Relaxation of Variational Problems Under Trace Constraints

Guy Bouchitté
Département de Mathématiques
Université de Toulon et du Var-BP 132
83957 La Garde Cedex, France

Irene Fonseca
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

and

Luisa Mascarenhas
Departemento de Matemática da Faculdade de Ciêcias
e C.M.A.F., Universidade de Lisboa, Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal

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GUY BOUCHITTÉ
Département de Mathématiques, Université de Toulon et du Var-BP 132
83957 La Garde Cedex, France

IRENE FONSECA
Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA 15213, USA

LUÍSA MASCARENHAS
Departamento de Matemática da Faculdade de Ciências
e C.M.A.F., Universidade de Lisboa, Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal

Abstract. The global method for relaxation in BV spaces recently introduced by the authors is used to address certain variational models involving both bulk and interfacial energies, and when no apriori growth bounds are available. Integral representations for relaxed energies involving surface energy contributions on a fixed hypersurfaces of discontinuity are obtained. Examples are given where the relaxed interfacial energy is explicitly characterized, such as in the case of variational problems under Dirichlet boundary conditions.

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1. Introduction.

In recent years there has been considerable effort invested in the search of integral representations for relaxed (or effective) energy functionals, motivated in part by the study of problems in physical and materials sciences where interesting phenomena and properties of equilibria result from lack of convexity.

Often, lower energy contributions are either present in the model from the onset or are created at metastable states, and one is naturally led to the study of free discontinuity problems where the jump set of the admissible fields is not a priori specified. A wealth of literature dedicated to such variational issues is available nowadays (see e.g. [ADG], [BBBF], [BC], [BF], [BFM], [FF], [FM2]).

Here we are concerned with the situation where a fixed surface energy contribution is assigned from the start in order to take into account the contact energy on part of the boundary of an open, bounded container $\Omega \subset \mathbb{R}^N$, or to charge a smooth surface Σ which decomposes Ω into several connected components. Precisely, let $F : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\mathbb{R}^N) \to [0,+\infty]$ be defined by

$$F(u;A) := \begin{cases} \displaystyle\int_{A\cap\Omega} f_0\,(x,u,\nabla u)\,dx & \\ \displaystyle + \int_{A\cap\Sigma} \beta_0(x,u^+,u^-)\,d\mathcal{H}^{N-1} & \text{if} \quad u \in W^{1,1}(\Omega\setminus\Sigma;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.0}$$

where $\mathcal{A}(\mathbb{R}^N)$ denotes the class of bounded open sets of $\mathbb{R}^N$, $\Sigma \subset \overline{\Omega}$ is a prescribed $N-1$ dimensional oriented interface, and u^+, u^-, represent the traces of u on both sides of Σ; whenever Σ coincides with $\partial\Omega$, we consider $u^+ = u^-$.

The main goal of this paper is to characterize the inner regularization $\mathcal{F} : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\mathbb{R}^N) \to [0,+\infty]$,

$$\mathcal{F}(u;A) := \sup\left\{\mathcal{F}_0(u;A') : A' \subset\subset A, A' \in \mathcal{A}(\mathbb{R}^N)\right\}, \tag{1.1}$$

of the energy $\mathcal{F}_0 : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\mathbb{R}^N) \to [0,+\infty]$ given by

$$\mathcal{F}_0(u;A) := \inf\left\{\liminf_{n\to+\infty} F(u;A) \,|u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d),\, u_n \in BV(\Omega;\mathbb{R}^d)\right\}. \tag{1.2}$$

In order to accommodate several applications, ranging from the study of nucleation (see Subsection 4.2 and [FL]) to equilibria problems under Dirichlet boundary conditions, we will be forced to assume that the surface energy density may take the value $+\infty$ at places, and so no upper bound will be available. This sets these variational problems outside the traditional framework of the Calculus of Variations.

Immediate applications of this setting include the study of parametric minimal surfaces where the relaxation of the boundary condition plays a fundamental role (see [G], [MM]) and, in a similar way, the vectorial treatment of elasto-plastic energies (see [Te]). Recently energies of the form (1.0) were considered by Auber, Deriche and Kornprobst [ADK] in the context of optical flow modelization when β_0 depends on the traces u^+, u^- on both sides of a time dependent interface. Also in their case β_0 does not depend only on the jump vector $u^+ - u^-$.

