Twin Balance Laws for Bodies Undergoing Structured Motions

David R. Owen

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

March 5, 2001

Contents

To Gianfranco Capriz, with admiration and gratitude

1 Introduction

The purpose of this article is the derivation of two pairs of balance laws for a continuous body undergoing structured motions; each pair will be called a pair of "twin balance laws" for reasons that will become apparent in what follows. Specifically, twin balance laws for forces and twin balance laws for moments will be derived by adapting postulates that require or imply invariance of the power [1], [2] to the non-classical setting of structured motions. The key features provided by structured motions in this analysis are (i) a natural and unambiguous additive decomposition of the power that reflects two different types of submacroscopic changes and (ii) a broadening of the collection of classical motions that permits a strengthening of the consequences implied by invariance of the power.

Structured motions [3] provide a setting in which the influences on macroscopic deformations and velocities of both smooth and non-smooth sub-macroscopic geometrical changes can be identified in a simple, rigorous and direct manner. The velocity and deformation due to *non-smooth* sub-macroscopic changes are called the velocity and deformation *due to disarrangments (*literally— *due to the upsetting of an arrangement)*, and both the macroscopic velocity and the macroscopic deformation gradient can be decomposed into a sum of a term due to disarrangements and a term without disarrangements. In Section 2, decompositions and identification relations obtained in an earlier study of structured motions [3] are recorded and discussed. These relations justify the attributives "without disarrangements" and "due to disarrangements" and permit the introduction in Section 2 of both "rigid motions due to disarrangments" and "rigid motions without disarrangements:" the former represent macroscopically rigid motions that arise from time-like jumps in a sequence of approximating piecewise rigid motions, while the latter represent classical rigid motions. The manner in which either kind of rigid motion can be superposed on a given structured motion also is recorded in Section 2, and the effects of these superpositions on the velocities and deformations with and without disarrangements are noted.

Earlier studies of structured deformations and refinements of classical balance laws have established additive decompositions and corresponding identification relations for not only kinematical quantitites as discussed above but also for measures of contact forces [3],[4],[5],[6]. These decompositions for contact forces resolve the limit of volume-averages of contact forces, or the "volume density of contact forces," into the *volume density of contact forces due to disarrangements,* a term due to the contact forces on the geometrical sites where approximating piecewise smooth motions experience space-like or time-like jumps, plus the *volume density of contact forces without disarrangements,* a term due to the contact forces on the boundary of a preassigned part of a body - even when that boundary is broken into pieces by the approximating piecewise smooth motions. This decomposition is recorded in Section 3, where it becomes clear that the volume density of contact forces without disarrangements always can be written as the divergence of the *stress without disarrangements*, a tensor field that represents the stress in a virgin configuration differing from the given reference configuration by a purely microscopic deformation ([3], Part Two, Section 1), [6]. Although the volume density of contact forces due to disarrangements generally does not take the form of the divergence of a tensor field, the analysis in Section 3 shows that there is a natural way of writing the volume density of contact forces due to disarrangements as the divergence of a tensor field, the *stress due to disarrangements*, plus a body force term, the *body force due to disarrangements* that reflects the macroscopic effects of spatial variations in voids appearing through approximating piecewise-smooth motions.

These considerations lead in Section 3 to a decomposition of the total body force (including inertial forces) in the virgin configuration into a sum of the total body force without disarrangements (including inertial forces without disarrangements) plus the total body force due to disarrangements (including inertial forces due to disarrangments). In this paper, the specific forms of the inertial forces with and without disarrangements are not derived, because they rest on notions of "acceleration with and without disarrangements" that require the notion of a "second-order structured deformation" [7] and go well beyond the scope of the present analysis. It is worth noting that the classical analysis of Noll [1] also leaves the form of the inertial forces unspecified. At the end of Section 3 transformation laws under the superposed rigid body motions considered in Section 2 are obtained for the stresses with and without disarrangements and for the total body forces with and without disarrangements.

In Section 4 the power expended in a structured motion is defined in terms of the kinematical and dynamical quantities discussed in Sections 2 and 3. The power expended is assumed to be the sum of the "power without disarrangements" and the "power due to disarrangements," each term having an explicit expression in terms of quantities introduced in Sections 2 and 3. With this extension to structured motions of the power expended, it is natural to extend the postulate of invariance of the power — imposed directly in [1], while following from invariance of certain terms in the equation of energy balance in [2]— to include both the cases of superposed rigid motions without disarrangements and superposed rigid motions due to disarrangments. It should be noted that in the classical arguments that motivated the present work Noll [1] employed "changes of observers," whereas Green and Rivlin [2] employed superposed rigid motions.

