Scalar minimizers with fractal singular sets

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Abstract

Lack of regularity of local minimizers for convex functionals with non-standard growth conditions is considered. It is shown that for every $\varepsilon > 0$ there exists a function $a \in C^{\alpha}(\Omega)$ such that the functional

$$\mathcal{F} : u \mapsto \int_{\Omega} (|Du|^p + a(x)|Du|^q)\, dx$$

admits a local minimizer $u \in W^{1,p}(\Omega)$ whose set of non-Lebesque points is a closed set Σ with $\dim_{\mathcal{H}}(\Sigma) > N - p - \varepsilon$, and where $1 < p < N < N + \alpha < q < +\infty$.

Key words. Gap, non standard growth conditions, regularity, Hausdorff dimension

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1. Introduction

The aim of this paper is to present an example of a convex regular integral in the Calculus of Variation which admits a minimizer with a very wild singular set. The striking features of the example are that the problem is scalar and the Hausdorff dimension of the singular set is quite big; in fact, the set of non-Lebesgue points is of Cantor type.

We consider the functional

$$\mathcal{F} :\ u \mapsto \int_{\Omega} f(x, Du)\, dx\,, \quad u \in W^{1,p}(\Omega), \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative Carathéodory function.

A classical result due to Giaquinta & Giusti ([GG]), resting on De Giorgi's iteration method ([DG1]), states that any local minimizer $u \in W^{1,p}(\Omega)$ of a functional of type (1) satisfying, for some $p > 1$ and $1 < L < +\infty$, the growth assumptions

$$L^{-1}|\xi|^p - L \le f(x,\xi) \le L(|\xi|^p + 1), \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N, \tag{2}$$

is locally Hölder continuous. This result holds true even without any convexity assumption on f. Observe that this is not the case in the vectorial setting $u : \Omega \to \mathbb{R}^k$, $k > 1$ (see for instance the counterexamples in [DG2], [N], [SY1], [SY2]). If the functional does not meet the condition (2), but satisfies only the more flexible (p,q)-growth conditions (following the terminology introduced by Marcellini in [M2])

$$L^{-1}|\xi|^p - L \le f(x,\xi) \le L(|\xi|^q + 1), \qquad 1 \le L, \qquad 1 < p < q < +\infty, \tag{3}$$

then the continuity of minimizers generally fails, provided p and q are not near enough, as shown by counterexamples (see [Gia], [H], [M1], [M2]). In particular, in the paper [M2], Marcellini presents a minimizer of an autonomous functional within the structure (3) exhibiting a singular set which is a line. In [M3] he raised the question of finding a minimizer of a regular functional (that is, with f being a Hölder or Lipschitz continuous function) with an isolated singularity. This problem was solved in a sharp way in [ELM]. All these examples require that the ratio q/p is not very close to 1. Indeed, in [ELM] it was shown that if $f(\cdot,\xi)$ is Hölder continuous with an exponent $\alpha \in (0,1]$, then a sufficient condition for a minimizer (which, by definition, is a priori only in $W^{1,p}_{\mathrm{loc}}(\Omega)$) to be in $W^{1,q}_{\mathrm{loc}}(\Omega)$ is that

$$\frac{q}{p} < 1 + \frac{\alpha}{N} \,. \tag{4}$$

This condition is sharp, as proved in [ELM], what shows that for this type of functionals the regularity of minimizers depends on a subtle interplay between the size of the ratio q/p and the regularity of $f(x,\xi)$ with respect to the variable x.

In what follows, we will be particularly interested on functionals of type (1) of the form

$$f(x,\xi) = |\xi|^p + a(x)|\xi|^q, \qquad a(x) \ge 0.$$

In this setting it is possible to conctruct examples of singular minimizers of functionals with coefficients having a cone type singularity. In the two dimensional case, Zhikov [Z] constructed an example exhibiting the Lavrentiev phenomenon. A similar geometry has been used in [ELM] to produce sharp examples of minimizers with isolated singularities in arbitrary dimension. This is the starting point of this paper where, based on the construction in [ELM] and via an iterative process, we build local minimizers having fractal singular sets.

