n L_{kj}\left(y_0, \mathcal{D}\right) \overline{u}^j\left(y\right) \tag{2.5}$$

$$:= \frac{d}{dt} F_k\left(y_0, \mathbf{u}(y_0) + t\overline{\mathbf{u}}, \mathcal{D}\mathbf{u}\left(y_0\right) + t\mathcal{D}\overline{\mathbf{u}}, \ldots, \mathcal{D}^\ell\mathbf{u}(y_0) + t\mathcal{D}^\ell\overline{\mathbf{u}})\right)\Big|_{t=0} = 0$$

constitute an elliptic system at y_0 as defined above. Again this notion is invariant under smooth change of coordinates.

Assume now that Ω is of class C^1 and let $B_{hj}\left(y, \mathcal{D}\right)$, $1 \leq h \leq \mu$, $1 \leq j \leq n$, be a linear differential operator with continuous coefficients. We say that the set of boundary conditions

$$\sum_{j=1}^n B_{hj}\left(y, \mathcal{D}\right) u^j\left(y\right) = g_h(y) \text{ on } S \subset \partial\Omega, \ 1 \leq h \leq \mu$$

is *complementing at* $y_0 \in S$ for the system (2.1) if

(i) the system (2.1) is elliptic at y_0 and

$$2\mu = \sum_{j=1}^n (s_j + t_j) \geq 0;$$

(ii) there exist integers r_h, $1 \leq h \leq \mu$, such that the order of $B_{hj}\left(y_0, \mathcal{D}\right)$ is at most $r_h + t_j$;

(iii) the homogeneous boundary value problem

$$\sum_{j=1}^n L'_{kj}\left(y_0, \mathcal{D}\right) u^j\left(y\right) = 0 \text{ in } \left\{y \in \mathbb{R}^N : \ \left(y - y_0\right) \cdot \nu\left(y_0\right) > 0\right\},$$

$$\sum_{j=1}^n B'_{hj}\left(y_0, \mathcal{D}\right) u^j\left(y\right) = 0 \text{ on } \left(y - y_0\right) \cdot \nu\left(y_0\right) = 0,$$

where $1 \leq k \leq n$, $1 \leq h \leq \mu$, and B'_{hj} is the part of B_{hj} of order $r_h + t_j$, admits no nontrivial bounded exponential solutions of the form

$$u^j(y) = e^{i\xi \cdot (y-y_0)} \varphi_j((y-y_0) \cdot \nu(y_0)), \ 1 \leq j \leq n,$$

for $\xi \in \mathbb{R}^N \setminus \{0\}$ orthogonal to the unit normal $\nu(y_0)$ to $\partial\Omega$ at y_0.

This notion of complementing boundary conditions depends only on the principal symbols of the operator and of the boundary conditions. It is again invariant under smooth changes of coordinates.

A set of (nonlinear) boundary conditions

$$G_h(y, \mathbf{u}(y), \mathcal{D}\mathbf{u}(y), \ldots, \mathcal{D}^s \mathbf{u}(y)) = 0 \quad \text{on } S, \quad 1 \leq h \leq \mu,$$

is *complementing for the system* (2.4) *for the solution* $\mathbf{u}$ at the point $y_0 \in S$ if there exist weights $r_1, \ldots, r_\mu$ such that the set of linearized boundary conditions

$$\begin{aligned}
&\sum_{j=1}^{n} B_{hj}(y_0, \mathcal{D}) \overline{u}^j(y) \\
&:= \frac{d}{dt} G_h(y_0, \mathbf{u}(y_0) + t\overline{\mathbf{u}}, \mathcal{D}\mathbf{u}(y_0) + t\mathcal{D}\overline{\mathbf{u}}, \ldots, \mathcal{D}^s\mathbf{u}(y_0) + t\mathcal{D}^s\overline{\mathbf{u}}) \Big|_{t=0} = 0
\end{aligned}$$

is complementing at y_0 for the linearized system (2.5).

The following classical theorem may be found in [30] (see also [22]).

Theorem 2.2 *Let U be a neighborhood of* 0 *in* $\mathbb{R}^N_+$ *and* $S = \partial U \cap \{y_N = 0\}$. *Assume that the system*

$$F_k\left(y, \mathbf{u}(y), \mathcal{D}\mathbf{u}(y), \ldots, \mathcal{D}^\ell \mathbf{u}(y)\right) = 0 \quad \text{in } U, \quad 1 \leq k \leq n,$$

is elliptic at 0 *and the boundary conditions*

$$G_h(y, \mathbf{u}(y), \mathcal{D}\mathbf{u}(y), \ldots, \mathcal{D}^s \mathbf{u}(y)) = 0 \quad \text{on } S, \quad 1 \leq h \leq \mu,$$

are complementing at 0 *for the solution* $\mathbf{u}$*, with weights* s_k, t_j, r_h, $1 \leq j, k \leq n$, $1 \leq h \leq \mu$.

Suppose also that F_k *is of class* $C^{-s_k+r,\alpha}$ *and* G_h *is of class* $C^{-r_h+r,\alpha}$ *for some* $\alpha > 0$ *and where* r *is an integer such that*

$$r \geq r_0 = \max_h (0, 1 + r_h).$$

If $u^j \in \mathbf{C}^{t_j+r_0}(U \cup S)$*, then* $u^j \in \mathbf{C}^{t_j+r,\alpha}((U \cup S) \cap B(0,\varepsilon))$*,* $1 \leq j \leq n$*, for some* $\varepsilon > 0$. *Moreover if* $r_0 \geq 1$ *then the same conclusion holds if the functions* u^j *are only assumed to be in* $\mathbf{C}^{t_j+r_0-1,\alpha}(U \cup S)$*,* $1 \leq j \leq n$. *Finally if* F_k *and* G_h *are analytic and* $u^j \in \mathbf{C}^{t_j+r_0}(U \cup S)$ *(respectively* $u^j \in \mathbf{C}^{t_j+r_0-1,\alpha}(U \cup S)$ *if* $r_0 \geq 1$*) then the functions* u^j *are analytic in* $(U \cup S) \cap B(0,\varepsilon)$.

3 General free boundary problems

In what follows, given a smooth $(N-1)$-dimensional manifold Γ in $\mathbb{R}^N$ and a vector field $\varphi \in C^1\left(\Gamma; \mathbb{R}^N\right)$, we define the tangential gradient $D_\tau \varphi$ at a point $x \in \Gamma$ as

$$D_\tau \varphi(x) := D\tilde{\varphi}(x)(I - \nu(x) \otimes \nu(x)), \tag{3.1}$$

where $\nu(x)$ is the unit normal vector to Γ at x and $\tilde{\varphi}$ is any smooth extension of φ to some open set Ω containing Γ. It may be verified that this definition does not depend on the particular extension of φ.

In this section we study the optimal regularity and analyticity of one-phase free boundary problems of the general form

$$F_k\left(x, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}\right) = 0 \quad \text{in } \Omega, \quad 1 \leq k \leq n, \tag{3.2}$$
$$G_h\left(x, \mathbf{u}, D\mathbf{u}, \nu\right) = 0 \quad \text{on } \Gamma, \quad 1 \leq h \leq \mu, \tag{3.3}$$
$$H\left(x, \mathbf{u}, D\mathbf{u}, \nu, D_\tau \nu\right) = 0 \quad \text{on } \Gamma, \tag{3.4}$$

where $\Gamma \subset \partial\Omega$ is the free boundary, ν is the normal to the free boundary, and $D_\tau \nu$ denotes the tangential gradient. Note that $D_\tau \nu$ is a symmetric $N \times N$ matrix; it defines a quadratic form whose restriction to the tangent space of Γ is the second fundamental form. Define

$$\hat{G}_h\left(x, \mathbf{u}(x), D\mathbf{u}(x)\right) := G_h\left(x, \mathbf{u}(x), D\mathbf{u}(x), \nu(x)\right).$$

In what follows $H = H\left(x, \mathbf{u}, P, \nu, M\right)$ and, to avoid cumbersome notation, we denote

$$[x_0]_{\mathbf{u},F_k} := \left(x_0, \mathbf{u}(x_0), D\mathbf{u}(x_0), D^2\mathbf{u}(x_0)\right),$$
$$[x_0]_{\mathbf{u},\Gamma,G_h} := \left(x_0, \mathbf{u}(x_0), D\mathbf{u}(x_0), \nu(x_0)\right),$$
$$[x_0]_{\mathbf{u},\Gamma,H} := \left(x_0, \mathbf{u}(x_0), D\mathbf{u}(x_0), \nu(x_0), D_\tau\nu(x_0)\right).$$

A similar notation will be used throughout the paper.

We now present the main result of the paper.