In Section 5, the classical derivations of balance of forces and moments are carried out with a new outcome: invariance of the power under superposed rigid motions without disarrangements yields both a law of balance of forces without disarrangements $\text{BF}_\backslash$ and a law of balance of moments without disarrangements $\text{BM}_\backslash$, while invariance under superposed rigid motions due to disarrangements yields both a law of balance of forces due to disarrangements BF_d and a law of balance of moments due to disarrangements BM_d. The typographical expressions of the two laws of balance of forces, $\text{BF}_\backslash$ and BF_d, are identical except for interchange of the symbols $\backslash$ and d, as is also the case for the equations expressing the two laws of balance of moments, $\text{BM}_\backslash$ and BM_d. Accordingly, the pair of balance laws ($\text{BF}_\backslash$,BF_d) are called the *twin balance laws for forces*, while the pair of balance laws ($\text{BM}_\backslash$,BM_d) are called the *twin balance laws for moments.* With the exception of the inertial forces, all of the terms appearing in the four balance laws are expressed directly in terms of the Piola-Kirchhoff stress, the external body force, and the structured motion under consideration. The detailed form of the inertial forces with and without disarrangements will be the subject of a subsequent analysis, based on second-order structured deformations [7].

It is customary in theories that treat motions at different length scales to introduce balance laws separately at each length scale, so that one postulates or derives, for example, both a law of balance of "macroforces" and a law of balance of "microforces." However, in the present theory, geometrical changes at smaller length scales affect *both* twins in each pair ($\cdot_\backslash$, $\cdot_d$) of twin balance laws, with smooth changes at smaller scales reflected in the first twin $\cdot_\backslash$ and non-smooth changes in the second twin $\cdot_d$. Because the macroscopic motions treated here are assumed to be smooth, the disarrangements covered by the twin balance laws are "sub-macroscopic." The inclusion of macroscopic disarrangements can be accomplished by means of methods already well-established in the context of classical motions.

Included in Section 5 are derivations of the reduced formula for the power expended and of the classical balance of forces and moments. The reduced formula for the power gives the power as the sum of the *stress-power without disarrangements* and the *stress-power due to disarrangements*. The decomposition of the stress-power will play a central role in the analysis of dissipation that arises in structured motions.

2 Structured motions

Specification of a structured motion from a region $\mathcal{A}$ —in the Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ — during a time interval $(0,T)$ includes the specification of three fields $\chi : \mathcal{A}\times(0,T) \to \mathcal{E}$, $G : \mathcal{A}\times(0,T) \to Lin\mathcal{V}$, and $\dot{\chi}_\backslash : \mathcal{A}\times(0,T) \to \mathcal{V}$ called the *macroscopic motion*, the *deformation without disarrangements*, and the *velocity without disarrangements*, respectively. The notation w instead of $\dot{\chi}_\backslash$ was used when structured motions were introduced in [3]; in addition, the "disarrangement site" $\kappa^{(4)}$ in [3] here is assumed to be the empty set. This assumption rules out in the present study discontinuities in the fields χ, G , and $\chi_\backslash$, although discontinuites are permitted in the piecewise-smooth approximating motions χ_n introduced below (2.1). To avoid technical issues that are not central to the present study, the precise smoothness assumptions on these fields and on the region $\mathcal{A}$ will not be specified here, but sufficient smoothness requirements on the fields can be inferred from the context. Other than smoothness requirements, the only conditions imposed on the fields χ, G, and $\dot{\chi}_\backslash$ is the injectivity of $x \mapsto \chi(x,t)$ for every $t \in (0,T)$, as well as the existence of a positive number m such that the inequalities

$$m < \det G(x,t) \leq \det \nabla\chi(x,t) \tag{2.1}$$

hold for all $(x,t) \in \mathcal{A}\times(0,T)$. The Approximation Theorem for structured motions ([3], Section 6.5) assures that there is a sequence $n \mapsto \chi_n$ of piecewise smooth motions defined on $\mathcal{A}\times(0,T)$ such that

$$\chi = \lim_{n\to\infty} \chi_n, \quad G = \lim_{n\to\infty} \nabla\chi_n, \quad \dot{\chi}_\backslash = \lim_{n\to\infty} \dot{\chi}_n, \tag{2.2}$$