Since until now all known examples provided singularities which were either isolated or concentrated on a line, we found it natural to ask how "bad" the singular set of a minimizer of a functional with (p,q) growth can be. Our aim is to show that under natural assumptions (that is, as soon as the sufficient condition for regularity in (4) is violated) minimizers may have very wild singular sets of Cantor type, with Hausdorff dimension "touching" the borderline case. Indeed, we have

Theorem 1. *For every choice of the parameters:*

$$2 \le N\,, \qquad \alpha \in (0,+\infty)\,, \qquad 1 < p < N < N+\alpha < q < +\infty\,, \qquad \varepsilon > 0\,, \tag{5}$$

there exist a functional

$$\mathcal{F}:\; u \mapsto \int_\Omega \big(|Du|^p + a(x)|Du|^q\big)\,dx, \qquad u \in W^{1,p}(\Omega), \tag{6}$$

with $\Omega \subset \mathbb{R}^N$ being a bounded Lipschitz domain, $a \in C^\alpha(\Omega)$, $a \ge 0$, a local minimizer $u \in W^{1,p}(\Omega)$ of $\mathcal{F}$ and a closed set $\Sigma \subset \Omega$ with

$$\dim_{\mathcal{H}}(\Sigma) > N - p - \varepsilon$$

such that all the points of Σ are non-Lebesgue points of the precise representative of u.

In the previous statement, and using standard notation, C^α denotes the space of all functions with continuous derivatives up to the order $[\alpha]$ (the integer part of α, whereas $\{\alpha\} := \alpha - [\alpha]$ is the noninteger part of α), with the $[\alpha]$-th derivative being $\{\alpha\}$-Hölder continuous. We use the symbol $\dim_{\mathcal{H}}(A)$ for the Hausdorff dimension of a set $A \subset \mathbb{R}^N$.

Observe that the results of the previous theorem are sharp in more than one respect. Indeed, the choice of the parameters made in (5) is necessary (at least when $\alpha \in (0,1]$) in order to violate the condition (4). Moreover, because u is a Sobolev function in $W^{1,p}(\Omega)$, the Hausdorff dimension of the singular set cannot exceed $N-p$, and here we can reach $N-p-\varepsilon$ for every $\varepsilon > 0$, while all the counterexamples presented up to now exhibited singular sets of dimension at most 1. Note that from condition (6) it follows that the more we want the integrand function f to be smooth the more we need p and q to be far apart. Furthermore, the Hausdorff dimension of the singular set Σ does not depend on the choice of the parameters α and q (but, of course, it does depend on N, p and ε). Finally, if we allow the function f to be only Hölder continuous with respect to x, then we can ensure that u is regular outside Σ, in the sense that $u \in W^{1,q}_{\mathrm{loc}}(\Omega \setminus \Sigma)$. Precisely,

Theorem 2. *For every choice of the parameters*

$$2 \le N\,, \quad \alpha \in (0,1)\,, \quad 1 < p < N < N+\alpha < q < \Big(1 + \frac{1}{N}\Big)\,p\,, \quad \varepsilon > 0\,,$$

there exist a functional

$$\mathcal{F}:\ u \mapsto \int_\Omega \left(|Du|^p + a(x)|Du|^q\right) dx\,, \qquad u \in W^{1,p}(\Omega),$$

with $\Omega \subset \mathbb{R}^N$ *being a bounded Lipschitz domain,* $a \in C^\alpha(\Omega)$*,* $a \geq 0$*, a local minimizer* $u \in W^{1,p}(\Omega)$ *of* $\mathcal{F}$ *and a closed set* $\Sigma \subset \Omega$ *with*

$$\dim_{\mathcal{H}}(\Sigma) > N - p - \varepsilon,$$

such that all the points of Σ *are non-Lebesgue points of the precise representative of* u*. Moreover,*

$$u \in W^{1,q}_{\text{loc}}(\Omega \setminus \Sigma),$$

and, in particular, the function u *is continuous at every point of* $\Omega \setminus \Sigma$*.*

2. Preliminaries

Recall that a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is said to be a *local minimizer* of a functional $\mathcal{F}$ of type (6) if

$$\int_{\text{supp }(u-v)} f(x, Du)\ dx \leq \int_{\text{supp }(u-v)} f(x, Dv)\ dx$$

for every function $v \in W^{1,1}_{\text{loc}}(\Omega)$ with supp $(u - v) \subset \Omega$.

