COMPOSITES OF HYPERELASTIC MATERIALS WITH PRESCRIBED LAVRENTIEV GAP FUNCTIONS

Dedicated to Walter Noll and Jerry Ericksen in Honor of Their 80th Birthdays

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In 1985, J. M. Ball and V. J. Mizel raised the question of whether there exist nonlinearly elastic materials possessing a physically natural stored energy density [one which is independent of an observer's coordinate frame (objective) and is invariant under a group of linear transformations (isotropic)] as well as physically reasonable boundary value problems for such materials, such that the infimum of the total stored energy for continuous deformations of the material meeting the boundary conditions (admissible deformations) and belonging to a Sobolev space W^{1,p_2} for some $p_2 > 1$ is strictly greater than its infimum for those admissible deformations which belong to a Sobolev space W^{1,p_1} with $p_1 < p_2$, despite the density of the former Sobolev space in the latter. The question was motivated by M. Lavrentiev's 1926 demonstration of such a gap for 1-dimensional variational boundary value problems on a bounded interval whose smooth integrand satisfied the conditions of Tonelli's existence theorem - as well as improved versions developed in the 1980's. Thereafter, M. Foss demonstrated in 2000 that there are (nonphysical) model problems in which the infimum over $W^{1,p}$ varies continuously with p. The positive (2-dimensional) resolution of the 1985 Ball/Mizel question was achieved by Foss, Hrusa and Mizel in 2003.

The present article constructs for each given positive continuous increasing function i on a compact interval $[p^0, p^1]$ of the half axis $(1, \infty)$ a "composite" of 2-dimensional hyperelastic materials such that for a certain continuous deformation of the composite satisfying prescribed boundary conditions the relation

$$(*) \qquad E|_{W^{1,p}} = i(p) \text{ for all } p \text{ in } [p^0,\ p^1].$$

for the minimal stored energy E holds. The continuous deformation $\boldsymbol{u}$ will be constructed as the sum of an infinite series of mappings associated with $i(\cdot)$, each of which is defined on a proper subdomain of the domain Ω_α of $\boldsymbol{u}$. That is, for specified $\alpha > \beta > 0$ (with $\alpha < 2\pi$) we present examples in 2-dimensions of deformations $\boldsymbol{u} \in A_1^{\alpha,\beta} = \bigoplus_{\alpha' \in (0,\infty]} W^{1,1}(\Omega_{\alpha'}, \mathbb{R}^2)$ where (in polar coordinates) the domains $\Omega_\alpha, \Omega_{\alpha'}$ and Ω_β are given by

$\Omega_\alpha = \{\boldsymbol{x} \in \mathbb{R}^2 \backslash \{(0,0)\} : r(\boldsymbol{x}) < 1,\ \theta(\boldsymbol{x}) \in (0, \alpha)\}$,

$\Omega_{\alpha'} = \{\boldsymbol{x} \in \mathbb{R}^2 \backslash \{(0,0)\} :\ r(x) < 1,\ \theta(\boldsymbol{x}) \in (0, \alpha')\}$,

$\Omega_\beta = \{\boldsymbol{u} \in \mathbb{R}^2 \backslash \{(0,0)\} : r(\boldsymbol{u}) < 1,\ \theta(\boldsymbol{u}) \in (0, \beta)\}$,

and $\Gamma_{3,\alpha}, \Gamma_{3,\alpha'}, \Gamma_{3,\beta}$ denote the curvilinear boundaries of these domains. The deformations under consideration depend on a parameter q with $q \in [q^0, q^{00}]$ for some specified $q^{00} > q^0 > 1$ and, using complex value notation, are defined by

$$(1) \qquad \begin{cases} \boldsymbol{u}_{PM,q}(\boldsymbol{x}) = r(\boldsymbol{x})^{\delta q} e^{i\gamma_q \theta(\boldsymbol{x})} \\ \boldsymbol{u}_{AM,q}(\boldsymbol{x}) = r(\boldsymbol{x})^{\gamma_q} e^{i\gamma_q \theta(\boldsymbol{x})} \end{cases}$$