An integral representation formula for (1.1) will be obtained under the following hypotheses :

(H1) $f_0 : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty]$, $\beta_0 : \Sigma \times (\mathbb{R}^d)^2 \to [0,+\infty]$ are Borel integrands ;

(H2) there exists $C > 0$ and $a, \overline{a} \in L^1(\Omega;[0,+\infty[)$ such that

$$\frac{1}{C}|\xi| - \overline{a}(x) \leq f_0(x,u,\xi) \leq a(x) + C|\xi|$$

$\mathcal{L}^N$ a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d\times N}$;

(H3) there exist $C > 0$, $0 < m < 1$, $L > 0$ such that

$$\left|f_0^\infty(x,u,\xi) - \frac{f_0(x,u,t\xi)}{t}\right| \leq \frac{C}{t^m}$$

for all $\xi \in \mathbb{R}^{d\times N}$, $\|\xi\| = 1$, $t > L$, $u \in \mathbb{R}^d$ and for $\mathcal{L}^N$ a.e. $x \in \Omega$, where the *recession function* f_0^∞ is defined by

$$f_0^\infty(x,u,\xi) := \limsup_{t\to+\infty} \frac{f_0(x,u,t\,\xi)}{t};$$

(H4) there exist $C > 0$ such that

$$\frac{1}{C}|\lambda - \theta| - C \leq \beta_0(x, \lambda, \theta),$$

for $\mathcal{H}^{N-1}$ a.e. $x \in \Sigma$ and for all $\lambda, \theta \in \mathbb{R}^d$;

(H5) there exists $u_0 \in W^{1,1}(\Omega \setminus \Sigma; \mathbb{R}^d)$ such that $\lambda_0 := u_0^+$ and $\theta_0 := u_0^-$ satisfy

(i) $\beta_0(\cdot, \lambda_0, \theta_0) \in L^1(\Sigma)$,

(ii) for every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|x-y| < \varepsilon \Rightarrow |\beta_0(x, \lambda_0(x)+r, \theta_0(x)+s) - \beta_0(y, \lambda_0(y)+r, \theta_0(y)+s)| \leq \delta(1+|r|+|s|),$$

for all $(x, y, r, s) \in \Sigma^2 \times (\mathbb{R}^d)^2$, under the convention $+\infty - \infty = 0$.

Note that (H4) and (H5) do not provide an apriori control from above on β_0. This flexibility turns out to be crucial in the study of the Dirichlet problem (see Subsection 4.1). Precisely, for fixed $\Phi_0 \in W^{1,1}(\Omega; \mathbb{R}^d)$ consider the energy functional

$$F(u; A) := \begin{cases} \displaystyle\int_{A\cap\Omega} f_0(\nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega \cap A; \mathbb{R}^d) \text{ and } u = \Phi_0 \text{ on } \Sigma \cap A, \\ +\infty & \text{otherwise,} \end{cases}$$

where f_0 satisfies hypotheses (H1)-(H3). It is clear that F may be re-written as in (1.0), with

$$\beta_0(x, \lambda, \theta) := \begin{cases} 0 & \text{if } \lambda = \theta = \Phi_0(x), \\ +\infty & \text{otherwise,} \end{cases}$$

and hypotheses (H4) and (H5) are verified with $u_0 := \Phi_0$.

Here we adopt the classical localization procedure by introducing the relaxed functional $\mathcal{F}(u; \cdot)$ as in (1.1). The necessity to consider the inner regularization of $\mathcal{F}_0$ (see (1.2)) appears in Section 2, where we prove that $\mathcal{F}(u; \cdot)$ is actually the trace on open sets of a measure supported on $\overline{\Omega}$. Note, however, that considering $\mathcal{F}$ instead of $\mathcal{F}_0$ has no implications from the practical viewpoint since $\mathcal{F}_0(u; \mathbb{R}^N)$ coincides with $\mathcal{F}(u; \mathbb{R}^N)$. We establish also that $\mathcal{F}(\cdot; A)$ satisfies coercivity and upper bounds. These properties allow us in Section 3 to exploit the global method of relaxation developed by the authors in [BFM] and adapted to the present setting. We obtain a representation formula for $\mathcal{F}$ (see Theorem 3.4), and, in particular, we are able to give an explicit characterization of the surface energy density on the fixed hypersurface Σ, even in the case where part of Σ lies on the boundary of the domain Ω.