Theorem 3.1 *Let $(\mathbf{u}, \Gamma)$ be a solution of (3.2) − (3.4) and assume that the system (3.2) is elliptic and the set of boundary conditions $\hat{G}_h$ are complementing for (3.2) for $\mathbf{u}$ at a point $x_0 \in \Gamma$, with weights s_k, t_j, r_h, $1 \leq j,k \leq n$, $1 \leq h \leq \mu$. Suppose that $u^j \in C^{t_j+r_0-1,\alpha}\left(\overline{\Omega}\right)$, Γ is of class $C^{t_0+r_0-1,\alpha}$, where $\alpha > 0$ and*

$$r_0 := 1 + \max_{1\leq h\leq\mu}\left(0, r_h\right), \quad t_0 := \max_{1\leq j\leq n}\left(2, t_j\right)$$

and that the functions F_k, G_h, and H are of class $C^{-s_k+r,\alpha}$, $C^{-r_h+r,\alpha}$, and $C^{r,\alpha}$ for some integer $r \geq r_0$ and $\alpha > 0$ in a neighborhood of the points $[x_0]_{\mathbf{u},F_k}$, $[x_0]_{\mathbf{u},\Gamma,G_h}$, and $[x_0]_{\mathbf{u},\Gamma,H}$, respectively. Finally assume that $D_M H\left([x_0]_{\mathbf{u},\Gamma,H}\right)$ is positive definite. Then near x_0 the functions u^j are of class $C^{t_j+r,\alpha}$ up to the boundary and Γ is of class $C^{2+r,\alpha}$.

If, in addition, the functions F_k, G_h, and H are analytic in a neighborhood of the points $[x_0]_{\mathbf{u},F_k}$, $[x_0]_{\mathbf{u},\Gamma,G_h}$, and $[x_0]_{\mathbf{u},\Gamma,H}$ respectively, then near x_0 the functions u^j and Γ are analytic.

Proof. Since the result is local it is enough to consider the special case where the free boundary is given by

$$\Gamma \subset \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N = f(x')\right\} \tag{3.5}$$

and Ω lies above. In this case the boundary conditions become

$$\bar{G}_h\left(x', f, \mathbf{u}, D\mathbf{u}, \nabla_{x'} f\right) = 0 \quad \text{on } \Gamma, \quad 1 \leq h \leq \mu,$$
$$\bar{H}\left(x', f, \mathbf{u}, D\mathbf{u}, \nabla_{x'} f, \nabla^2_{x'} f\right) = 0 \quad \text{on } \Gamma,$$

where

$$\bar{G}_h(x,\mathbf{u},P,q) := G_h(x,\mathbf{u},P,\nu(q)),$$
$$\bar{H}(x,\mathbf{u},P,q,Z) := H(x,\mathbf{u},P,\nu(q),M(q,Z)),$$

with

$$v(q) := \left(\frac{q}{\sqrt{1+|q|^2}}, \frac{-1}{\sqrt{1+|q|^2}}\right), \tag{3.6}$$
$$M(q,Z) := \begin{pmatrix} \frac{Z}{\sqrt{1+|q|^2}} - \frac{Z(q\otimes q)}{\left(1+|q|^2\right)^{\frac{3}{2}}} & 0 \\ \frac{qZ}{\left(1+|q|^2\right)^{\frac{3}{2}}} & 0 \end{pmatrix} (I - v(q)\otimes v(q)).$$

Here the variables q and Z correspond to the $\nabla_{x'} f$ and to the $(N-1)\times(N-1)$ Hessian matrix $\nabla^2_{x'} f$. Note that

$$M(0,Z) := \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

Without loss of generality we may assume that

$$x_0 = 0 \quad \text{and} \quad \nabla_{x'} f(0) = 0. \tag{3.8}$$

In addition, we make the substitution $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{p}$, where the components of $\mathbf{p}$ are polynomials of degree less than or equal to 2 such that $D^l\mathbf{p}(0) = D^l\mathbf{u}(0)$, $l = 0,1,2$. We now define

$$\tilde{F}_k\left(x,\tilde{\mathbf{u}},D\tilde{\mathbf{u}},D^2\tilde{\mathbf{u}}\right) := F_k\left(x,\tilde{\mathbf{u}}+\mathbf{p},D(\tilde{\mathbf{u}}+\mathbf{p}),D^2(\tilde{\mathbf{u}}+\mathbf{p})\right),$$
$$\tilde{G}_h(x',f,\tilde{\mathbf{u}},D\tilde{\mathbf{u}},\nabla_{x'}f) := \bar{G}_h(x',f,\tilde{\mathbf{u}}+\mathbf{p},D(\tilde{\mathbf{u}}+\mathbf{p}),\nabla_{x'}f) = 0,$$
$$\tilde{H}\left(x',f,\tilde{\mathbf{u}},D\tilde{\mathbf{u}},\nabla_{x'}f,\nabla^2_{x'}f\right) := \bar{H}\left(x',f,\tilde{\mathbf{u}}+\mathbf{p},D(\tilde{\mathbf{u}}+\mathbf{p}),\nabla_{x'}f,\nabla^2_{x'}f\right).$$

where $1 \le k \le n$ and $1 \le h \le \mu$. Since by construction $D^l\tilde{\mathbf{u}}(0) = 0$ for $l = 0,1,2$, it is clear that the principal symbol of the modified system and of the modified boundary conditions at $x = 0$ remains unchanged.

Using (3.7) it follows from the hypotheses on H that $\tilde{H}$ is analytic and $D_Z\tilde{H}(0,u,p,0,Z)$ is positive definite.

For simplicity in what follows we will write F_k, G_h, H, and u_j in place of $\tilde{F}_k$, $\tilde{G}_h$, $\tilde{H}$, and $\tilde{u}_j$, where

$$D^l\mathbf{u}(0) = 0 \quad \text{for } l = 0,1,2. \tag{3.9}$$

Let $\phi \in C^2(B(0,1))$ satisfy

$$\begin{aligned} &\Delta\phi = 0 && \text{in } B^+, \\ &\phi(y',0) = f(y') && \text{for } |y'| \le 1, \end{aligned}$$

where $B^+ := \{y = (y', y_N) \in B(0,1) : y_N > 0\}$. We consider the following map

$$\begin{aligned} &\Phi : B \to \mathbb{R}^N \\ &y \mapsto (y', \phi(y) + \lambda y_N), \end{aligned}$$

where $\lambda > 0$ is chosen so large that $\det D\Phi(0) = \phi_{y_N}(0) + \lambda > 0$. By the Inverse Function Theorem there exists r positive such that

$$\Phi : B^+(0,r) \to \Phi\left(B^+(0,r)\right) \subset \Omega$$

is invertible with inverse denoted by Ψ. Define for $y \in B^{+}(0,r)$

$$\mathbf{v}(y) := \mathbf{u}(\Phi(y)), \quad \varphi(y) := \phi(y) + \lambda y_N.$$

A straightforward calculation shows that $\mathbf{v}$ satisfies the following system in $B^{+}(0,r)$ [1]

$$F_k\Big(\Phi(y), \mathbf{v}, D\mathbf{v}D_x\Psi(\Phi(y)), (D_x\Psi(\Phi(y)))^T D^2\mathbf{v}D_x\Psi(\Phi(y)) \\ + D\mathbf{v}D_x^2\Psi(\Phi(y))\Big) = 0,$$

together with the boundary conditions

$$G_h(y', f, \mathbf{v}, D\mathbf{v}D_x\Psi(\Phi(y',0)), \nabla_{y'}f) = 0 \quad \text{on } \{y_N = 0\} \cap B(0,r),$$

and the transmission condition on $\{y_N = 0\} \cap B(0,r)$

$$H\left(y', f, \mathbf{v}, D\mathbf{v}D_x\Psi(\Phi(y',0)), \nabla_{y'}f, \nabla_{y'}^2 f\right) = 0.$$

Hence $(\mathbf{v}, \varphi)$ is a solution of the following system in $B^{+}(0,r)$

$$F_k\Big(y', \varphi, \mathbf{v}, D\mathbf{v}T(\nabla\varphi), (T(\nabla\varphi))^T D^2\mathbf{v}T(\nabla\varphi) + D\mathbf{v}T_1\left(\nabla\varphi, \nabla^2\varphi\right)\Big) = 0, \tag{3.10}$$

$$\Delta\varphi = 0 \tag{3.11}$$

together with the following boundary conditions on $\{y_N = 0\} \cap B(0,r)$

$$G_h(y', \varphi, \mathbf{v}, D\mathbf{v}T(\nabla\varphi), \nabla_{y'}\varphi) = 0, \tag{3.12}$$

and the transmission conditions on $\{y_N = 0\} \cap B(0,r)$

$$H\left(y', \varphi, \mathbf{v}, D\mathbf{v}T(\nabla\varphi), \nabla_{y'}\varphi, \nabla_{y'}^2\varphi\right) = 0. \tag{3.13}$$

Here

$$T(\nabla\varphi(y)) := D_x\Psi(\Phi(y)), \quad T_1\left(\nabla\varphi, \nabla^2\varphi\right) := D_x^2\Psi(\Phi(y)).$$

We now assign to the dependent variable φ the weight $t_n + 1 := 2$ and to the equations (3.11) and (3.13) the weights $s_n + 1 = 0$ and $r_\mu + 1 := 0$ respectively. Then

$$\max_{1 \le h \le \mu+1}(0, 1 + r_h) = r_0 = 1 + \max_{1 \le h \le \mu}(0, r_h).$$

Since $v^j \in C_j^t + r_0 - 1, \alpha\left(\overline{B^{+}(0,r)}\right)$, $\varphi \in C_n^t + 1 + r_0 - 1, \alpha\left(\overline{B^{+}(0,r)}\right)$, to conclude the proof, in view of Theorem 2.2 it is enough to show in $y = 0$ ellipticity of (3.10)-(3.11) and the complementing condition for (3.12)-(3.13).