where

$\gamma_q = 1 - \frac{1}{q} - \epsilon_q,\ \delta_q = 1 - \frac{1+\gamma_q}{2q-1} = 1 - \frac{1}{q} + \frac{\epsilon_q}{2q-1}$ for a small $\epsilon_q > 0$. We put $\alpha'_q = \alpha$ when $\gamma_q \alpha > \beta$, and we put $\alpha'_q = \beta/\gamma_q$ otherwise, so $\alpha'_q < \alpha$.

Thus the associated stored energy E $:= J_q[\boldsymbol{u}]$ is expressed in terms of the first derivatives of $\boldsymbol{u} = u + iv$ [employing complex notation] with $\frac{\partial}{\partial \bar{z}}(u+iv) = (u_x - v_y) + i(v_x + u_y)$ as

$$J_q[\boldsymbol{u}] = \int_{\Omega_{\alpha'_q}} \left(\left[\left| \left(\cos\theta(\boldsymbol{x}) \frac{\partial u(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{\sin\theta(\boldsymbol{x})}{r(\boldsymbol{x})} \frac{\partial u(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} - \sin\theta(\boldsymbol{x}) \frac{\partial v(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{\cos\theta(\boldsymbol{x})}{r(\boldsymbol{x})} \frac{\partial v(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} \right) \right. \right. \right.$$

$$\left. \left. \left. + i \left(\cos\theta(\boldsymbol{x}) \frac{\partial v(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{\sin\theta(\boldsymbol{x})}{r(\boldsymbol{x})} \frac{\partial v(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} + \sin\theta(\boldsymbol{x}) \frac{\partial u(\boldsymbol{x})}{\partial r(\boldsymbol{x})} + \frac{\cos\theta(\boldsymbol{x})}{r(\boldsymbol{x})} \frac{\partial u(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} \right) \right|^2 \right]^q \right) dxdy$$

which leads to

(2)

$$J_q[\boldsymbol{u}] = \int_{\Omega\alpha'_q} \left[\left(\frac{\partial u(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{1}{r(\boldsymbol{x})} \frac{\partial v(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} \right)^2 + \left(\frac{1}{r(\boldsymbol{x})} \frac{\partial u(\boldsymbol{x})}{\partial \theta(\boldsymbol{x})} + \frac{\partial v(\boldsymbol{x})}{\partial r(\boldsymbol{x})} \right)^2 \right]^q dxdy$$

where $\alpha'_q = \alpha \wedge \dfrac{\beta}{1 - \frac{1}{q} - \epsilon_q}$ for $\epsilon_q > 0$ sufficiently small. Hereafter we refer to the stored energy density integrand in (2) as $W_q(\nabla \boldsymbol{u})$, where $\boldsymbol{u} = (u, v)$.

It follows from (1) that

$$u_{PM,q} \in A_p^{\alpha,\beta},\ \forall p \in \left[1, \frac{2}{1-\delta_q}\right) = \left[1, \frac{2q}{1 - \epsilon_q \frac{q}{2q-1}}\right),$$
$$u_{AM,q} \in A_p^{\alpha,\beta},\ \forall p \in [1, \frac{2}{1-\gamma_q}) = \left[1, \frac{2q}{1+q\epsilon_q}\right).$$

Thus if we put $p_q^* = \frac{2q}{1+q\epsilon_q},\ p_q^{**} = \frac{2q}{1-\epsilon_q \frac{q}{2q-1}}$ it follows that $p_q^* < p_q^{**}$

with

$$p_q^{**} = p_q^* + 2q\left(\frac{1}{1-\epsilon_q\frac{q}{2q-1}} - \frac{1}{1+q\epsilon_q}\right) \approx p_q^* + \frac{4q^3}{2q-1}\epsilon_q. \tag{3}$$