As it is usual, with this generality, it is difficult to explicitly characterize in closed form the local densities of the effective energies, as these arise naturally by solving local variational problems. So it is worth it to pause for a moments time and to search for friendlier formulas in simple examples of interest. Two such examples are treated in Section 4, and the first concerns a Dirichlet condition $u^+ = u^- = \Phi_0$ assigned on Σ , where it is shown that

$$\begin{aligned}\mathcal{F}(u;A) = &\int_{A\cap\Omega} Qf_0(\nabla u)\,dx + \int_{(A\setminus\Sigma)\cap S(u)} (Qf_0)^\infty([u]\otimes\nu_u)\,d\mathcal{H}^{N-1}\\ &+ \int_{A\cap\Omega} (Qf_0)^\infty\left(\frac{dC(u)}{d|C(u)|}\right) d|C(u)|\\ &+ \int_{A\cap\Sigma} \beta(x,u^+,u^-)d\mathcal{H}^{N-1},\end{aligned}$$

with

$$\beta(x_0,\lambda,\theta) := \begin{cases} (Qf_0)^\infty\Big((\lambda-\Phi_0(x_0))\otimes\nu_0\Big) + (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) & \text{if } x_0\in\Omega\cap\Sigma,\\ (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) & \text{if } x_0\in\partial\Omega\cap\Sigma.\end{cases}$$

2. Notations and preliminary results. In what follows $BV(\Omega;\mathbb{R}^d)$, $W^{1,p}(\Omega;\mathbb{R}^d)$ and $L^p(\Omega;\mathbb{R}^d)$ denote, respectively, the spaces of functions of bounded variation, Sobolev and p-integrable functions mapping Ω into $\mathbb{R}^d$, where Ω is an open, bounded, Lipschitz domain of $\mathbb{R}^N$ (see [EG], [F], [G], [Z]). The Lebesgue measure and the Hausdorff (N-1)-dimensional measure in $\mathbb{R}^N$ are designated by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively. C will stand for a generic constant which may vary from line to line.

To each $\nu \in S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ we associate a rotation R_ν such that $R_\nu(e_N) = \nu$, where $(e_i)_{i=1,\cdots,N}$ is the canonical basis in $\mathbb{R}^N$. We may choose $\nu \mapsto R_\nu$ so that R_{e_N} is the identity and $\nu \mapsto R_\nu(e_i)$ is continuous on $S^{N-1}\setminus\{e_N\}$, for all $i = 1,\cdots,N-1$. We define $Q_\nu := R_\nu(Q)$, where $Q := \{x \in \mathbb{R}^N \mid |x\cdot e_i| < 1/2,\ i = 1,\cdots,N\}$, and we set $Q_\nu(x,\varepsilon) := x + \varepsilon Q_\nu$, for $\varepsilon > 0$. We will omit the subscript ν whenever ν coincides with e_N.

As usual we represent by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure (or Radon Nikodym derivative), and $S(u)$ is the *jump set*, i.e. the set of points x where the *approximate upper limit* $u_i^+(x)$ is different from the *approximate lower limit* $u_i^-(x)$, for some $i \in \{1,\ldots,d\}$, namely

$$S(u) := \bigcup_{i=1}^d \Big\{x\in\Omega \mid u_i^-(x) < u_i^+(x)\Big\}.$$

It can be shown that $S(u)$ and the complement of the set of Lebesgue points of u differ by a set of $\mathcal{H}^{N-1}$ measure zero.

Choosing a normal $\nu_u(x)$ to $S(u)$ at x (defined uniquely, up to sign, for $\mathcal{H}^{N-1}$ a.e. x), we set $[u](x) := u^+(x) - u^-(x)$ the difference between the traces of u at $x \in S(u)$, oriented by $\nu_u(x)$. Representing by $C(u)$ the Cantor part of the measure Du, the following decomposition holds :

$$Du = \nabla u \, \mathcal{L}^N \lfloor \Omega + ([u] \otimes \nu_u) \, \mathcal{H}^{N-1} \lfloor S(u) + C(u).$$

We represent by $SBV(\Omega; \mathbb{R}^d)$ the space of *special functions of bounded variation* introduced by De Giorgi and Ambrosio (see [ADG]), i.e. the space of all functions in $BV(\Omega; \mathbb{R}^d)$ such that $C(u) = 0$.