We will use the following result from [ELM, Theorem 3.1 and Sections 4 and 5]

Theorem 3. *Let* $u \in W^{1,p}_{\text{loc}}(\Omega)$ *be a local minimizer of the functional*

$$u \mapsto \int_\Omega \left(|Du|^p + a(x)|Du|^q\right) dx, \qquad u \in W^{1,p}_{\text{loc}}(\Omega)\,,$$

where $a \in \text{Lip}(\Omega)$*,* $a \geq 0$*. Suppose that the exponents* $1 < p < q < +\infty$*, are such that*

$$\frac{q}{p} < 1 + \frac{1}{N}\,.$$

Then

$$u \in W^{1,q}_{\text{loc}}(\Omega).$$

We say that u is *precisely represented* if

$$u(x) = \lim_{r\to 0+} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy$$

whenever the limit on the right exists.

In what follows, C denotes a generic constant which may vary from expression to expression.

3. Construction

3.1. Underlying geometry

We will work in the space $\mathbb{R}^N$, and we set

$$K_0 = (-1,1)^N, \qquad \mathbb{V} = \{-1,1\}^{N-1} \times \{0\}.$$

We will construct sequences of sets and functions on the unit cube K_0. Fix a parameter $\lambda \in (0,1/2)$ and use the powers λ^n as radii in our construction. We define inductively a countable family of finite sets $\mathcal{Z}_n \subset (-1,1)^{N-1} \times \{0\}$, for $n = 0,1,2,\dots$, whose elements will be used as centers for cubes in our construction. We start by letting

$$\mathcal{Z}_0 := \{0_{\mathbb{R}^N}\}.$$

Let $n \in \mathbb{N}$ and suppose that we have constructed $\mathcal{Z}_k$ for $k = 0,\dots,n-1$. Define

$$\mathcal{Z}_n^z := \{z + \lambda^n V : V \in \mathbb{V}\}, \quad z \in \mathbb{R}^N, \quad n \in \mathbb{N},$$
$$\mathcal{Z}_n := \bigcup_{z \in \mathcal{Z}_{n-1}} \mathcal{Z}_n^z.$$

Since $\mathcal{Z}_k \cap \mathcal{Z}_n = \emptyset$ for $k < n$, without any ambiguity and in order to simplify the notation, we will use often $z \in \mathcal{Z}_n$ also as a subscript, and the information about the integer n under consideration will be implicitly contained in the symbol z. We set

$$K_z := z + (-\lambda^n, \lambda^n)^N,$$
$$K_z' := z + (-2\lambda^{n+1}, 2\lambda^{n+1})^{N-1} \times (-\lambda^{n+1}, \lambda^{n+1}).$$

If $z \in \mathcal{Z}_n$ then we consider in each cube K_z a concentric block K_z' which we split into 2^{N-1} cubes $K_{z'}$, $z' \in \mathcal{Z}_{n+1}^z$.

For any $x \in K_z$ we write

$$\phi_z(x) := \max\Big\{\lambda^{n+1}, \tfrac{\lambda^n - \lambda^{n+1}}{\lambda^n - 2\lambda^{n+1}} \left|\hat{x} - \hat{z}\right|_\infty - \tfrac{\lambda^n \lambda^{n+1}}{\lambda^n - 2\lambda^{n+1}}\Big\},$$

where we denote by $\hat{z}$ the projection of z onto $(-1,1)^{N-1}$, i.e. $\hat{z} := (z_1,\dots,z_{N-1})$, and

$$\left|x\right|_\infty := \max\{|x_1|,|x_2|,...,|x_k|\}, \qquad \text{for } x = (x_1,x_2,...,x_k) \in \mathbb{R}^k, \quad k \in \mathbb{N}.$$