In addition (cf. [15]):

$$\nabla u_{PM,q} = r(\boldsymbol{x})^{-\frac{1+\gamma_q}{2q-1}} \begin{pmatrix} \left(1-\frac{1+\gamma_q}{2q-1}\right)\cos\gamma_q\theta(\boldsymbol{x}) & -\gamma_q\sin\gamma_q\theta(\boldsymbol{x}) \\ \left(1-\frac{1+\gamma_q}{2q-1}\right)\sin\gamma_q\theta(\boldsymbol{x}) & \gamma_q\cos\gamma_q\theta(\boldsymbol{x}) \end{pmatrix} \tag{4}$$

$$\det \nabla u_{PM,q} = \gamma_q\left(1-\frac{1+\gamma_q}{2q-1}\right) r(\boldsymbol{x})^{-\frac{2(1+\gamma_q)}{2q-1}} > 0$$

Now it is easy to verify that (cf. [15]) the form of J_q for each $q \in [q^0, q^{00}] \subset (1, \infty)$ is given by

$$\begin{cases} J_q[u_{PM,q}] = \int_{\Omega_{\alpha'_q}} [r(\boldsymbol{x})^{2(\delta_q-1)} \left[(\delta_q-\gamma_q)^2\right]^q \; dxdy = \\ \alpha'_q \left[(\delta_q-\gamma_q)^2\right]^q \dfrac{1}{2q(\delta_q-1)+2} =: K_q \quad \forall p \in [1, p_q^{**}) \\ J_q[\boldsymbol{u}_{AM,q}] = 0, \;\; \forall p \in [1, p_q^*), \end{cases} \tag{5}$$

whereby, as indicated in [15], the infimum of J_q, for $q \in [q^0, q^{00}]$ for some $q^{00} > q^0 > 1$ is given by

(6)
$$\begin{cases} \inf_{\tilde{A}p,\alpha'_q} J_q[\boldsymbol{u}] = J_q[\boldsymbol{u}_{AM,q}] = 0, \;\; \forall p \in [1, p_q^*) \\ \inf_{\tilde{A}p,\alpha'_q} J_q[\boldsymbol{u}] = J_q[\boldsymbol{u}_{PM,q}] = K_q, \forall p \in [p_q^*, p_q^{**}) \text{ [actually, for } \forall p \in (p_q^*, \infty)]. \end{cases}$$

In view of the constancy of "$J_q[u_{PM,q}]$"$(\cdot)$ we will henceforth express this function in terms of the characteristic function $I_{[c,\infty)}$ of an associated half axis. Now suppose that $[p^0, p^1] \subset (1, \infty)$ is a prescribed closed interval and that $i : [p^0, p^1] \to \mathbb{R}$ is a prescribed positive increasing continuous function. Our goal is to produce a "composite hyperelastic material of type I" such that its minimal stored energy function E satisfies

$$(**) \qquad E|_{W^{1,p}} = i(p) \quad \forall p \in [p^0, p^1].$$

The strategy to be utilized involves constructing an infinite series of functions with terms of the form

$$\frac{1}{q_k} \int_{\Omega_{\alpha'}} \left(|\nabla u_{PM,q_k}|^2\right)^{q_k} dx\, dy \quad q_k \in \left[q^0, q^{00}\right] \subset (1, \infty),$$

where it is known (cf. [15]) that for each $q > 1$

$$\text{“}\int_{\Omega_{\alpha'}} |\nabla u_{PM,q}|^{2q}\, dx\, dy\text{”}(\cdot)$$

is constant along the half axis $p \in [q, \infty)$. Thus, our method is along the lines of an exercise in chapter 7 of *Principles of Mathematical Analysis* (3rd edition) by W. Rudin.