with the limits taken in the sense of essentially uniform convergence (i.e., L^∞-convergence) . The spatial derivatives $\nabla\chi_n$ and the time derivatives $\dot{\chi}_n$ are taken in the classical sense, and the limits G and $\dot{\chi}_\backslash$ of derivatives in $(2.2)_{2.3}$ need not equal the corresponding derivatives $\nabla\chi$ and $\dot{\chi}$ of the macroscopic motion (nor need G even be the gradient of some motion). Specific quantitative information about the differences $M := \nabla\chi - G$ and $\dot{\chi}_d := \dot{\chi} - \dot{\chi}_\backslash$ are provided in the first subsection below and justify the terminology *deformation due to disarrangements* for M and *velocity due to disarrangements* for $\dot{\chi}_d$.

2.1 Decompositions and identification relations

The algebraically trivial *additive decompositions*

$$\nabla\chi = G + M, \quad \dot{\chi} = \dot{\chi}_\backslash + \dot{\chi}_d \tag{2.3}$$

for the macroscopic deformation gradient and velocity are given deeper significance by means of the limit relations (2.2) for G and $\dot{\chi}_\backslash$ and by means of the following *identification relations* obtained in [3], Part Two, Section 6.6, for M and $\dot{\chi}_d$:

$$M(x,t) = \lim_{\delta\to0}\lim_{n\to\infty} vol\mathcal{B}(x,t;\delta)^{-1} \int_{\Gamma(\chi_n)\cap\mathcal{B}(x,t;\delta)} [\chi_n](y,\tau)\otimes\nu^{(3)}(y,\tau)dA_{(y,\tau)} \tag{2.4}$$

and

$$\dot{\chi}_d(x,t) = \lim_{\delta\to 0}\lim_{n\to\infty} vol\mathcal{B}(x,t;\delta)^{-1}\int_{\Gamma(\chi_n)\cap\mathcal{B}(x,t;\delta)} \nu^{(1)}(y,\tau)[\chi_n](y,\tau)dA_{(y,\tau)}. \tag{2.5}$$

In these relations, $n \mapsto \chi_n$ is an arbitrary sequence of piecewise smooth motions that satisfies the limit relations (2.2). The symbol $\mathcal{B}(x,t;\delta) \subset \mathcal{E}\times\mathbb{R}$ denotes the space-time ball of radius $\delta > 0$ centered at a point (x,t) in $\mathcal{A}\times(0,T)$, and $\Gamma(\chi_n) \subset \mathcal{E}\times\mathbb{R}$, $[\chi_n](y,\tau) \in \mathcal{V}$, and $(\nu^{(3)}(y,\tau), \nu^{(1)}(y,\tau)) \in \mathcal{V}\times\mathbb{R}$ denote respectively the jump set of the piecewise smooth motion χ_n, the jump of χ_n at a point $(y,\tau) \in \Gamma(\chi_n)$, and the unit normal to the jump set $\Gamma(\chi_n)$ at the point (y,τ). The integrand in (2.4) is the tensor product of $[\chi_n](y,\tau)$ and $\nu^{(3)}(y,\tau)$, both vectors in $\mathcal{V}$, while the integrand in (2.5) is the scalar $\nu^{(1)}(y,\tau)$ multiplied by the vector $[\chi_n](y,\tau) \in \mathcal{V}$.

The precise interpretations now available for G, $M = \nabla\chi - G$, $\dot{\chi}_\backslash$, and $\dot{\chi}_d = \dot{\chi} - \dot{\chi}_\backslash$ permit us to understand and interpret various examples of structured motions as illustrated in the next two subsections.

2.2 Rigid motions without disarrangements

If $r : \mathcal{A}\times(0,T) \to \mathcal{E}$ is a smooth mapping that at each time preserves distances, then $\nabla r : \mathcal{A}\times(0,T) \to Lin\mathcal{V}$ has orthogonal values and $\nabla\dot{r}\,\nabla r^T : \mathcal{A}\times(0,T) \to Lin\mathcal{V}$ has skew values. The structured motion

$$\chi := r, \quad G := \nabla r,\ \dot{\chi}_\backslash := \dot{r} \tag{2.6}$$

satisfies $M = \nabla\chi - G = 0$ and $\dot{\chi}_d = \dot{\chi} - \dot{\chi}_\backslash = 0$. Here, there is no deformation due to disarrangements and no velocity due to disarrangements, and the identification relations above tell us that no space-like or time-like jumps in approximating motions contribute to the macroscopic deformation and velocity. Accordingly, the triple $(r, \nabla r, \dot{r})$ will be called a *classical rigid motion* or a *rigid motion without disarrangements.*