Now, we partition each cube K_z into the "p-domain" P_z and the "q-domain" Q_z, precisely

$$P_z := \{x \in K_z : \ |x_N| < \phi_z(x)\},$$
$$Q_z^+ := \{x \in K_z : \ x_N > \phi_z(x)\},$$
$$Q_z^- := \{x \in K_z : \ -x_N > \phi_z(x)\},$$
$$Q_z := Q_z^+ \cup Q_z^- .$$

3.2. Construction of the functional

We consider the functional (6) with $\Omega = K_0$ and

$$f(x,\xi) := |\xi|^p + a(x)|\xi|^q\,, \qquad x \in K_0\,,\ \xi \in \mathbb{R}^N\,,$$

where the function a is constructed below. Let $z \in \mathcal{Z}_n$. We set

$$\begin{aligned}
L_z^+ &:= z + (-\lambda^{n+1}, \lambda^{n+1})^{N-1} \times (2\lambda^{n+1}, 2\lambda^n),\\
M_z^+ &:= z + (-2\lambda^{n+1}, 2\lambda^{n+1})^{N-1} \times (\lambda^{n+1}, \lambda^{n-1}),\\
L_z^- &:= z + (-\lambda^{n+1}, \lambda^{n+1})^{N-1} \times (-2\lambda^n, -2\lambda^{n+1}),\\
M_z^- &:= z + (-2\lambda^{n+1}, -2\lambda^{n+1})^{N-1} \times (-\lambda^{n-1}, -\lambda^{n+1}),
\end{aligned}$$

and, finally,

$$L_z := L_z^+ \cup L_z^-\,, \qquad\qquad M_z := M_z^+ \cup M_z^-\,.$$

Let η_z be a C^∞ cut-off function between L_z and M_z, i.e. $0 \le \eta_z \le 1$, $\eta_z = 1$ in L_z, the support of η_z is contained in M_z, and

$$|\nabla^s \eta_z| \le C\lambda^{-ns} \qquad \forall\, s \in \mathbb{N}\,, \tag{7}$$

where $C \equiv C(N,\lambda)$ is independent of $n \in \mathbb{N}$. We set

$$a(x) := \sum_{n\in\mathbb{N}} \sum_{z\in\mathcal{Z}_n} \lambda^{n\alpha}\eta_z(x)\,, \qquad x \in K_0. \tag{8}$$

3.3. Hölder estimates for the coefficient a

We claim that

$$a(x) \in C^\alpha(\Omega)\,. \tag{9}$$

We first check that $a(x) \in C^k(\Omega)$, where $k := [\alpha]$. Note that by the definition of η_z, the sum (8) locally reduces to the sum of no more than three functions, so it suffices we reduce ourself to prove a uniform (with respect to $z \in \mathcal{Z}_n$, $n \in \mathbb{N}$) bound for $\lambda^{n\alpha}\|\nabla^k\eta_z\|_\infty$. This follows immediately from (7) applied to $s = k$, precisely

$$\|\lambda^{n\alpha}\eta_z\|_{C^k(\Omega)} \le C\lambda^{n(\alpha-k)} \le C \qquad \forall\, n \in \mathbb{N}\,.$$

In the case where α is not an integer we must prove that $\nabla^k a(x)$ is Hölder continuous with exponent $\{\alpha\}$. For any $x, y \in K_0$, we have

$$|\nabla^k a(x) - \nabla^k a(y)| \le \sum_{n\in\mathbb{N}} \sum_{z\in\mathcal{Z}_n} \lambda^{n\alpha}|\nabla^k\eta_z(x) - \nabla^k\eta_z(y)|\,, \tag{10}$$

where, arguing as before, there are at most six nonvanishing terms in the sum (10). Therefore, it suffices to prove that there exists an absolute constant such that for every $z \in \mathcal{Z}_n$, $n \in \mathbb{N}$,

$$\lambda^{n\alpha}|\nabla^k\eta_z(x) - \nabla^k\eta_z(y)| \le C|x-y|^{\{\alpha\}}\,.$$