Utilizing the fact that $\boldsymbol{u}_{PM,q}$ is a solution of the Euler-Lagrange system for our variational problem we obtain

$$(7)\qquad \begin{aligned} &\inf_{\tilde{A}_{p,\alpha'_q}} J_q[\boldsymbol{u}] = \text{“}J_q[\boldsymbol{u}_{PM,q}]\text{”} = K_q, \quad \forall p \in [p^*_q, \infty);\\ &\inf_{\tilde{A}p,\alpha'_q} J_q(\boldsymbol{u}) = J_q[\boldsymbol{u}_{AM,q}] = 0, \qquad \forall p \in [1, p^*_q) \qquad \text{where } p^*_q = \frac{2q}{1+q\epsilon_q}. \end{aligned}$$

We proceed as follows under the assumption (to avoid a trivial conclusion):

$$(\#) \qquad r = i(p^*_{q^0}) < i(p^*_{q^{00}}) := r + S \text{ with } r \geq 0,\ S > 0.$$

Step 1 Select an integer $n_0 > 1$ and let $q_{0,1}$ be the smallest value in $[q^0, q^{00}]$ satisfying

(i)$_0$ $r + \frac{1}{2^{n_0}} SI_{[p^*_{q_{0,1}},\infty)}(\cdot) \equiv i(p^*_{q_{0,1}})$,

(ii)$_0$ Next let $q_{0,2}$ denote the smallest value in $[q_{0,1}, q^{00}]$ satisfying

$$r + \tfrac{2}{2^{n_0}} SI_{[p^*_{q_{0,2}},\infty)}(\cdot) \equiv i(p^*_{q_{0,2}}),$$

(iii)$_0$ Next let $q_{0,3}$ denote the smallest value in $[q_{0,2}, q^{00}]$ satisfying

$$r + \tfrac{3}{2^{n_0}} SI_{[p^*_{q_{0,3}},\infty)}(\cdot) \equiv i(p^*_{q_{0,3}})$$

and continue this procedure until we arrive at

$(2^{n_0} - 1)_0$. Let $q_{0,2^{n_0}-1}$ denote the smallest value in $[q_{0,2^{n_0}-2}, q^{00}]$ satisfying $r + \frac{2^{n_0}-1}{2^{n_0}} SI_{[p^*_{q_{0,2^{n_0}-1}},\infty)}(\cdot) \equiv i(p^*_{q_{0,2^{n_0}-1}})$.

It is readily seen that the function

$$f^0 = r + \sum_{k=1}^{2^{n_0}-1} \frac{k}{2^{n_0}} SI_{[p^*_{q_{0,k}},\infty)}(\cdot)$$

is a step function on $[p^0, p^1]$ which agrees with $i(\cdot)$ at the points $\{p^*_{q_{0,k}}\}$ and that

$$(C_0) \qquad 0 \le i(\cdot) - f^0(\cdot) \le \frac{1}{2^{n_0}} S;$$

Step 2 Select an integer $n_1 > n_0$ and let $q_{1,1}$ be the smallest value in $[q^0, q^{00}]$ satisfying

(i)$_1$ $r + \frac{1}{2^{n_1}} SI_{[p^*_{q_{1,1}},\infty)}(\cdot) \equiv i(p^*_{q_{1,1}})$

(ii)$_1$ Next let $q_{1,2}$ denote the smallest value in $[q_{1,1}, q^{00}]$ satisfying

$r + \frac{2}{2^{n_1}} SI_{[p^*_{q_{1,2}},\infty)}(\cdot) \equiv i(p^*_{q_{1,2}})$

(iii)$_1$ Next let $q_{1,3}$ denote the smallest value in $[q_{1,2}, q^{00}]$ satisfying

$r + \frac{3}{2^{n_1}} SI_{[p^*_{q_{1,3}},\infty)}(\cdot) \equiv i(P^*_{q_{1,3}})$,

and continue this process until arriving at step $(2^{n_1} - 1)_1$:

Let $q_{1,2^{n_1}-1}$ denote the smallest value in $[q_{0,2^{n_1}-2}, q^{00}]$ satisfying

$r + \frac{2^{n_1}-1}{2^{n_1}} SI_{[p^*_{q_{1,2^{n_1}-1}},\infty)}(\cdot) \equiv i(p^*_{q_{1,2^{n_1}-1}})$.