Using (3.9) it is easy to see that the principal symbol of (3.10) at 0 is given by

$$\tilde{L}'_{kj}(0,\xi) := L'_{kj}\left(0, D_x\Psi(0)^T\xi\right) \quad 1 \le k, j \le n,$$

[1] With the notation

$$(D_x\Psi(\Phi(y)))^T D^2\mathbf{v}D_x\Psi(\Phi(y)) + D\mathbf{v}D_x^2\Psi(\Phi(y))$$

we mean the third order tensor of components

$$\left((D_x\Psi)^T D^2\mathbf{v}D_x\Psi + D\mathbf{v}D_x^2\Psi\right)_{ijk} := \left((D_x\Psi)^T D^2\mathbf{v}_i D_x\Psi + D\mathbf{v}_i D_x^2\Psi\right)_{jk}.$$

Here the superscript T indicates transpose.

where $L'_{kj}(x,\xi)$ is the principal symbol of (3.2), and so, by assumption, the rank of $\tilde{L}'_{kj}(0,\xi)$ is n for every $\xi \in \mathbb{R}^N \setminus \{0\}$. Hence the principal symbol of the full system (3.10)-(3.11) is given by

$$\begin{pmatrix} \tilde{L}'_{kj}(0,\xi) & 0 \\ 0 & -\sum_{i=1}^{N} (\xi_i)^2 \end{pmatrix},$$

which has clearly rank $n+1$. Moreover, when $N=2$ for each pair of independent vectors ξ, $\eta \in \mathbb{R}^2$, the polynomial

$$p(z) = -\sum_{i=1}^{2} (\xi_i + z\eta_i)^2 \det \tilde{L}'_{kj}(0, \xi + z\eta)$$

has exactly $\frac{1}{2}\deg p$ roots with positive imaginary part and $\frac{1}{2}\deg p$ roots with negative imaginary part, since the root condition is satisfied by hypothesis for $\tilde{L}'_{kj}$. Hence the system (3.10)-(3.11) is elliptic at $y=0$ for $(\mathbf{v},\varphi)$.

To prove the complementing condition for (3.12)-(3.13), we consider the homogeneous boundary value problem in $\mathbb{R}^N_+$ obtained by linearization (according to our choice of weights) of (3.10)-(3.11) and (3.12)-(3.13), and which takes the form

$$\sum_{j=1}^{n} \tilde{L}'_{kj}(0,\mathcal{D})\, \bar{u}^j(y) = 0 \text{ in } \mathbb{R}^N_+,\ 1 \le k \le n, \tag{3.14}$$

$$\Delta\bar{\varphi} = 0 \text{ in } \mathbb{R}^N_+, \tag{3.15}$$

$$\sum_{j=1}^{n} \tilde{B}'_{hj}(0,\mathcal{D})\, \bar{u}^j(y) + \tilde{B}'_{h(n+1)}(0,\mathcal{D})\, \bar{\varphi}(y) = 0 \text{ on } y_N = 0,\ 1 \le h \le \mu, \tag{3.16}$$

$$D_Z H\left(0,0,0,0,\nabla^2_{y'}\varphi(0)\right)\cdot \nabla^2_{y'}\bar{\varphi} = 0 \text{ on } y_N = 0. \tag{3.17}$$

Note that

$$\tilde{B}'_{hj}(0,\xi) := B'_{hj}\left(0, D_x\Psi(0)^T \xi\right) \quad 1 \le h \le \mu,\ 1 \le j \le n,$$

with B'_{hj} the principal symbol of (3.3).

To check the complementing condition at 0, we need to show that it admits no nontrivial bounded exponential solutions of the form

$$\bar{u}^j(y) = e^{i\xi' y'} w_j(y_N),\ 1 \le j \le n, \quad \bar{\varphi}(y) = e^{i\xi' y'}\psi(y_N),$$

where as usual $y' = (y_1, \ldots, y_{N-1})$ and $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$. From (3.17) it follows

$$e^{i\xi'\cdot y'}\psi(0)\left(D_Z H \xi' \cdot \xi'\right) = 0,$$

and since $D_Z H$ is by assumption positive definite, we obtain $\psi(0) = 0$. On the other hand, from (3.15), we get $\frac{d^2\psi}{dt^2} - |\xi'|^2 \psi = 0$. Consequently, $\psi(t) = c\left(e^{|\xi'|t} - e^{-|\xi'|t}\right)$, which is bounded only if $c = 0$. Thus the system simplifies to

$$\sum_{j=1}^{n} \tilde{L}'_{kj}(0,\mathcal{D})\, \bar{u}^j(y) = 0 \text{ in } \mathbb{R}^N_+,\ 1 \le k \le n,$$

$$\sum_{j=1}^{n} \tilde{B}'_{hj}(0,\mathcal{D})\, \bar{u}^j(y) = 0 \text{ on } y_N = 0,\ 1 \le h \le \mu,$$

and since by assumption $\tilde{G}_h$ satisfy the complementing condition we conclude that $\bar{u}^j = 0$ for $1 \le j \le n$. This shows the complementing condition for (3.12)-(3.13).

We can now apply Theorem 2.2 to obtain the desired regularity for $(\mathbf{v},\varphi)$ and, in turn, for $(\mathbf{u},\Gamma)$. This concludes the proof of the theorem. ■

Remark 3.2 Along the same lines one can prove a result analogous to Theorem 3.1 for higher order systems of the form

$$\begin{aligned} F_k\left(x, \mathbf{u}, D\mathbf{u}, \ldots, D^{\ell}\mathbf{u}\right) &= 0 \quad \text{in } \Omega, \quad 1 \le k \le n, \\ G_h\left(x, \mathbf{u}, D\mathbf{u}, , \ldots, D^{s}\mathbf{u}, D_\tau\nu, \ldots, D_\tau^{t}\nu\right) &= 0 \quad \text{on } \Gamma, \quad 1 \le h \le \mu, \\ H\left(x, \mathbf{u}, D\mathbf{u}, \ldots, D^{s}\mathbf{u}, \nu, D_\tau\nu, \ldots, D_\tau^{m}\nu\right) &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $t \le s \le m$ and similar hypotheses apply. We leave the details to the interested reader. The condition $s \le m$ allows us to decouple the boundary conditions of the system from the transmission condition H (see the final part of the proof of Theorem 3.1). However when the boundary conditions G_h are of special form such a decoupling is possible even when $s > m$, for example for second-order elliptic equations of the form

$$F\left(x, u, \nabla u, \nabla^2 u\right) = 0 \quad \text{in } \Omega,$$

one can allow the transmission condition

$$H\left(x, u, \nabla u, \nabla^2 u, \nu, D_\tau\nu\right) = 0 \quad \text{on } \Gamma$$

with the following Neumann or Dirichlet boundary conditions:

$$\frac{\partial u}{\partial \nu} = b\,(x, u) \quad \text{or } u = g\,(x)\,.$$

We next consider two-phase free boundary problems of the general form

$$F_k^{\pm}\left(x, \mathbf{u}^{\pm}, D\mathbf{u}^{\pm}, D^2\mathbf{u}^{\pm}\right) = 0 \quad \text{in } \Omega^{\pm}, \quad 1 \le k \le n, \tag{3.18}$$

$$G_h^{\pm}\left(x, \mathbf{u}^{+}, \mathbf{u}^{-}, D\mathbf{u}^{\pm}, \nu\right) = 0 \quad \text{on } \Gamma, \quad 1 \le h \le \mu, \tag{3.19}$$

$$H\left(x, \mathbf{u}^{+}, D\mathbf{u}^{+}, \mathbf{u}^{-}, D\mathbf{u}^{-}, \nu, D_\tau\nu\right) = 0 \quad \text{on } \Gamma, \tag{3.20}$$

where $\Gamma := \partial\Omega^{+} \cap \partial\Omega^{-}$ is the free boundary. Define

$$\begin{aligned} \hat{G}_h^{+}\left(x, \mathbf{u}^{+}(x), D\mathbf{u}^{+}(x)\right) &:= G_h^{+}\left(x, \mathbf{u}^{+}(x), \mathbf{u}^{-}(x), D\mathbf{u}^{+}(x), \nu\,(x)\right), \\ \hat{G}_h^{-}\left(x, \mathbf{u}^{-}(x), D\mathbf{u}^{-}(x)\right) &:= G_h^{-}\left(x, \mathbf{u}^{+}(x), \mathbf{u}^{-}(x), D\mathbf{u}^{-}(x), \nu\,(x)\right). \end{aligned}$$