2.3 Rigid motions caused by disarrangements

If r again is again smooth and distance-preserving, the structured motion

$$\chi := r, \quad G := \nabla r, \ \dot{\chi}_{\backslash} := 0 \tag{2.7}$$

satisfies $M = \nabla\chi - G = 0$ and $\dot{\chi}_d = \dot{\chi} - \dot{\chi}_{\backslash} = \dot{r}$. Here, there is no deformation due to disarrangements, so that no space-like jumps in approximating motions contribute to the macroscopic deformation. However, the relation $\dot{r} = \dot{\chi}_d$ and the identification relations tell us that the entire macroscopic velocity field arises due to time-like jumps in approximating piecewise smooth motions. The triple $(r, \nabla r, 0)$ will be called a *rigid motion caused by (time-like) disarrangements.* An example in [3] (Part Two, Section 6.3: the "blinking motion") of approximating piecewise smooth motions provides instances of the time-like disarrangements required for a rigid motion caused by disarrangements.

2.4 Superposed rigid motions

The definition of composition of two structured deformations in [4] was shown in [3], Part Two, Section 6.5 to lead to the following notion of composition for structured motions:

$$(\eta, H, \dot{\eta}_{\backslash}) \circ (\chi, G, \dot{\chi}_{\backslash}) = \left(\eta \diamond \chi, (H \diamond \chi)G, (H \diamond \chi)\dot{\chi}_{\backslash} + \dot{\eta}_{\backslash} \diamond \chi\right). \tag{2.8}$$

Here, the symbol $\diamond$ denotes the "composition in the first variable", for example: $(\eta \diamond \chi)(x,t) := \eta(\chi(x,t),t)$ for all $(x,t) \in \mathcal{A}\times(0,T)$. This formula provides the following formulas for a structured motion $(\chi, G, \dot{\chi}_\backslash)$ followed by rigid motions of the types just considered:

$$(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash) = \big(r \diamond \chi, (\nabla r \diamond \chi)G, (\nabla r \diamond \chi)\dot{\chi}_\backslash + \dot{r} \diamond \chi\big) \tag{2.9}$$

$$(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash) = \big(r \diamond \chi, (\nabla r \diamond \chi)G, (\nabla r \diamond \chi)\dot{\chi}_\backslash\big). \tag{2.10}$$

We interpret the compositions in (2.9) and (2.10) as *superpositions* of a rigid motion (without disarrangements or caused by disarrangements) on the given structured motion $(\chi, G, \dot{\chi}_\backslash)$. The superpositions result in new structured motions given by the right-hand sides of (2.9) and (2.10), and the velocity without disarrangements for $(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$ is the third entry on the right-hand side of (2.9): $(\nabla r \diamond \chi)\dot{\chi}_\backslash + \dot{r} \diamond \chi$, while the velocity without disarrangements for $(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash)$ is the third entry of the right-hand side of (2.10): $(\nabla r \diamond \chi)\dot{\chi}_\backslash$. Moreover, we may compute the velocity due to disarrangements for each of the superpositions by subtracting the third entry from the time derivative of the first. For the superposition $(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$ the velocity due to disarrangements is

$$\begin{aligned}(r \diamond \chi)^{\cdot} - (\nabla r \diamond \chi)\dot{\chi}_\backslash - \dot{r} \diamond \chi &= \dot{r} \diamond \chi + (\nabla r \diamond \chi)(\dot{\chi} - \dot{\chi}_\backslash) - \dot{r} \diamond \chi \\ &= (\nabla r \diamond \chi)\dot{\chi}_d \end{aligned} \tag{2.11}$$

while for $(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash)$ the velocity due to disarrangements is

$$\begin{aligned}(r \diamond \chi)^{\cdot} - (\nabla r \diamond \chi)\dot{\chi}_\backslash &= \dot{r} \diamond \chi + (\nabla r \diamond \chi)(\dot{\chi} - \dot{\chi}_\backslash) \\ &= \dot{r} \diamond \chi + (\nabla r \diamond \chi)\dot{\chi}_d. \end{aligned} \tag{2.12}$$