It is clear that the function $f^1 = r + \sum_{k=1}^{2^{n_1}-1} \frac{k}{2^{n_1}} SI_{p^*_{q_{1,k}},\infty}(\cdot)$ is a step function on $[p^0, p^1]$ such that $f^1 \ge f^0$ and f^1 agrees with $i(\cdot)$ at the points $p^*_{q_{1,k}}$ and satisfies

$$(C_1) \qquad 0 \le i(\cdot) - f^1(\cdot) \le \frac{1}{2^{n_1}} S.$$

By continuing in this fashion we produce a sequence of step functions $f^j(\cdot)$ which converge uniformly to $i(\cdot)$ on $[p^0, p^1]$ and for which the associated points

$p^*_{q_{s,j}}$ are densely distributed outside intervals of constancy of the monotone continuous function $i(\cdot)$. Now the results described in [14] and [15] imply that for each $q > 1$ the associated "pseudominimizer" is given by

$$\boldsymbol{u}_{PM,q} = r(\boldsymbol{x})^{\frac{2q-2-\gamma_q}{2q-1}} \begin{pmatrix} \cos\gamma_q\theta(\boldsymbol{x}) \\ \sin\gamma_q\theta(\boldsymbol{x}) \end{pmatrix} = r(\boldsymbol{x})^{1-\frac{1+\gamma_q}{2q-1}} \begin{pmatrix} \cos\gamma_q\theta(\boldsymbol{x}) \\ \sin\gamma_q\theta(\boldsymbol{x}) \end{pmatrix}$$

which [avoiding complex-valued notation] we abbreviate as

(8)
$$\boldsymbol{u}_{PM,q} = r(\boldsymbol{x})^{\delta_q} \begin{pmatrix} \cos\gamma_q\theta(\boldsymbol{x}) \\ \sin\gamma_q\theta(\boldsymbol{x}) \end{pmatrix}, \text{ with}$$

$$\gamma_q = 1 - \frac{1}{q} - \epsilon_q, \delta_q = 1 - \frac{1+\gamma_q}{2q-1} = 1 - \frac{1}{q} + \frac{\epsilon_q}{2q-1}, \text{for a small } \epsilon_q > 0.$$

Thus

$$\nabla\boldsymbol{u}_{PM,q} = (r(\boldsymbol{x}))^{\frac{-(1+\gamma_q)}{2q-1}} \begin{pmatrix} \delta_q\cos\gamma_q\theta(\boldsymbol{x}) & -\gamma_q\sin\gamma_q\theta(\boldsymbol{x}) \\ \delta_q\sin\gamma_q\theta(\boldsymbol{x}) & \gamma_q\cos\gamma_q\theta(\boldsymbol{x}) \end{pmatrix}$$

whence by equation (2) for $\boldsymbol{u} = u + iv$ we have

(9)
$$W_q(\nabla\boldsymbol{u}_{PM,q}) = \left[\left(\frac{\partial u(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{1}{r(\boldsymbol{x})}\frac{\partial v(\boldsymbol{x})}{\partial\theta(\boldsymbol{x})}\right)^2 + \left(\frac{1}{r(\boldsymbol{x})}\frac{\partial u(\boldsymbol{x})}{\partial\theta(\boldsymbol{x})} + \frac{\partial v(\boldsymbol{x})}{\partial r(\boldsymbol{x})}\right)^2\right]^q.$$

$$= [||\nabla\boldsymbol{u}||^2 - 2\det\,\nabla(\boldsymbol{u})]^q = r(\boldsymbol{x})^{\frac{-2q(1+\gamma_q)}{2q-1}}\,(\delta_q - \gamma_q)^{2q}.$$