Theorem 3.3 *Let* $(\mathbf{u}^{\pm}, \Gamma)$ *be a solution of* (3.18) $-$ (3.20) *and assume that the system* (3.18) *is elliptic and the set of boundary conditions* $\hat{G}_h^{\pm}$ *are complementing for* (3.18) *for* $\mathbf{u}^{\pm}$ *at a point* $x_0 \in \Gamma$*, with weights* $s_k^{\pm}$*,* $t_j^{\pm}$*,* $r_h^{\pm}$*,* $1 \le j, k \le n$*,* $1 \le h \le \mu$*. Suppose that* $(u^{\pm})^j \in C^{t_j^{\pm}+r_0-1,\alpha}\left(\overline{\Omega}\right)$*,* Γ *is of class* $C^{t_0+r_0-1,\alpha}$*, where* $\alpha > 0$ *and*

$$r_0 := 1 + \max_{1 \le h \le \mu}\left(0, r_h^{+}, r_h^{-}\right), \quad t_0 := \max_{1 \le j \le n}\left(2, t_j^{+}, t_j^{-}\right)$$

and that the functions $F_k^{\pm}$*,* $G_h^{\pm}$*, and* H *are of class* $C^{-s_k^{\pm}+r,\alpha}$*,* $C^{-r_h^{\pm}+r,\alpha}$*, and* $C^{r,\alpha}$ *for some integer* $r \ge r_0$ *and* $\alpha > 0$ *in a neighborhood of the points* $[x_0]_{\mathbf{u}^{\pm}, F_k^{\pm}}$*,* $[x_0]_{\mathbf{u}^{\pm}, \Gamma, G_h^{\pm}}$ *, and* $[x_0]_{\mathbf{u}^{+}, \mathbf{u}^{-}, \Gamma, H}$*, respectively. Finally assume that* $D_M H\left([x_0]_{\mathbf{u}^{+}, \mathbf{u}^{-}, \Gamma, H}\right)$ *is positive definite. Then near* x_0 *the functions* $(u^{\pm})^j$ *are of class* $C^{t_j^{\pm}+r,\alpha}$ *up to the boundary and* Γ *is of class* $C^{2+r,\alpha}$*.*

If, in addition, the functions $F_k^{\pm}$*,* $G_h^{\pm}$*, and* H *are analytic in a neighborhood of the points* $[x_0]_{\mathbf{u}^{\pm}, F_k^{\pm}}$*,* $[x_0]_{\mathbf{u}^{\pm}, \Gamma, G_h^{\pm}}$ *, and* $[x_0]_{\mathbf{u}^{+}, \mathbf{u}^{-}, \Gamma, H}$*, respectively, then near* x_0 *the functions* $\mathbf{u}^{\pm}$ *and* Γ *are analytic.*

Proof. The proof is similar to that of Theorem 3.1 and we only indicate the main changes. As before it is enough to consider the special case where the free boundary is given by

$$\Gamma := \partial\Omega^+ \cap \partial\Omega^- \subset \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, x_N = f(x')\right\}.$$

Let $\varphi \in C^2(B(0,1))$ satisfy

$$\begin{aligned} \Delta\varphi &= 0 && \text{in } B^+, \\ \varphi(y', 0) &= f(y') && \text{for } |y'| \leq 1, \end{aligned}$$

where $B^+ := \{y = (y', y_N) \in B(0,1) : \, y_N > 0\}$. We consider the following maps

$$\begin{aligned} \Phi^\pm &: B \to \mathbb{R}^N \\ y &\mapsto (y', \varphi(y) \pm \lambda y_N), \end{aligned}$$

where $\lambda > 0$ is chosen so large that $\det D\Phi^+(0) = \varphi_{y_N}(0) + \lambda > 0$ and $\det D\Phi^-(0) = \varphi_{y_N}(0) - \lambda < 0$.

For $y \in B^+(0,r)$ let

$$\mathbf{v}^\pm(y) := \mathbf{u}^\pm\left(\Phi^\pm(y)\right), \quad \varphi^\pm(y) := \varphi(y) \pm \lambda y_N,$$

where $\varphi^\pm$, $\Phi^\pm$ and $\Psi^\pm$ are defined as in the previous section. A straightforward calculation shows that $(\mathbf{v}^+, \mathbf{v}^-, \varphi^+, \varphi^-)$ is a solution of the following system in $B^+(0,r)$

$$\begin{aligned} &F_k^\pm\Big(y', \varphi^\pm, \mathbf{v}^\pm, D\mathbf{v}^\pm T^\pm\left(\nabla\varphi^\pm\right), \left(T^\pm\left(\nabla\varphi^\pm\right)\right)^T D^2\mathbf{v}^\pm T^\pm\left(\nabla\varphi^\pm\right) \\ &\qquad + D\mathbf{v}^\pm T_1^\pm\left(\nabla\varphi^\pm, \nabla^2\varphi^\pm\right)\Big) = 0, \\ &\Delta\varphi^\pm = 0 \end{aligned}$$

together with the following boundary conditions on $\{y_N = 0\} \cap B(0,r)$

$$G_h^\pm\left(y', \varphi^\pm, \mathbf{v}^+, \mathbf{v}^-, D\mathbf{v}^\pm T^\pm\left(\nabla\varphi^\pm\right), \nabla_{y'}\varphi^\pm, \nabla_{y'}^2\varphi^\pm\right) = 0,$$

and the transmission conditions on $\{y_N = 0\} \cap B(0,r)$

$$\begin{aligned} &H\left(y', \varphi^+, \mathbf{v}^+, \mathbf{v}^-, D\mathbf{v}^+T^+\left(\nabla\varphi^+\right), D\mathbf{v}^-T^-\left(\nabla\varphi^-\right), \nabla_{y'}\varphi^+, \nabla_{y'}^2\varphi^+\right) = 0, \\ &H\left(y', \varphi^-, \mathbf{v}^+, \mathbf{v}^-, D\mathbf{v}^+T^+\left(\nabla\varphi^+\right), D\mathbf{v}^-T^-\left(\nabla\varphi^-\right), \nabla_{y'}\varphi^-, \nabla_{y'}^2\varphi^-\right) = 0. \end{aligned}$$

Here, as in the previous section,

$$T^\pm\left(\nabla\varphi^\pm(y)\right) := D_x\Psi^\pm\left(\Phi^\pm(y)\right), \quad T_1^\pm\left(\nabla\varphi^\pm, \nabla^2\varphi^\pm\right) := D_x^2\Psi^\pm\left(\Phi^\pm(y)\right).$$

We can now continue essentially as before. We omit the details. ■

4 Applications

In this section we present some applications of the main theorems in Section 3.

4.1 Strongly and very strongly elliptic systems

In this subsection we begin by showing that when the system (3.2) satisfies the strict Legendre-Hadamard condition, or strong ellipticity condition, then our regularity result holds with Dirichlet boundary data, namely

$$F_k\left(x,\mathbf{u},D\mathbf{u},D^2\mathbf{u}\right)=0 \quad \text{in } \Omega, \quad 1\le k\le n, \tag{4.1}$$

$$\mathbf{u}=\mathbf{b}(x) \quad \text{on } \Gamma, \tag{4.2}$$

$$H\left(x,\mathbf{u},D\mathbf{u},\nu,D_\tau\nu\right)=0 \quad \text{on } \Gamma. \tag{4.3}$$

We assume that the strict Legendre-Hadamard condition is satisfied at some point $x_0\in\Gamma$, that is,

$$(\eta\otimes\xi)\cdot D_Q\mathbf{F}\left([x_0]_{\mathbf{u},\mathbf{F}}\right)[\eta\otimes\xi]\ge\lambda\left|\eta\right|^2\left|\xi\right|^2 \tag{4.4}$$

for all $(\eta,\xi)\in\mathbb{R}^N\times\mathbb{R}^n$ and for some $\lambda>0$, where

$$\mathbf{F}\left(x,\mathbf{u},P,Q\right):=\left(F_1\left(x,\mathbf{u},P,Q\right),\dots,F_n\left(x,\mathbf{u},P,Q\right)\right).$$