We distinguish two cases. If $|x-y| \leq \lambda^n$ then, using the Mean Value Theorem and (7) with $s = k+1$, we have

$$\begin{aligned}\lambda^{n\alpha}|\nabla^k \eta_z(x) - \nabla^k \eta_z(y)| &\leq \lambda^{n\alpha}||\nabla^{(k+1)}\eta_z||_\infty |x-y| \\ &= \lambda^{n\alpha}||\nabla^{(k+1)}\eta_z||_\infty |x-y|^{1-\{\alpha\}}|x-y|^{\{\alpha\}} \\ &\leq C\lambda^{n\alpha}\lambda^{-n(k+1)}|x-y|^{1-\{\alpha\}}|x-y|^{\{\alpha\}} \\ &\leq C|x-y|^{\{\alpha\}}\ .\end{aligned}$$

If $|x-y| > \lambda^n$ then, again by the Mean Value Theorem and (7) with $s = k+1$, we obtain

$$\begin{aligned}\lambda^{n\alpha}|\nabla^k \eta_z(x) - \nabla^k \eta_z(y)| &\leq \lambda^{n\alpha}||\nabla^{(k+1)}\eta_z||_\infty |x-y| \\ &\leq C\lambda^{n\alpha}\lambda^{-n(k+1)}|x-y| \\ &\leq C\lambda^{n(\alpha-k)} \\ &\leq C|x-y|^{\{\alpha\}}\ ,\end{aligned}$$

where C is the absolute constant appearing in (7). The proof of (9) is now complete.

3.4. Low energy competitor

We construct recursively a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$. First we set

$$u_0(x) := \begin{cases} \frac{x_N}{\phi_0(x)} & \text{if } x \in P_0, \\ 1 & \text{if } x \in Q_0^+, \\ -1 & \text{if } x \in Q_0^-. \end{cases}$$

For a general $n \in \mathbb{N}$, and assuming that u_1, u_2, ..., u_{n-1} have been determined, we define the corrector function c_z as

$$c_z(x) := \begin{cases} \frac{x_N}{\phi_z(x)} & \text{if } x \in P_z, \\ 1 & \text{if } x \in Q_z^+, \\ -1 & \text{if } x \in Q_z^-. \end{cases}$$

We now correct u_{n-1} with c_z in each of the blocks K_z, $z \in \mathcal{Z}_n$, as

$$u_n(x) := \begin{cases} u_{n-1}(x) & \text{if } x \notin \cup_{z\in\mathcal{Z}_n} K_z, \\ c_z(x) & \text{if } x \in K_z\ , \end{cases} \qquad z \in \mathcal{Z}_n.$$

Note that $u_0 \equiv c_0$ and that $||u_n||_\infty \leq 1$. With $n \in \mathbb{N}$ fixed and $z \in \mathcal{Z}_n$, a direct calculation shows that

$$|\nabla c_z(x)| \leq C\lambda^{-n}, \qquad x \in K_z$$

where $C \equiv C(N,p,\lambda)$ is an absolute constant. Hence

$$\int_{K_z} |\nabla c_z|^p \, dx \leq C\lambda^{n(N-p)},$$

and thus

$$\int_{K_z} |\nabla u_{n-1}|^p \, dx \leq C\lambda^{n(N-p)}\,, \qquad n \geq 1. \tag{11}$$

Taking into account the cardinality of the construction, by (11) we obtain

$$\int_{K_0} |\nabla u_n - \nabla u_{n-1}|^p \, dx \leq C\, 2^{n(N-1)}\lambda^{n(N-p)}.$$

This shows that $u_n \to u$ in $W^{1,p}$ for some $u \in W^{1,p}(\Omega)$ provided

$$\lambda < 2^{\frac{1-N}{N-p}}\ . \tag{12}$$

Moreover, we observe that by the definitions of $a(x)$ and c_z it follows that

$$a(x)|\nabla c_z(x)|^q = 0 \quad \text{for each} \quad x \in K_z \setminus K'_z\ .$$