Thus

$$J_q[\boldsymbol{u}_{PM,q}] = \int_{\Omega\alpha'_q}\left[\left(\frac{\partial u(\boldsymbol{x})}{\partial r(\boldsymbol{x})} - \frac{1}{r(\boldsymbol{x})}\frac{\partial v(\boldsymbol{x})}{\partial\theta(\boldsymbol{x})}\right)^2 + \left(\frac{1}{v(\boldsymbol{x})}\frac{\partial u}{\partial\theta(\boldsymbol{x})} + \frac{\partial r(\boldsymbol{x})}{\partial r(\boldsymbol{x})}\right)^2\right]^q dxdy$$

so that by (8)

$$E(\boldsymbol{u}_{PM,q}) = \text{``}\int_{\Omega_{\alpha'_q}} W_q(\nabla u_{PM,q})dxdy\text{''} = \tag{10}$$
$$(\delta_q-\gamma_q)^{2_q}\alpha'_q\left(2-\frac{1}{q}+\frac{2}{2q-1}\epsilon_q\right)^{-1} =: M_q \text{ with } \alpha'_q > \frac{\beta}{\gamma_q},$$

whence $\alpha'_q < \alpha$ for $\epsilon_q > 0$ small enough.

It follows from (9) that for all $p \geq \frac{2}{1+\gamma_q}$ one has

$$E(u_{PM,q})|_{W^{1,p}} \equiv M_q.$$

We now consider a final issue in demonstrating the continuity of the mapping from the composite to the region Ω_β. For all exponents $p^*_{q_{j,k}} \in [p^0, p^1]$ which arose in connection with the construction associated with $i(\cdot)$, we wish to ascertain those values $q \in [q^0, q^{00}]$ which are associated with these exponents. Recall that for a given $q \geq q^0 > 1$ we have $\gamma_q = 1 - \frac{1}{q} - \epsilon_q,\ \delta_q =$ $1-\frac{1}{q}\left(1-\frac{\epsilon_q}{2q-1}\right)$ for some $\epsilon_q > 0$ so that $u_{PM,q} = r(\boldsymbol{x})^{\delta_q}\begin{pmatrix}\cos\gamma_q\theta(\boldsymbol{x})\\ \sin\gamma_q\theta(\boldsymbol{x})\end{pmatrix}$ must be such that the p values in $[p^0, p^1]$ for which $E|_{W^{1,p}} \neq 0$ must include $p^*_q = \frac{2}{1-\gamma_q} = \frac{2q}{1+q\epsilon_q}$.

Thus for a given $p^*_l \in [p^0, p^1]$ one obtains

$$q_{p^*_l} = \frac{p^*_l}{2}(1-\frac{p^*_l}{2}\epsilon_q)^{-1} \text{ so that } q_{p^*_l} = \frac{p^*_l}{2\left(1-\frac{p^*_l}{2}\epsilon_{q_{p^*_l}}\right)} \tag{11}$$

This ensures that on $\Gamma_{3,\alpha}\quad \alpha q_{p^*_l} > \beta$, so that on choosing $\alpha'_q\gamma_{q_{p^*_l}} = \beta$ one finds $\alpha'_q < \alpha$. It follows that $W_q(\nabla u_{PM,q}) \in L^1$ if and only if

$\gamma_q = 1 - \frac{1}{q} - \epsilon_q, \delta_q = 1 - \frac{1}{q} + \frac{\epsilon_q}{2q-1}$, whence $\delta_q - \gamma_q = \frac{2q}{2q-1}\epsilon_q$.

Thus $E|_{W^{1,p}} \neq 0$ for all those values γ_{q_l} with $p^*_{ql} = \frac{1}{1-\frac{1}{q-1}\epsilon}$ corresponding to $q_{p^*_l} \sim \frac{p^*_l}{2}\left(1 + \epsilon_{q_{p^*_l}}\right)$. This ensures that $p_l \mapsto q^*_{p_l}$ is a continuous mapping from $\Gamma_{3,\alpha'}$ onto $\Gamma_{3,\beta}$. Thus [for the case where $i(\cdot)$ has no interval of constancy] the mapping described here when applied to the dense sequence of exponents in $[p^0, p^1]$ yields a dense sequence on a subinterval of $[q^0, q^{00}]$, from which the continuity result for the mapping from the composite to Ω_β follows. [An adaptation to the case where $i(\cdot)$ has intervals of constancy can be easily achieved.]