Theorem 4.1 *Let $\mathbf{u}\in C^{2,\alpha}\left(\overline{\Omega};\mathbb{R}^n\right)$, $\alpha>0$, be a solution of the free-boundary problem* $(4.1)-(4.3)$, *where Γ is an $(N-1)$-manifold of class $C^{2,\alpha}$. Assume that that the functions $\mathbf{F}$, $\mathbf{b}$, and H are of class $C^{r,\alpha}$, $C^{2+r,\alpha}$, and $C^{r,\alpha}$ for some integer $r\ge1$ and $\alpha>0$ in a neighborhood of the points $[x_0]_{\mathbf{u},\mathbf{F}}$, x_0, and $[x_0]_{\mathbf{u},\Gamma,H}$, respectively. Finally, assume that $D_MH\left([x_0]_{\mathbf{u},\Gamma,H}\right)$ is positive definite and that* (4.4) *holds. Then near x_0 the function $\mathbf{u}$ is of class $C^{2+r,\alpha}$ up to the boundary and Γ is of class $C^{2+r,\alpha}$. If, in addition, the functions $\mathbf{F}$, $\mathbf{b}$, and H are analytic in a neighborhood of the points $[x_0]_{\mathbf{u},\mathbf{F}}$, x_0, and $[x_0]_{\mathbf{u},\Gamma,H}$, respectively, then near x_0 the function $\mathbf{u}$ and Γ are analytic.*

Proof. By Theorem 6.5.5 in [30] the Dirichlet conditions are complementing provided we assign weights $t_j:=2$ to the dependent variables u^j, $s_k:=0$ to the k-th equation, and $r_h=-2$ to the boundary conditions, for $1\le j,k,h\le n$. The thesis follows now from Theorem 3.1. ■

Remark 4.2 In general, Neumann boundary conditions are not complementing for the system (4.1). Indeed, consider the system corresponding to the equilibrium equations of an isotropic elastic body in the framework of linear elasticity:

$$\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\left(\operatorname{div}\mathbf{u}\right)=0 \quad \textit{in } \Omega, \tag{4.5}$$

$$\left[\mu\left(D\mathbf{u}+D\mathbf{u}^T\right)+\lambda\left(\operatorname{div}\mathbf{u}\right)I\right]\nu=0 \quad \textit{on } \partial\Omega. \tag{4.6}$$

If $\mu>0$, $\lambda+2\mu>0$, then it can been shown that (4.4) holds. In this case the complementing condition is satisfied if and only if $\lambda+\mu\neq0$ (see [29], [33], [32]).

If we strengthen the strict Legendre-Hadamard condition to the Legendre condition, then it is possible to prove that the natural Neumann boundary conditions are complementing for second-order quasilinear strongly elliptic systems of the form:

$$\sum_{j=1}^{n}\sum_{l,m=1}^{N}a_{kj}^{lm}\left(x,\mathbf{u},D\mathbf{u}\right)D_{lm}^2u^j=g_k\left(x,\mathbf{u},D\mathbf{u}\right) \quad \text{in } \Omega,\ 1\le k\le n, \tag{4.7}$$

$$\sum_{j=1}^{n}\sum_{l,m=1}^{N}a_{kj}^{l,m}\left(x,\mathbf{u},D\mathbf{u}\right)u_{x_m}^j\nu_l=0 \quad \text{on } \Gamma,\ k=1,\dots,n, \tag{4.8}$$

$$H\left(x,\mathbf{u},D\mathbf{u},\nu,D_\tau\nu\right)=0 \quad \text{on } \Gamma, \tag{4.9}$$

where $\Gamma\subset\partial\Omega$ is the free boundary.

We assume that the strict Legendre condition is satisfied at some point $x_0 \in \Gamma$, that is

$$\sum_{j,k=1}^{n} \sum_{l,m=1}^{N} a_{kj}^{l,m} \left([x_0]_{\mathbf{u}} \right) p^{lk} p^{mj} \geq \gamma \left| P \right|^2, \tag{4.10}$$

for every $P = (p^{ij}) \in \mathbb{R}^{N \times n}$ and for some $\gamma > 0$, where

$$[x_0]_{\mathbf{u}} := (x_0, \mathbf{u}(x_0), D\mathbf{u}(x_0)).$$

Theorem 4.3 *Let $\mathbf{u} \in C^{2,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$, $\alpha > 0$, be a solution of the free-boundary problem (4.7) – (4.9), where Γ is an $(N-1)$-manifold of class $C^{2,\alpha}$. Assume that that the functions $a_{kj}^{l,m}$, g_k and H are of class $C^{r,\alpha}$, for some integer $r \geq 1$ and $\alpha > 0$ in a neighborhood of the points $[x_0]_{\mathbf{u}}$, and $[x_0]_{\mathbf{u},\Gamma,H}$, respectively. Finally, assume that $D_M H\left([x_0]_{\mathbf{u},\Gamma,H}\right)$ is positive definite and that (4.10) holds. Then near x_0 the function $\mathbf{u}$ is of class $C^{2+r,\alpha}$ up to the boundary and Γ is of class $C^{2+r,\alpha}$. If, in addition, the functions $a_{kj}^{l,m}$, g_k and H are analytic in a neighborhood of the points $[x_0]_{\mathbf{u}}$, and $[x_0]_{\mathbf{u},\Gamma,H}$, respectively, then near x_0 the function $\mathbf{u}$ and Γ are analytic.*

Proof. Assign to the equations (4.7) weights $t_k := 2$, to the dependent variables u^j the integers $s_j := 0$, and to the boundary conditions (4.8) weights $r_h := -1$, $1 \leq k, j, h \leq n$.

It is clear that the equation is elliptic. In order to apply Theorem 3.1 it is enough to check the complementing condition. This follows as in Theorem 1 in [32]. ■

Remark 4.4 It is easy to see that Theorems 4.1 and 4.3 can be extended to two-phase free boundary problems.

Theorem 4.3 can be significantly improved for elliptic equation $(n = 1)$. Indeed, consider the following one-phase free boundary problem

$$F\left(x, u, \nabla u, \nabla^2 u\right) = 0 \quad \text{in } \Omega, \tag{4.11}$$

$$G(x, u, \nabla u, \nu) = 0 \quad \text{on } \Gamma, \tag{4.12}$$

$$H(x, u, \nabla u, \nu, D_\tau \nu) = 0 \quad \text{on } \Gamma. \tag{4.13}$$

In what follows $F = F(x, u, p, S)$.

Theorem 4.5 *Let $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$, $\alpha > 0$, be a solution of the free-boundary problem (4.11) – (4.13), where Γ is an $(N-1)$-manifold of class $C^{2,\alpha}$. Let $x_0 \in \Gamma$ and assume that the functions F, G and H are of class $C^{r,\alpha}$, $C^{1+r,\alpha}$ and $C^{r,\alpha}$ for some integer $r \geq 1$ and $\alpha > 0$ in a neighborhood of the points $[x_0]_{u,F}$, $[x_0]_{u,\Gamma,G}$ and $[x_0]_{u,\Gamma,H}$, respectively. Finally assume that $D_S F\left([x_0]_{u,F}\right)$ and $D_M H\left([x_0]_{u,\Gamma,H}\right)$ are positive definite and that $D_p G\left([x_0]_{u,\Gamma,G}\right) \cdot \nu(x_0) \neq 0$. Then near x_0 the function u is of class $C^{2+r,\alpha}$ up to the boundary and Γ is of class $C^{2+r,\alpha}$. If in addition the functions F, G and H are analytic in a neighborhood of the points $[x_0]_{u,F}$, $[x_0]_{u,\Gamma,G}$ and $[x_0]_{u,\Gamma,H}$, respectively, then near x_0 the function u and Γ are analytic.*

Proof. Assign to the equation (4.11) weight $t_1 := 2$, to the dependent variables u the integer $s_1 := 0$, and to the boundary condition (4.12) weight $r_1 := -1$.

It is clear that the equation is elliptic. In order to apply Theorem 3.1 it is enough to check the complementing condition. For this purpose we consider the linearized homogeneous boundary value problem

$$A \cdot \nabla^2 \bar{u}(x) = 0 \quad \text{in } x \cdot \nu > 0,$$

$$e \cdot \nabla \bar{u}(x) = 0 \quad \text{on } x \cdot \nu = 0,$$

where

$$A := D_S F\left([x_0]_{u,F}\right), \quad e := D_p G\left([x_0]_{u,\Gamma,G}\right), \quad \nu := \nu(x_0).$$

We need to show that the only bounded solution of the form

$$\bar{u}(x) = e^{ix\cdot\xi} w(x\cdot\nu),$$

where $x\cdot\nu \geq 0$, and $\xi \in \mathbb{R}^N \setminus \{0\}$ is orthogonal to ν, is identically zero. It is easy to see that $w(t)$ satisfies the Cauchy Problem

$$(A\nu\cdot\nu)\frac{d^2w}{dt^2} + 2i(A\xi\cdot\nu)\frac{dw}{dt} - (A\xi\cdot\xi)w = 0, \tag{4.14}$$

$$e\cdot(i\xi w(0) + \nu w'(0)) = 0. \tag{4.15}$$

The general solution of the ODE is given by

$$\begin{aligned} w(t) &= c_1 \exp\left[\frac{t}{(A\nu\cdot\nu)}\left(-i(A\xi\cdot\nu) + \sqrt{-(A\xi\cdot\nu)^2 + (A\nu\cdot\nu)(A\xi\cdot\xi)}\right)\right] \\ &+ c_2 \exp\left[\frac{t}{(A\nu\cdot\nu)}\left(-i(A\xi\cdot\nu) - \sqrt{-(A\xi\cdot\nu)^2 + (A\nu\cdot\nu)(A\xi\cdot\xi)}\right)\right] \end{aligned}$$

The argument of the square root is equal to $A\bar{\xi}\cdot\bar{\xi}$, where

$$\bar{\xi} := \left(\sqrt{(A\nu\cdot\nu)}\xi - \frac{1}{\sqrt{(A\nu\cdot\nu)}}(A\xi\cdot\nu)\nu\right),$$

therefore, since A is positive definite and w is bounded we obtain that $c_1 = 0$.