If r and $t_o \in (0, T)$ are such that $\nabla r(x, t_o) = I$ for all $x \in \mathcal{A}$, then the formulas just obtained may be simplified as follows when evaluated at t_o :

structured motion	*vel. without disarrangements*	*vel. due to disarrangements*
$(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$	$\dot{\chi}_\backslash(\cdot, t_o) + \dot{r}(\chi(\cdot, t_o), t_o)$	$\dot{\chi}_d(\cdot, t_o)$
$(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash)$	$\dot{\chi}_\backslash(\cdot, t_o)$	$\dot{\chi}_d(\cdot, t_o) + \dot{r}(\chi(\cdot, t_o), t_o)$.

In particular, if for every time t the mapping $r(\cdot,t)$ is a translation of $\mathcal{E}$, then $\nabla r = I$ holds at all times. The deformation without disarrangements for the superpositions in (2.9) and (2.10) are the second entries in each triple of the right-hand side, and the deformation due to disarrangements is obtained by subtracting the second entry on the right-hand sides from the gradient of the first. At a time $t_o \in (0,T)$ such that $\nabla r(x,t_o) = I$ for all $x \in A$, we find:

structured motion	*def. without disarrangments*	*def. due to disarrangments*
$(r,\nabla r,\dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$	$G(\cdot,t_o)$	$M(\cdot,t_o)$
$(r,\nabla r,0) \circ (\chi, G, \dot{\chi}_\backslash)$	$G(\cdot,t_o)$	$M(\cdot,t_o).$

3 Contact forces and body forces

3.1 Decompositions

Earlier studies of balance laws for bodies undergoing structured *deformations* ([5], Part Two, Section 1, and [6]) —corresponding in the present context to the conditions $\dot{\chi} = \dot{\chi}_\backslash = 0$ at all times— showed that the classical law of balance of forces in the reference configuration is equivalent to the "refined balance law":

$$div(SK^*) - S\, divK^* + \nabla S[(\det K)I - K^*] + (\det K)\, b_{ref} = 0. \tag{3.1}$$

Here, $S : \mathcal{A}{\times}(0,T) \to Lin\mathcal{V}$ is the Piola-Kirchhoff stress field, $K := (\nabla\chi)^{-1}G$, b_{ref} is the body force per unit volume in the reference configuration, and $A^* := (\det A)A^{-T}$ for all invertible $A \in Lin\mathcal{V}$. Moreover, identification relations derived in the earlier studies permit us to call SK^* the stress without disarrangements, $div(SK^*)$ the volume density of contact forces without disarrangments, and $-S\, divK^* + \nabla S[(\det K)I - K^*]$ the volume density of contact forces due to disarrangments. The availability through structured deformations of a broader class of configurations to use as reference permits one to view (3.1) as balance of forces in a "virgin configuration" that differs from the given reference configuration by a "purely microscopic deformation" and that contains none of the disarrangements associated with a given structured deformation. Moreover, the scalar field $\det K$ may be thought of as the volume fraction associated with the given structured deformation.

As noted in [6], when $\det K$ =const $= 1$ the volume density of contact forces due to disarrangements is the divergence of the tensor field $S(I - K^*)$, but when $\det K$ is not a constant, the volume density of contact forces due to disarrangements has no obvious representation as a divergence. We note that the contact forces due to disarrangements always obey the identity

$$-S\, div K^* + \nabla S[(\det K)I - K^*] = div(S[(\det K)I - K^*]) - S\nabla \det K \tag{3.2}$$

which permits us to write the refined balance of forces as:

$$div(SK^*) + div(S[(\det K)I - K^*]) - S\,\nabla \det K + (\det K)\, b_{ref} = 0. \tag{3.3}$$

We call $S[(\det K)I - K^*]$ the *stress due to disarrangments,* while we call $-S\,\nabla \det K$ the *body force due to disarrangements* and $(\det K)\, b_{ref}$ the *body force without disarrangements.* It is immediate that when $M = 0$, the case in which there is no deformation due to disarrangements, the stress and the body force due to disarrangements vanish, the body force without disarrangements reduces to b_{ref}, and the stress without disarrangements reduces to S. Actually, the body force due to disarrangements vanishes whenever the volume fraction $\det K$ does not vary with position.