In view of the brevity of the description provided for the construction which was referred to as a "final issue" it seems worthwhile to discuss matters in a bit more detail. First, given a small number $e_1 > 0$ it follows from our construction of the step functions $f^0, f^1, \ldots$ approximating the monotone continuous function $i(\cdot)$ on $[p^0, p^1]$ (cf. $(C_0), (C_1)$) that for some sufficiently large k the step function f^k will satisfy the requirement

$$(12)\quad f^k(p) = \sum_{l=1}^{2^k-1}(r + \frac{l}{2^{n_k}} SI_{[p_{qk,l} - \infty)}(p) = i(p) \text{ for } p_{qk,1} \in [p^0, p^1], 1 \leq l \leq 2^k - 1$$

as well as

$$0 \leq i(\cdot) - f^k(\cdot) \leq \frac{1}{2^{n_k}} S \text{ in } [p^0, p^1],\ q_{k,l+1} - q_{k,l} < e_1$$

Since by (8)

$$\boldsymbol{u}_{PM,q} = r(\boldsymbol{x})^{\delta_q} \begin{pmatrix} \cos\gamma_q\theta(\boldsymbol{x}) \\ \sin\gamma_q\theta(\boldsymbol{x}) \end{pmatrix}, \ \nabla\boldsymbol{u}_{PM,q} = r(\boldsymbol{x})^{\delta q-1} \begin{pmatrix} \delta_q\cos\gamma_q\theta(x) & - & \gamma_q\sin\gamma_q\theta(x) \\ \delta_q\sin\gamma_q\theta(x) & & \gamma_q\cos\gamma_q\theta(x) \end{pmatrix}$$

with $\gamma_q = 1 - \dfrac{1}{2} - \epsilon_q, \delta_q = 1 - \dfrac{1+\gamma_q}{2q-1} = 1 - \dfrac{1}{q} + \dfrac{\epsilon_q}{2q-1}$, for some $\epsilon_q > 0$, (9) yields

$$W_q(\nabla\boldsymbol{u}_{PM,q}) = [||\nabla\boldsymbol{u}_{PM,q}||^2 - 2\det\nabla\boldsymbol{u}_{PM,q}]^q = r(\boldsymbol{x})^{2q(\delta_q-1)}(\delta_q-\gamma_q)^{2q} \in L^1(\Omega_{\alpha'};\Omega_\beta)$$

so that (10) yields

(13)

$J_q[\boldsymbol{u}_{pM,q}] = \alpha'_q\dfrac{2q-1}{2q\epsilon_q}\left(\dfrac{2q}{2q-1}\epsilon_q\right)^{2q} = \alpha'_q\left(\dfrac{2q}{2q-1}\epsilon_q\right)^{2q-1} =: K_q$. Thus by (∗∗) and (7)

$E|_{W^{1,p}} =$ "$J_q[\boldsymbol{u}_{PMq}](p)$" $= K_1$ for all $p \in [p_q^*, p_q^{**})$ [actually for all $p \in [p_q^*, \infty)$]

We now consider one further item. For those exponents $p_{q_{k,l}} \in [p^0, p^1]$ associated with the entries $q_{k,l} \in [q^0, q^{00}]$ involved in constructing the function f^k we find $p \geq p_q^* = \frac{2q}{1+q\epsilon_q} = \frac{2}{1-\gamma_q}$ so that $\frac{1+\epsilon_{q_{k,l}}}{q_{k,l}} < \frac{2}{p_{q_{k,l}}}$

This leads to $q_{kl} \approx \frac{p_{q_{k,l}}}{2}$ so that the q values corresponding to the given p values are close to one another – thus ensuring the claimed **continuity** of the q mappings.

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