Finally, using (4.15) we obtain

$$c_2\left[\frac{-e\cdot\nu}{(A\nu\cdot\nu)}\sqrt{-(A\xi\cdot\nu)^2 + (A\nu\cdot\nu)(A\xi\cdot\xi)} + ie\cdot\left(\xi - \frac{(A\xi\cdot\nu)}{(A\nu\cdot\nu)}\nu\right)\right] = 0$$

and since the real part of the complex number inside the square brackets is different from zero we must have that $c_2 = 0$ and in turn $w = 0$. ∎

Under additional assumptions on the regularity and the structure of F, b, and H, it is possible to weaken the initial regularity of u and Γ. Specifically, we have the following:

Corollary 4.6 *In addition to the hypotheses of Theorem 4.5 assume that $F(x,u,p,\cdot)$ and $H(x,u,p,\nu,\cdot)$ are concave. Let $u \in C^2(\overline{\Omega})$ be a solution of the free-boundary problem (4.11) − (4.13), where Γ is an $(N-1)$-manifold of class C^2. Then the conclusions of Theorem 4.5 continue to hold.*

Proof. As in the proof of Theorem 4.5, without loss of generalit,y we can assume Γ to have the form (3.5), in turn the transmission condition becomes

$$\hat{H}\left(x', \nabla^2_{x'} f\right) = 0 \quad \text{on } \Gamma,$$

where

$$\hat{H}(x', Z) := H(x', f, u, \nabla u, \nu(\nabla_{x'} f), M(\nabla_{x'} f, Z)),$$

and ν and M are defined in (3.6). Since M is linear in Z it is clear that the function $\hat{H}$ is concave in Z and of class C^1 (by the regularity assumptions on u and Γ). Hence we may apply classical interior regularity results (see Theorem 3 and the remarks following it in [10], see also [16], [25]) to conclude that f is of class $C^{2,\alpha}$, for some $\alpha > 0$. We may now apply Theorem 5.4 in [27] (see also the remarks on pages 533 and 544 in the same reference) to obtain that u is of class $C^{2,\alpha}$ up to the boundary. ∎

Remark 4.7 Under suitable growth conditions on the functions F and G, we can relax the initial regularity of u in the previous corollary down to $C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha > 0$ (see Theorem 1.1 in [27]).

Moreover if we consider solutions in the viscosity sense (see [11]) then we can assume both u and Γ to be of class $C^{1,\alpha}$, $\alpha > 0$ (to see this, one can argue as in the corollary using the regularity results for viscosity solutions contained in [10]).

Example 4.8 *1.* (The p-Laplacian operator)

$$\begin{array}{ll} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = g(x,u,\nabla u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = b(x,u) \quad \text{or} \quad u = b(x) & \text{on } \Gamma, \\ \mathcal{K} = h(x,u,\nabla u) & \text{on } \Gamma. \end{array}$$

In this case when $p \neq 2$ the equation is elliptic only in the set $\left\{x \in \overline{\Omega} : \nabla u \neq 0\right\}$ and thus we obtain analyticity in the set

$$\{x \in \Gamma : \nabla u \neq 0\}.$$

Note that it is possible to construct nonzero solutions which vanish in an open set.

2. The following problem related to extremal domains for eigenvalues was considered by Garabedian and Schiffer [19] in dimension $N = 2$ and by Kinderlehrer, Nirenberg and Spruck [23] in arbitrary dimension:

$$\begin{aligned} \Delta u + \lambda u = 0 &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \Gamma, \\ \left(\frac{\partial u}{\partial \nu}\right)^2 = c\mathcal{K} &\quad \text{on } \Gamma, \end{aligned}$$

where $\mathcal{K}$ is the mean curvature of the unknown free boundary Γ and $c \neq 0$. In this case we can prove the analyticity of the free boundary Γ without the non-degeneracy condition $\mathcal{K} \neq 0$ which was needed in Theorem 6.1 of [23].

It is easy to show that all the results of this subsection can be extended to two-phase free boundary problems of the form

$$\begin{array}{ll} F^{\pm}\left(x, u^{\pm}, \nabla u^{\pm}, \nabla^2 u^{\pm}\right) = 0 & \text{in } \Omega^{\pm}, \\ b^{\pm}\left(x, u^+, u^-, \nabla u^{\pm}, \nu\right) = 0 & \text{on } \Gamma, \\ H\left(x, u^+, \nabla u^+, u^-, \nabla u^-, \nu, D_\tau \nu\right) = 0 & \text{on } \Gamma, \end{array} \tag{4.16}$$

where $\Gamma := \partial\Omega^+ \cap \partial\Omega^-$ is the free boundary. We omit the details.

4.2 General free discontinuity problems

In this subsection we apply our regularity results to general free discontinuity functionals of the form

$$F(\mathbf{u}, \Gamma) = \int_{\Omega\setminus\Gamma} f(x, \mathbf{u}, D\mathbf{u})\, dx + \int_{\Gamma} g(x, \mathbf{u}^+, \mathbf{u}^-, \nu)\, d\mathcal{H}^{N-1}, \tag{4.17}$$

where $\Gamma \subset \overline{\Omega}$ is a closed $(N-1)$-rectifiable set (see [4]),

$$\mathbf{u} \in W^{1,1}(\Omega \setminus \Gamma; \mathbb{R}^n) \cap L^\infty(\Omega \setminus \Gamma; \mathbb{R}^n),$$

and $\mathbf{u}^+$ and $\mathbf{u}^-$ denote the traces of $\mathbf{u}$ on both sides of Γ. The existence of the traces is guaranteed by Theorem 3.77 and Proposition 4.4 in [4].

Theorem 4.9 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Assume that:*

(i) $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a function of class $C^{2+r,\alpha}$ (respectively analytic) for some integer $r \geq 1$ and some $\alpha > 0$, with $D^2_P f(x, \mathbf{u}, P)$ positive definite for every $(x, \mathbf{u}, P) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times N}$;

(ii) $g : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times S^{N-1} \to \mathbb{R}$ is a function of class $C^{2+r,\alpha}$ (respectively, analytic) such that[2]

$$D^2_{\nu^\perp}\left(|w|\, g\left(x, \mathbf{u}_1, \mathbf{u}_2, \frac{w}{|w|}\right)\right) \text{ is positive definite}$$

for every $(x, \mathbf{u}_1, \mathbf{u}_2, w) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \left(\mathbb{R}^N \setminus \{0\}\right)$.

Let $(\mathbf{u}, \Gamma)$ be a local minimizer of the functional F. Assume that $\Gamma \cap \Omega$ is a C^1 manifold which divides Ω into two connected components and that $\mathbf{u}$ is of class $C^{2,\alpha}$ up to the boundary in each component of $\Omega \setminus \Gamma$. Then $\mathbf{u}$ is of class $C^{2+r,\beta}$ (respectively, analytic) up to the boundary, and $\Gamma \cap \Omega$ is of class $C^{2+r,\beta}$ (respectively, analytic), for some $\beta > 0$.