If we define the body force per unit volume in the virgin cofiguration b_v to be the sum of the body force without disarrangements and the body force due to disarrangements, the considerations above lead us to the following decompositions of stress and body force:

$$(\det K)S = SK^* + S[(\det K)I - K^*] \tag{3.4}$$

$$b_v = (\det K)\, b_{ref} - S\,\nabla \det K. \tag{3.5}$$

The stress tensor $(\det K)S$ is an analogue of the "weighted Cauchy tensor" $(\det \nabla\chi)T \circ \chi$ discussed in [8], and (3.4) shows that it is this measure of stress that readily decomposes into a part without disarrangements plus a part due to disarrangements.

When the special case $\dot{\chi} = \dot{\chi}_\backslash = 0$ is set aside in favor of the general case of structured motions, the measures of stress and body force and their decompositions remain relevant, but inertial forces must be taken into account. In the context of second-order structured deformations [7], it will be possible to decompose additively the acceleration into a part due to disarrangements and a part without disarrangements, so that an analogous decomposition of inertial forces is possible. Unfortunately, the derivation of such a decomposition is beyond the scope of the present article. Consequently, we procede simply by postulating a decomposition of the inertial force (measured per unit volume in the virgin configuration)

$$b_v^{in} = (b_v^{in})_\backslash + (b_v^{in})_d \tag{3.6}$$

into an inertial force without disarrangements $(b_v^{in})_\backslash$ plus an inertial force due disarrangements $(b_v^{in})_d$.

3.2 Transformation laws

Consider now a rigid motion r and a time t_o at which $\nabla r = I$. From the analysis in Section 2.4, both $\nabla\chi$ and G are unchanged at time t_o, and it follows that the tensor field $K = (\nabla\chi)^{-1}G$ is unchanged at time t_o under superposition of $(r, \nabla r, \dot{r})$ and $(r, \nabla r, 0)$.

Moreover, we assume that the Piola-Kirchhoff stress S at time t_o also is unchanged under these superpositions, and we conclude that the stress without disarrangements SK^*, the stress due to disarrangements $S[(\det K)I - K^*]$, and the body force due to disarrangements $-S\,\nabla \det K$ in a given structured motion all remain unchanged at time t_o under superposition of $(r, \nabla r, \dot{r})$ and $(r, \nabla r, 0)$.

We specify the transformation law for the *total body force* $b^* := b_v + b_v^{in}$ first by noting that the decompositions (3.5) and (3.6) yield

$$\begin{aligned} b^* &= b_v + b_v^{in} \\ &= (\det K)\, b_{ref} - S\,\nabla \det K + (b_v^{in})_\backslash + (b_v^{in})_d \\ &= (\det K)\, b_{ref} + (b_v^{in})_\backslash - S\,\nabla \det K + (b_v^{in})_d, \end{aligned} \tag{3.7}$$

then by defining $b^*_\backslash$, the *total body force without disarrangements*, and b^*_d, the *total body force due to disarrangements,* through the formulas

$$b^*_\backslash \;:\; = (\det K)\, b_{ref} + (b_v^{in})_\backslash \tag{3.8}$$

$$b^*_d \;:\; = -S\,\nabla \det K + (b_v^{in})_d, \tag{3.9}$$

and finally by requiring that *both* $b^*_\backslash$ *and* b^*_d *are unchanged at time* t_o *under superposition of* $(r, \nabla r, \dot{r})$ *and* $(r, \nabla r, 0)$.

4 Power expended

4.1 Postulate on the form of the power

Equipped now with decompositions of velocity, of contact forces, and of total body forces that reflect smooth versus non-smooth geometrical changes at smaller length scales, we postulate that the power expended at each time $t \in (0,T)$ on each subbody $\mathcal{S} \subset \mathcal{A}$ of the body in a structured motion $(\chi, G, \dot{\chi}_\backslash)$ be given by the formula

$$\begin{aligned} P(\mathcal{S},t) &= \textstyle\int_{bdy\mathcal{S}} \left(S_\backslash(x,t)\nu(x)\cdot\dot{\chi}_\backslash(x,t) + S_d(x,t)\nu(x)\cdot\dot{\chi}_d(x,t)\right) dA_x + \\ & \textstyle\int_{\mathcal{S}} \left(b^*_\backslash(x,t)\cdot\dot{\chi}_\backslash(x,t) + b^*_d(x,t)\cdot\dot{\chi}_d(x,t)\right) dV_x. \end{aligned} \tag{4.1}$$