Since

$$\left| K_0 \setminus \left(\bigcup_n \bigcup_{z \in \mathcal{Z}_n} (K_z \setminus K'_z) \right) \right| = 0\ ,$$

it follows that

$$a(x)|\nabla u(x)|^q = 0 \quad \text{for each} \quad x \in K_0\ .$$

Therefore if we set

$$C_1 := \int_{K_0} |\nabla u|^p \, dx = \int_{K_0} \big(|\nabla u|^p + a(x)|\nabla u|^q\big)\, dx\,,$$

then for any $c \in \mathbb{R}$ we have

$$\int_{K_0} \big(|\nabla (cu)|^p + a(x)|\nabla (cu)|^q\big)\, dx = c^p C_1\ . \tag{13}$$

The latter equality will be used in the sequel.

3.5. The minimizer and its singular set.

Set

$$v_0(x) := cx_N$$

where $c > 0$ is a constant to be suitably chosen below. We define $v \in v_0 + W_0^{1,p}(\Omega)$ as the unique solution to the following Dirichlet problem:

$$\begin{cases} \text{Min } \mathcal{F}(w) \\ w \in v_0 + W_0^{1,p}(\Omega) \ . \end{cases}$$

The existence of v follows via the direct methods of the Calculus of Variations. The uniqueness follows by the fact that the functional $\mathcal{F}$ is strictly convex. The function v is obviously a local minimizer of $\mathcal{F}$. We are going to show that for suitable large values of the parameter c, the function v is the local minimizer we are looking for and its singular set is given by

$$\Sigma := \bigcap_n \bigcup_{z \in \mathcal{Z}_n} K_z \ ,$$

which is a closed set. Let us note that in the proof of the singular behavior we do not use the full strength of the information that v is minimizer, but only the boundary data and the property that v has lower energy than the competitor u.

Fix $y \in \Sigma$. Then there exists a sequence $z_n(y) \to y$ such that $z_n(y) \in \mathcal{Z}_n$ and $y \in K_{z_n(y)}$, and with $z_n := z_n(y)$ we define

$$T_y^+ := \bigcup_n L_{z_n}^+(y), \qquad T_y^- := \bigcup_n L_{z_n}^-(y) \ .$$

Set

$$\ell_+ := \liminf_{x \to x_0, x \in T_y^+} v(x), \qquad \ell_- := \limsup_{x \to x_0, x \in T_y^-} v(x) \ .$$

Considering the precise representative of v, we have, by Morrey's theorem (see eg. [B, Theorem IX.12])

$$\operatorname*{osc}_{L_{z_n}^+} v := \operatorname*{ess\,sup}_{x,y \in L_{z_n}^+} |v(x) - v(y)| \leq C \Big(\int_{L_{z_n}^+} |\nabla v|^q \, dx \Big)^{1/q} \lambda^{n\left(1 - \frac{N}{q}\right)}$$

with C independent of $n \in \mathbb{N}$. Notice that the closures $\overline{L}^+_{z_n}$ of $L^+_{z_n}$ overlap the corresponding subsequent ones and thus

$$\begin{aligned}
c - \ell_+ &\le C \sum_n \operatorname*{osc}_{\overline{L}^+_{z_n}} v \\
&\le C \sum_n \Big(\lambda^{n(q-N)} \int_{L^+_{z_n}} |\nabla v|^q \, dx \Big)^{1/q} \\
&\le C \Big(\sum_n \lambda^{n\alpha} \int_{L^+_{z_n}} |\nabla v|^q \, dx \Big)^{1/q} \Big(\sum_n \lambda^{\frac{n(q-N-\alpha)}{q-1}} \Big)^{1/q'} \\
&\le CA \Big(\int_{Q^+} a(x) |\nabla v|^q \, dx \Big)^{1/q} \\
&\le CA \Big(\int_{Q^+} |\nabla (cu)|^p + a(x) |\nabla (cu)|^q \, dx \Big)^{1/q} \\
&\le C_2 A C_1^{1/q} c^{p/q},
\end{aligned} \tag{14}$$