Proof. Let Ω^+ and Ω^- be the connected components of $\Omega \setminus \Gamma$ and denote $\mathbf{u}^\pm$ the restriction of $\mathbf{u}$ to $\Omega^\pm$. By a standard variation argument we deduce that $(\mathbf{u}^+, \mathbf{u}^-, \Gamma)$ solves the following free boundary system:

$$\operatorname{div}\left(D_P f\left(x, \mathbf{u}^\pm, D\mathbf{u}^\pm\right)\right) = D_{\mathbf{u}} f\left(x, \mathbf{u}^\pm, D\mathbf{u}^\pm\right) \quad \text{in } \Omega^\pm \tag{4.18}$$

together with the Neumann boundary conditions

$$D_P f\left(x, \mathbf{u}^\pm, D\mathbf{u}^\pm\right) \cdot \nu = \text{lower-order terms} \quad \text{on } \Gamma, \tag{4.19}$$

and the transmission condition

$$D^2_{\nu^\perp} \tilde{g}(x, \mathbf{u}^+, \mathbf{u}^-, \nu) \cdot D_\tau \nu = \text{lower-order terms} \quad \text{on } \Gamma, \tag{4.20}$$

where $\tilde{g} : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \left(\mathbb{R}^N \setminus \{0\}\right) \to \mathbb{R}$ is defined by

$$\tilde{g}\left(x, \mathbf{u}_1, \mathbf{u}_2, w\right) := |w|\, g\left(x, \mathbf{u}_1, \mathbf{u}_2, \frac{w}{|w|}\right),$$

and the right-hand sides of (4.19) and (4.20) are analytic functions that depend on lower-order terms. We only observe that to prove the transmission condition it is sufficient to assume that

$$\Gamma \cap \Omega = \{x = (x', x_N) \in \Omega' \times \mathbb{R} : \, x_N = h\left(x'\right)\},$$

where $\Omega' \subset \mathbb{R}^{N-1}$ is an open set, and to perform variations of h in the functional

$$\tilde{F}\left(\mathbf{u}, h\right) := \int\limits_{\Omega\setminus\Gamma} f(x, \mathbf{u}, D\mathbf{u})\, dx + \int\limits_{\Omega'} \hat{g}(x', h\left(x'\right), \mathbf{u}^+, \mathbf{u}^-, \nabla_{x'} h\left(x'\right))\, dx', \tag{4.21}$$

where

$$\hat{g}(x, \mathbf{u}_1, \mathbf{u}_2, q) := \tilde{g}\left(x, \mathbf{u}_1, \mathbf{u}_2, (q, -1)\right).$$

It is clear that (i) and (ii) imply the hypotheses of Theorem 4.3.

From (4.21) it is clear that the equation (4.20) can be written as an elliptic equation for the unknown h and so by classical regularity results we obtain that h is of class $C^{1,\gamma}$, and in turn, by standard Schauder estimates, of class $C^{2,\min\{\alpha,\gamma\}}$ (see [20]). ■

In the scalar case, that is when $n = 1$, we can weaken the initial regularity of minimizers u, thus obtaining the following result which includes, as a particular case, the proof of De Giorgi's conjecture quoted in the introduction.

[2] With the notation $D^2_{\nu^\perp}\left(|w|\, g\left(x, \mathbf{u}_1, \mathbf{u}_2, \frac{w}{|w|}\right)\right)$ we denote the Hessian of the function $|\cdot|\, g\left(x, \mathbf{u}_1, \mathbf{u}_2, \frac{\cdot}{|\cdot|}\right)$ restricted to the affine hyperplane tangent to ν.

Corollary 4.10 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Assume that all the hypotheses of the previous theorem are satisfied with $n = 1$.*

Let (u, Γ) be a local minimizer of the functional F. Assume that $\Gamma \cap \Omega$ is a C^1 manifold, which divides Ω into two connected components and that u is of class C^1 up to the boundary in each component of $\Omega \setminus \Gamma$. Then u is of class $C^{2+r,\beta}$ (respectively, analytic) up to the boundary, and $\Gamma \cap \Omega$ is of class $C^{2+r,\beta}$ (respectively, analytic), for some $\beta > 0$.

Proof. We only have to show that under our assumptions $u \in C^{2,\beta}\left(\overline{\Omega}^{\pm}\right)$, $\beta > 0$, and Γ is of class $C^{2,\beta}$. As in the previous theorem we obtain that Γ is of class $C^{1,\theta}$.

Since $u \in C^1\left(\overline{\Omega}^{\pm}\right)$ it is possible to construct two functions A and B satisfying all the assumptions of Theorem 2 in [28] and such that

$$\nabla_p f\left(x, u^{\pm}, \nabla u^{\pm}\right) = A\left(x, u^{\pm}, \nabla u^{\pm}\right), \quad f_u\left(x, u^{\pm}, \nabla u^{\pm}\right) = B\left(x, u^{\pm}, \nabla u^{\pm}\right)$$

in $\overline{\Omega}^{\pm}$ and thus by the same theorem there exists a positive constant γ such that $u \in C^{1,\gamma}\left(\overline{\Omega}^{\pm}\right)$. In turn, by standard Schauder estimates applied to the equation (4.20) one obtains $\Gamma \in C^{2,\min\{\alpha,\gamma\}}$(see [20]). By classical regularity results (see [27]) it now follows that $u \in C^{2,\min\{\alpha,\gamma\}}\left(\overline{\Omega}^{\pm}\right)$. ■

Under appropriate growth conditions on f the initial regularity of u can be significantly weakened provided we strengthen the initial regularity on Γ.

Corollary 4.11 *In addition to conditions* (i) *and* (ii) *in the previous theorem assume that for every L there exists $\gamma \in (0, 1]$, $0 < \lambda < \Lambda$, $m > -1$, and $k > 0$ such that*

$$\begin{aligned}
\nabla_p^2 f(x, u, p)\xi \cdot \xi &\geq \lambda\,(k + |p|)^m\,|\xi|^2\,,\\
\left|\nabla_p^2 f(x, u, p)\right| &\leq \Lambda\,(k + |p|)^m\,,\\
|\nabla_p f(x, u, p) - \nabla_p f(y, w, p)| &\leq \Lambda\,(1 + |p|)^{m+1}\left[|x - y|^\gamma + |u - w|^\gamma\right],\\
|f_u(x, u, p)| &\leq \Lambda\,(1 + |p|)^{m+2}\,,
\end{aligned}$$

for all x, $y \in \Omega$, u, $w \in [-L, L]$, p, $\xi \in \mathbb{R}^N$. Let (u, Γ) be as in the previous theorem, where u is assumed to be only in $W^{1,1}(\Omega \setminus \Gamma) \cap L^\infty(\Omega \setminus \Gamma)$ and $\Gamma \cap \Omega$ is a $C^{1,\alpha}$ manifold. Then the conclusions of the previous theorem continue to hold.

Proof. By Theorem 2 in [28] it follows (see also [4]) that $u \in C^{1,\beta}\left(\overline{\Omega}^{\pm}\right)$, for some $\beta > 0$. We now apply the previous theorem. ■

Remark 4.12 It goes almost without saying that the Mumford-Shah functional falls under the hypotheses of the previous corollary.

Finally we remark that functionals of the type (4.17) in the vectorial setting arise in the theory of brittle fracture developed by Griffith (see [21]). In this context it is worth mentioning the recent variational model of quasistatic crack growth evolution proposed by Francfort and Marigot (cite [17]). More precisely, they consider functionals of the form

$$F(\mathbf{u}, \Gamma) = \int_{\Omega\setminus\Gamma} \frac{1}{2} A(x)\mathbf{E}(\mathbf{u}) \cdot \mathbf{E}(\mathbf{u})\,dx + \int_{\Gamma} g(x, \mathbf{u}^+, \mathbf{u}^-, \nu)\,d\mathcal{H}^{N-1},$$

where $\mathbf{E}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$ is the strain tensor and $A(x)$ is a symmetric and positive definite fourth order tensor. Assuming that g satisfies the condition of Theorem 4.9, to apply our theory we have to show that the natural Neumann boundary conditions are complementing for the equilibrium equations corresponding to the bulk energy of F. Given a fixed $x_0 \in \Gamma$, this follows from Theorem 1 in [32] provided we show that

$$\int_{B^{\pm}(x_0,r)\setminus\Gamma} \frac{1}{2}A(x_0)\mathbf{E}(\mathbf{u})\cdot\mathbf{E}(\mathbf{u})\,dx \geq c \int_{B^{\pm}(x_0,r)\setminus\Gamma} |D\mathbf{u}|^2 + |\mathbf{u}|^2\,dx,$$

for all $\mathbf{u} \in C^{\infty}\left(\overline{B^{\pm}(x_0,r)};\mathbb{R}^N\right)$ with $\mathbf{u} = 0$ on a portion of $\partial B^{\pm}(x_0,r)\setminus\Gamma$. This is an immediate consequence of the definite positiveness of $A(x_0)$ and of Korn and Poincaré inequalities.

5 Stationary solutions of Navier-Stokes equations

In this section we study the analyticity of solutions of the stationary Navier-Stokes equations with a free-capillarity condition (see [7]). This problem is not included in the general framework of Section 4, but it can be treated in a similar way.