Suppose that Ω has Lipschitz boundary, and consider a closed and connected $\mathcal{C}^1$ hypersurface $\Sigma \subset \overline{\Omega}$. Select a unit normal vector ν_Σ so that if Σ intersects $\partial\Omega$ then ν_Σ agrees $\mathcal{H}^{N-1}$ almost everywhere with the exterior normal to $\partial\Omega$. For each $u \in BV(\Omega; \mathbb{R}^d)$ we define $u^+(x)$ and $u^-(x)$ as the traces of u on Σ, oriented by ν_Σ, with the convention $u^+ = u^- = \operatorname{tr} u$ on $\Sigma \cap \partial\Omega$. If $\mathcal{H}^{N-1}(S(u) \cap \Sigma) > 0$ we choose ν_u so that it coincides with ν_Σ $\mathcal{H}^{N-1}$ a.e. on $S(u) \cap \Sigma$. Let $\mathcal{A}(\mathbb{R}^N)$ be the class of bounded open subsets of $\mathbb{R}^N$, and denote by $\mathcal{A}_\infty(\mathbb{R}^N)$ the subclass of those open sets which are Lipschitz.

It is easy to see that $u \in W^{1,1}(\Omega \setminus \Sigma; \mathbb{R}^d)$ if and only if $u \in SBV(\Omega; \mathbb{R}^d)$ and $S(u) \subset \Sigma$. Moreover $Du = \nabla u \, \mathcal{L}^N \lfloor \Omega + ([u] \otimes \nu_\Sigma) \, \mathcal{H}^{N-1} \lfloor \Sigma$.

In order to establish the main result of this section, Proposition 2.5 below, first we need some technical lemmas.

Lemma 2.1. *Let $A \in \mathcal{A}_\infty(\mathbb{R}^N)$. For every $w \in BV(A; \mathbb{R}^d)$ and $\theta \in L^1(\partial A; \mathbb{R}^d)$, there exists a sequence (w_ε) in $W^{1,1}(A; \mathbb{R}^d)$ such that*

$$w_\varepsilon|_{\partial A} = \theta, \quad w_\varepsilon \to w \text{ in } L^1(A; \mathbb{R}^d)$$

and

$$\limsup_{\varepsilon \to 0} \int_A |\nabla w_\varepsilon| \, dx \leq |Dw|(A) + \int_{\partial A} |\theta - \operatorname{tr} w| \, d\mathcal{H}^{N-1},$$

where $\operatorname{tr} w$ *denotes the trace of* w *on* ∂A.

Proof. Using a refinement of [BFM, Lemma 2.4] due to L. Tartar [T] (see also [G]) applied to $\tilde{\theta} := \theta - \operatorname{tr} w$, we may find $h_\varepsilon \in W^{1,1}(A; \mathbb{R}^d)$ such that

$$h_\varepsilon = \tilde{\theta} \text{ on } \partial A, \int_A |h_\varepsilon| \, dx \leq \varepsilon \int_{\partial A} |\tilde{\theta}| \, d\mathcal{H}^{N-1}, \int_A |\nabla h_\varepsilon| \, dx \leq (1+\varepsilon) \int_{\partial A} |\tilde{\theta}| d\mathcal{H}^{N-1}.$$

By [BFM, Lemma 2.5] there exists a sequence $(v_\varepsilon) \in W^{1,1}(A; \mathbb{R}^d)$ such that

$$v_\varepsilon = w \text{ on } \partial A, v_\varepsilon \to w \text{ in } L^1(A; \mathbb{R}^d) \text{ and } \lim_{\varepsilon \to 0} \int_A |\nabla v_\varepsilon| \, dx = |Dw|(A).$$

Set $w_\varepsilon := v_\varepsilon + h_\varepsilon$. Then $w_\varepsilon \in W^{1,1}(A;\mathbb{R}^d)$, $\operatorname{tr} w = \theta$,

$$\limsup_{\varepsilon\to 0} \int_A |\nabla w_\varepsilon|\, dx \le |Dw|(A) + \int_{\partial A} |\theta - \operatorname{tr} w|\, d\mathcal{H}^{N-1},$$

and, since

$$\int_A |w_\varepsilon - w|\; dx \le \int_A |v_\varepsilon - w|\; dx + \varepsilon \int_{\partial A} |\theta - \operatorname{tr} w|\, d\mathcal{H}^{N-1},$$

we have that $w_\varepsilon \to w$, as $\varepsilon \to 0$.