where

$$A = \Big(\sum_n \lambda^{\frac{n(q-N-\alpha)}{q-1}} \Big)^{1/q'}$$

and u is the low energy competitor constructed in Subsection 3.4. Note that this last series converge because of our choice

$$N + \alpha < q \, .$$

This is the only point in the proof where we need to use this condition. Also, observe that in the last two estimates in (14) we have used the local minimality of v and (13). Now, if c is chosen suitably large, (14) implies that $\ell_+ > 0$. Taking into account a symmetric estimate for the lower part, we obtain for the same choice of c

$$\ell_+ - \ell_- \ge 2c - 2C_2 A C_1^{1/q} c^{p/q} > 0. \tag{15}$$

The estimate

$$|T^+_y \cap (y + (-2\lambda_n, 2\lambda_n)^N)| \ge |L^+_{z_n}| \ge \frac{1}{C} \, \lambda^{nN}$$

shows that the lower Lebesgue density of both sets T^+_y, T^-_y at y is strictly positive. Therefore the point y cannot be a Lebesgue point for v (no matter what is the value $v(y)$).

Remark 1. Observe that we actually proved that the size of the jump (see (15)) at singular points is bounded away from zero by a constant independent of $y \in \Sigma$, and depending only on the boundary data u_0.

If we take into account the convention on precise representative, we can evaluate the values $v(y)$ at the singular points, namely $v \equiv 0$ on Σ. This follows from the symmetry properties of the minimizer.

3.6. Estimate of $\dim_{\mathcal{H}}(\Sigma)$.

Let ψ_z be the linear function mapping K_0 onto K_z, $z \in \mathcal{Z}_1$. Then the restriction of ψ_z to Σ maps Σ onto $\Sigma \cap K_z$. Hence Σ is self-similar in the sense of the discussion in Section 8.3 of [F] and, in view of (8.21) in [F],

$$0 < \mathcal{H}^d(\Sigma) < \infty$$

for d computed from the equation

$$2^{N-1}\lambda^d = 1.$$

By (12), the only restriction we put on λ is that $0 < \lambda < 2^{\frac{1-N}{N-p}}$, which enables to make the Hausdorff dimension of Σ arbitrarily close to $N-p$, choosing λ in a suitable way.

Proof of Theorem 1. It suffices to summarize the considerations made above in this section.

Proof of Theorem 2. The only part of the statement that remains to be proven is the one asserting that the minimizer v, found in the proof of Theorem 1 is actually in $W^{1,q}_{\mathrm{loc}}(\Omega \setminus \Sigma)$. In order to do this, we consider an arbitrary open subset $A \subset\subset \Omega \setminus \Sigma$. There exists an integer $n \in \mathbb{N}$ such that

$$A \subset\subset B := \Omega \setminus \left(\bigcup_{s \geq n} \bigcup_{z \in \mathcal{Z}_s} K_z \right).$$

Now, we observe that a is smooth on B. Indeed, the sum appearing in (8) reduces on B to the following:

$$a(x) = \sum_{s=1}^{n+2} \sum_{z \in \mathcal{Z}_s} \lambda^{s\alpha} \eta_z(x)\,, \qquad x \in B\,,$$

therefore, on the set B the function a is a finite sum of smooth functions and so it is also smooth. We are now in position to apply Theorem 3 (with Ω replaced by the open subset A) to assert that the function v belongs to $W^{1,q}_{\mathrm{loc}}(A)$. Since A was arbitrary, this concludes the proof.

Remark 2. The functionals that we considered in Theorems 1 and 1 may be replaced by their non-degenerate analogs

$$\int_\Omega \left[(1+|Du|^2)^{\frac{p}{2}} + a(x)(1+|Du|^2)^{\frac{q}{2}}\right] dx$$

with a slight modification of the proofs given here (in particular, the choice of the coefficient $a(x)$ remains the same). Therefore, we can consider energy densities f which are smooth also with respect to the gradient variable.

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