More precisely, we consider the following free boundary problem:

$$\gamma\Delta\mathbf{u} - \nabla p = u_i D_i\mathbf{u} \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{5.2}$$

$$\mathbf{u}\cdot\nu = 0 \quad \text{on } \Gamma, \tag{5.3}$$

$$T_{ij}\nu_j = \mathcal{K}\nu_i \quad \text{on } \Gamma,\ i = 1,\ldots,3, \tag{5.4}$$

where $\gamma > 0$, $\Omega \subset \mathbb{R}^3$ is an open set, $\Gamma \subset \partial\Omega$ is the free boundary, and

$$T_{ij} := \left(\frac{D\mathbf{u} + (D\mathbf{u})^T}{2}\right)_{ij} - p\delta_{ij}.$$

Theorem 5.1 *Let* $\mathbf{u} \in C^1\left(\overline{\Omega};\mathbb{R}^3\right)$, $p \in C^{1,\alpha}\left(\overline{\Omega}\right)$, $\alpha > 0$, *be a solution with* Γ *a surface of class* C^1. *Then* Γ *is analytic*

Proof. Arguing as in the previous sections we deduce that $\mathbf{u} \in C^{2,\alpha}\left(\overline{\Omega};\mathbb{R}^3\right)$ and Γ is of class $C^{3,\alpha}$. Fix $x_0 \in \Gamma$. Since the problem is rotation-invariant, without loss of generality, we may assume that

$$x_0 = 0 \quad \text{and} \quad \nu(x_0) = (0,0,1). \tag{5.5}$$

Upon the usual change of variable and using the notations of Section 3 the system becomes

$$\operatorname{div}\left(A(\nabla\varphi)\,D\mathbf{v}\right) - \varphi_{y_3}\nabla\tilde{p}\,T(\nabla\varphi) = \varphi_{y_3}\mathbf{v}\cdot\left(D\mathbf{v}\,T(\nabla\varphi)\right), \tag{5.6}$$

$$D\mathbf{v}\cdot T(\nabla\varphi) = 0, \tag{5.7}$$

$$\Delta\varphi = 0, \tag{5.8}$$

together with the following boundary conditions on $\{y_N = 0\} \cap B(0,r)$

$$\mathbf{v}\cdot(\nabla_{y'}\varphi, -1) = 0, \tag{5.9}$$

$$\operatorname{div}_{y'}\left(\frac{\nabla_{y'}\varphi}{\sqrt{1+|\nabla_{y'}\varphi|^2}}\right)(\nabla_{y'}\varphi,-1)_i = \tilde{L}_{ij}(\tilde{p},\nabla\varphi,D\mathbf{v})\,(\nabla_{y'}\varphi,-1)_j\,, \tag{5.10}$$

$i = 1,\ldots,3$, where

$$A(\nabla\varphi) := |\varphi_{y_3}|\,D_x\Psi(\Phi(y))\,(D_x\Psi(\Phi(y)))^T$$

and

$$\tilde{L}_{ij}(\tilde{p}, \nabla\varphi, D\mathbf{v}) := \left(\frac{D\mathbf{v}T(\nabla\varphi) + (D\mathbf{v}T(\nabla\varphi))^T}{2} \right)_{ij} - \tilde{p}\delta_{ij}.$$

We assign to the three equations in (5.6), to (5.7), and to (5.8) respectively the weights $s_1 = s_2 = s_3 := 0$, $s_4 = s_5 := -1$, to the dependent variables v_1, v_2, v_3, $\tilde{p}$, and φ, respectively, the weights $t_1 = t_2 = t_3 := 2$, $t_4 := 1$, and $t_5 := 3$, and to (5.9) and to the three equations in (5.10) respectively the weights $r_1 := -2$, $r_2 = r_3 = r_4 := -1$. The principal part of the linearized system at 0 is

$$\operatorname{div}(A_0 D\bar{\mathbf{v}}(y)) - \varphi_{y_3}(0)\nabla\bar{p}(y)T_0, \tag{5.11}$$

$$D\bar{\mathbf{v}}(y) \cdot T_0, \tag{5.12}$$

$$\Delta\bar{\varphi}(y), \tag{5.13}$$

while the principal part of the linearized boundary conditions at 0 becomes, also by (5.5),

$$-\bar{\mathbf{v}}_3(y) + \mathbf{v}(0) \cdot i(\nabla_{y'}\bar{\varphi}, 0) \tag{5.14}$$

$$\left(\frac{D\bar{\mathbf{v}}(y)T_0 + (D\bar{\mathbf{v}}(y)T_0)^T}{2} \right)_{i3} \qquad i = 1, 2, \tag{5.15}$$

$$\Delta_{y'}\bar{\varphi}(y) - \left(\frac{D\bar{\mathbf{v}}(y)T_0 + (D\bar{\mathbf{v}}(y)T_0)^T}{2} \right)_{33} + \bar{p}, \tag{5.16}$$

where

$$A_0 := A(\nabla\varphi(0)) = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix},$$

$$T_0 := T(\nabla\varphi(0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

with $c := \varphi_{y_3}(0)$.

To prove that the system is elliptic for each $\xi \in \mathbb{R}^3 \setminus \{0\}$, we set $\eta = i\xi$ and consider the determinant

$$\begin{vmatrix} A_0\eta\cdot\eta & 0 & 0 & \eta_j(T_0)_{1j}\,c & 0 \\ 0 & A_0\eta\cdot\eta & 0 & \eta_j(T_0)_{2j}\,c & 0 \\ 0 & 0 & A_0\eta\cdot\eta & \eta_j(T_0)_{3j}\,c & 0 \\ \eta_j(T_0)_{1j} & \eta_j(T_0)_{2j} & \eta_j(T_0)_{3j} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sum_{i=1}^3 \eta_i^2 \end{vmatrix} = c^2 (A_0\eta\cdot\eta)^2 |T_0\eta|^2$$

which is different from zero since the matrix T_0 is non-singular and A_0 is positive definite.

To show that the boundary conditions are complementing, we consider the homogeneous system in $\mathbb{R}^3_+$ associated with (5.11)-(5.13) together with the homogeneous boundary conditions associated with (5.14)-(5.16) on $y_3 = 0$ and study the bounded solutions of the special form

$$(\bar{\mathbf{v}}, \bar{p}, \bar{\varphi})(y) = e^{i\xi'\cdot y'}(\mathbf{w}(y_3), p(y_3), \psi(y_3)),$$

where $\xi' \in \mathbb{R}^2 \setminus \{0\}$. Simple manipulations show that $\frac{d^2p}{dt^2} - c^2|\xi'|^2 p = 0$ whose general bounded solutions are given by $p(t) = Ce^{-c|\xi'|t}$. Hence we obtain the following system:

$$\frac{d^2 w_j}{dt^2} - c^2 |\xi'|^2 w_j - c^2 i\xi_j C e^{-c|\xi'|t} = 0, \quad j = 1, 2, \tag{5.17}$$

$$\frac{d^2 w_3}{dt^2} - c^2 |\xi'|^2 w_3 + c^2 |\xi'| C e^{-c|\xi'|t} = 0, \tag{5.18}$$

$$\frac{d^2 \psi}{dt^2} - |\xi'|^2 \psi = 0 \tag{5.19}$$

$$ci\xi_1 w_1 + ci\xi_2 w_2 + \frac{dw_3}{dt} = 0, \tag{5.20}$$

together with the initial conditions

$$-w_3(0) + \mathbf{v}(0) \cdot i(\xi', 0)\psi(0) = 0, \tag{5.21}$$

$$ci\xi_j w_3(0) + \frac{dw_j}{dt}(0) = 0, \quad j = 1, 2, \tag{5.22}$$

$$c|\xi'|^2 \psi(0) + \frac{dw_3}{dt}(0) - C = 0. \tag{5.23}$$

By differentiating (5.20), evaluating at $t = 0$ and using (5.22) we obtain $\frac{d^2 w_3}{dt^2}(0) = -c^2 |\xi'|^2 w_3(0)$ which, upon substitution in (5.18), yields $w_3(0) = \frac{C}{2|\xi'|}$ and, in turn, from (5.18), $w_3(t) = \frac{C}{2|\xi'|} e^{-c|\xi'|t} + \frac{cC}{2} t e^{-c|\xi'|t}$. Since $\frac{dw_3}{dt}(0) = 0$ the boundary conditions now reduce to the following linear system

$$-\frac{C}{2|\xi'|} + \mathbf{v}(0) \cdot i(\xi', 0)\psi(0) = 0, \quad C - c|\xi'|^2 \psi(0) = 0$$

$$ci\xi_j \frac{C}{2|\xi'|} + \frac{dw_j}{dt}(0) = 0, \quad j = 1, 2,$$

in the unknowns $\psi(0)$, $\frac{dw_1}{dt}(0)$, $\frac{dw_2}{dt}(0)$, C, whose determinant is equal to $\mathbf{v}(0) \cdot i(\xi', 0) - \frac{1}{2}|\xi'| c \neq 0$. Hence $\psi(0) = \frac{dw_1}{dt}(0) = \frac{dw_2}{dt}(0) = C = 0$ from which it is clear that $(\bar{\mathbf{v}}, \bar{p}, \bar{\varphi})$ is identically zero. ■

Acknowledgment

The authors would like to thank Gary Lieberman for useful bibliographic references. The authors thank the Center for Nonlinear Analysis (NSF Grant No. DMS–9803791) for its support during the preparation of this paper. The research of G. leoni was partially supported by the National Science Foundation under Grant No. DMS-0405423.

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