In this formula, $\nu(x)$ denotes the unit outer normal to $bdy\mathcal{S}$ at x, and

$$S_\backslash := SK^*, \quad S_d := S[(\det K)I - K^*] \tag{4.2}$$

are the stress without disarrangements and the stress due to disarrangements introduced in Section 3. If the structured motion is a classical motion, i.e., if $G = \nabla\chi$ and $\dot{\chi}_\backslash = \dot{\chi}$, and, further, if $(b^{in}_v)_\backslash = b^{in}_v$, then (3.6) - (4.2) imply the formula

$$P(\mathcal{S},t) = \int_{bdy\mathcal{S}} S(x,t)\nu(x)\cdot\dot{\chi}(x,t)dA_x + \int_{\mathcal{S}} b^*(x,t)\cdot\dot{\chi}(x,t)dV_x,$$

which is the standard formula for the power expended.

4.2 Postulate of invariance of the power

We broaden the classical invariance assumptions affecting the power [1],[2] by requiring: *for every rigid motion r and time t_o* satisfying $\nabla r(\cdot,t_o) = I$, *for every part $\mathcal{S}$, and for every structured motion $(\chi, G, \dot{\chi}_\backslash)$, not only is the power expended at time t_o in the superposition $(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$ equal to the the power expended in $(\chi, G, \dot{\chi}_\backslash)$ at time t_o, but also the power expended at time t_o in the superposition $(r, \nabla r, 0)\circ(\chi, G, \dot{\chi}_\backslash)$ equals the power expended at time t_o in $(\chi, G, \dot{\chi}_\backslash)$.*

5 Twin balance laws

Let $a \in \mathcal{V}$ be given, and put $r(x,t) := x + at$ for all $x \in \mathcal{E}$ and $t \in \mathbb{R}$. We then can use the results in the table below (2.12), with t_o an arbitrary time $t \in \mathbb{R}$, and apply the invariance of the power, of the stresses with and without disarrangements, and of the total body forces with and without disarrangements first for the superposition $(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$ to obtain the relation

$$\int_{bdy\mathcal{S}} S_\backslash(x,t)\nu(x) \cdot a \, dA_x + \int_{\mathcal{S}} b_\backslash^*(x,t) \cdot a \, dV_x = 0 \tag{5.1}$$

and next for the superposition $(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash)$ to obtain the relation

$$\int_{bdy\mathcal{S}} S_d(x,t)\nu(x) \cdot a \, dA_x + \int_{\mathcal{S}} b_d^*(x,t) \cdot a \, dV_x = 0. \tag{5.2}$$

Using the arbitrariness of a and of the part $\mathcal{S}$ as well as the Divergence Theorem, we obtain from (5.1) and (5.2) the *twin balance laws for forces* $(scBF_\backslash, scBF_d)$:

$$scBF_\backslash : \; divS_\backslash + b_\backslash^* = 0; \quad scBF_d : \; divS_d + b_d^* = 0. \tag{5.3}$$

In view of the twin balance laws for forces, the power (4.1) reduces to:

$$\begin{aligned} P(\mathcal{S},t) \;\; = \;\; & \int_{bdy\mathcal{S}} \big(S_\backslash(x,t)\nu(x) \cdot \dot{\chi}_\backslash(x,t) + S_d(x,t)\nu(x) \cdot \dot{\chi}_d(x,t)\big) \, dA_x \\ & - \int_{\mathcal{S}} \big(divS_\backslash(x,t) \cdot \dot{\chi}_\backslash(x,t) + divS_d \cdot \dot{\chi}_d(x,t)\big) \, dV_x. \end{aligned}$$

Applying the Divergence Theorem to the surface integral and the product rule

$$div(A^T w) = divA \cdot w + A \cdot \nabla w$$

to the integrand in the resulting volume integral, we obtain the *reduced formula for the power expended*:

$$P(\mathcal{S},t) = \int_{\mathcal{S}} \big(S_\backslash(x,t) \cdot \nabla\dot{\chi}_\backslash(x,t) + S_d(x,t) \cdot \nabla\dot{\chi}_d(x,t)\big) \, dV_x. \tag{5.4}$$

The integrand is the total stress power decomposed as the sum of the *stress power without disarrangements* $S_\backslash \cdot \nabla\dot{\chi}_\backslash$ and the *stress power due to disarrangements* $S_d \cdot \nabla\dot{\chi}_d$.

For each $t \in \mathbb{R}$, $x_o \in \mathcal{E}$, and skew tensor $W \in Lin\mathcal{V}$, we put

$$r(y,\tau) := x_o + e^{W(\tau - t)}(y - x_o) \qquad \text{forall} y \in \mathcal{E}, \tau \in \mathbb{R}, \tag{5.5}$$

and note that $\dot{r}(\chi(x,t),t) = W(\chi(x,t) - x_o)$ and $\nabla(\dot{r} \diamond \chi)(x,t) = W\ \nabla\chi(x,t)$ for all $x \in \mathcal{A}$. We again can use the results below (2.12), with t_o the given time $t \in \mathbb{R}$, and apply the invariance of the power in the reduced form (5.4), as well as the invariance of the stresses with and without disarrangements, first for the superposition $(r, \nabla r, \dot{r}) \circ (\chi, G, \dot{\chi}_\backslash)$ to obtain the relation

$$\int_{\mathcal{S}} S_\backslash(x,t) \cdot W\ \nabla\chi(x,t)\, dV_x = 0 \tag{5.6}$$

and next for the superposition $(r, \nabla r, 0) \circ (\chi, G, \dot{\chi}_\backslash)$ to obtain the relation

$$\int_{\mathcal{S}} S_d(x,t) \cdot W\ \nabla\chi(x,t)\, dV_x = 0. \tag{5.7}$$

The arbitrariness of the skew tensor W and of the part $\mathcal{S}$ imply that $S_\backslash(x,t)\nabla\chi(x,t)^T$ and $S_d(x,t)\nabla\chi(x,t)^T$ are symmetric for each $(x,t) \in \mathcal{A} \times (0,T)$ and yield the *twin balance laws for moments* $(scBM_\backslash, scBM_d)$:

$$scBM_\backslash :\ S_\backslash \nabla\chi^T = \nabla\chi\, S_\backslash^{\,T}; \quad scBM_d :\ S_d \nabla\chi^T = \nabla\chi\, S_d^{\,T}. \tag{5.8}$$

Because (3.4) and the definitions $S_\backslash = SK^*$ and $S_d = S[(\det K)I - K^*]$ yield the relation

$$(\det K)S = S_\backslash + S_d, \tag{5.9}$$

the twin balance laws for moments (5.8) then imply the *classical balance law for moments*:

$$S\,\nabla\chi^T = \nabla\chi\,S^{\,T}.$$

Adding the twin balance laws for force $divS_\backslash + b^*_\backslash = 0$ and $divS_d + b^*_d = 0$, we obtain — in view of (5.9), (3.8), (3.9), and (3.6) — the relation

$$\det K\,(divS + b_{ref}) + b_v^{in} = 0. \tag{5.10}$$

This represents the classical balance law of forces written with respect to the virgin configuration. When the inertial forces take the form $b_v^{in} = -(\det K\,)\rho_{ref}\,\ddot{\chi}$, we obtain the *classical balance of forces with respect to the given reference configuration*:

$$divS + b_{ref} - \rho_{ref}\,\ddot{\chi} = 0. \tag{5.11}$$

References

[1] Noll, W., La mécanique classique, basée sur un axiome d'objectivité, pp. 47 - 56 of *La Méthode Axiomatique dans les Mécaniques Classiques et Nouvelles* (Colloque International, Paris, 1959), Paris, Gauthier-Villars, 1963.

[2] Green, A. E., and Rivlin, R. S., On Cauchy's equations of motion, *Journal of Applied Mathematics and Physics*, **15**, 290-292, 1964.

[3] Del Piero, G., and Owen, D. R., *Structured Deformations*, Quaderni dell' Istituto Nazionale di Alta Matematica, Gruppo Nazionale di Fisica Matematica, No. 58, 2000.

[4] Del Piero, G., and Owen, D.R., Structured deformations of continua, *Archive for Rational Mechanics and Analysis* **124**, 99-155, 1993.

[5] Del Piero, G., and Owen, D.R., Integral-gradient formulae for structured deformations, *Archive for Rational Mechanics and Analysis* **131**, 121-138, 1995.

[6] Owen, D. R., Structured deformations and the refinements of balance laws induced by microslip, *International Journal of Plasticity*, **14**, 289-299, 1998.

[7] Owen, D. R., and Paroni, R., Second-order structured deformations, *Archive for Rational Mechanics and Analysis*, **155**, 215-235, 2000.

[8] Haupt, P., *Continuum Mechanics and Theory of Materials*, Springer Verlag, Berlin, etc., 2000.