□

Definition 2.2. We say that $A \in \mathcal{A}_\infty(\mathbb{R}^N)$ is *transversal* to Σ if $A \cap \Sigma \neq \emptyset$, $\mathcal{H}^{N-1}(\partial A \cap \Sigma) = 0$ and there exist two disjoint Lipschitz open sets $A_1, A_2 \subset A$ such that $A \setminus (A_1 \cup A_2)$ is a regular oriented hypersurface $\tilde{\Sigma}$, containing $A \cap \Sigma$.

We denote by $\mathcal{T}(\mathbb{R}^N)$ the set of all $A \in \mathcal{A}_\infty(\mathbb{R}^N)$ such that either A is transversal to Σ or $A \cap \Sigma = \emptyset$.

Remark 2.3. The set $\mathcal{T}(\mathbb{R}^N)$ is dense in $\mathcal{A}(\mathbb{R}^N)$ in the sense that, for all $A, A' \in \mathcal{A}(\mathbb{R}^N)$ such that $A' \subset\subset A$, there exists a $B \in \mathcal{T}(\mathbb{R}^N)$ satisfying $A' \subset\subset B \subset\subset A$. Recalling that f_0 and β_0 are nonnegative, the set function $A \mapsto \mathcal{F}_0(u; A)$ is monotone nondecreasing and therefore

$$\mathcal{F}(u;A) = \sup\left\{\mathcal{F}_0(u;B) : B \subset\subset A, B \in \mathcal{T}(\mathbb{R}^N)\right\} \le \mathcal{F}_0(u;A)\ , \qquad (2.0)$$

for every $A \in \mathcal{A}(\mathbb{R}^N)$.

The following slicing result proved on open subsets of the class $\mathcal{T}(\mathbb{R}^N)$ will play a crucial role in the matching of boundary conditions in an energetically economical way.

Lemma 2.4. *Let F, defined by (1.2), satisfy hypotheses (H1), (H2), (H4) and (H5)(i). Let $u \in BV(\Omega;\mathbb{R}^d)$ and let (v_n) be a sequence in $BV(\Omega;\mathbb{R}^d)$ such that $v_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. For every $A \in \mathcal{T}(\mathbb{R}^N)$, there exists a sequence (w_n) in $W^{1,1}(\Omega \setminus \Sigma;\mathbb{R}^d)$ such that*

$$\|w_n - u\|_{L^1(\Omega;\mathbb{R}^d)} \to 0\ ,\ w_n = u \ \text{ on } \ \partial A \cap \Omega,\ \limsup_{n\to+\infty} F(w_n;A) \le \liminf_{n\to+\infty} F(v_n;A).$$

Proof. Without loss of generality we may assume that

$$\liminf_{n\to+\infty} F(v_n;A) = \lim_{n\to+\infty} F(v_n;A) < +\infty$$

and, consequently, v_n belongs to $W^{1,1}(\Omega \setminus \Sigma;\mathbb{R}^d)$. It is convenient to consider an extension of u to a fixed neigbourhood V of $\overline{\Omega} \cup A$, still denoted by u, such that

$|Du|(\partial\Omega) = 0$. Thanks to Lemma 2.1, we may also extend each v_n to V in such a way that the extension $\tilde{v}_n$ satisfies $\tilde{v}_n \in W^{1,1}(V \setminus \overline{\Omega}; \mathbb{R}^d)$, $\tilde{v}_n \to u$ in $L^1(V \setminus \overline{\Omega}; \mathbb{R}^d)$, tr $\tilde{v}_n =$ tr v_n on $\partial\Omega$ and so that

$$F(v_n; A) = F(\tilde{v}_n; A).$$

For simplicity of notations, we still denote by v_n its extension $\tilde{v}_n$. For every $t > 0$ set $A_t := \{x \in A \mid \operatorname{dist}(x, \partial A) > t\}$ and define $L_\delta := A \setminus \overline{A_\delta}$. Clearly, we have $L_\delta \in \mathcal{T}(\mathbb{R}^N)$ for δ small enough.

Assume first that $L_\delta \cap \Sigma \neq \emptyset$ so that, in view of Definition 2.2, we may split L_δ into two sets L_δ^+ and L_δ^- with common boundary Σ. We may choose also δ so that

$$\lim_{n \to +\infty} \int_{\partial A_\delta} |\text{tr } v_n - \text{tr } u| = 0. \tag{2.1}$$

For fixed n we apply Lemma 2.1 in L_δ^+ and L_δ^-, with w replaced by u, and θ given by

$$\theta_n^+ := \begin{cases} u & \text{on} \quad \partial A \\ \lambda_0 & \text{on} \quad \Sigma \cap L_\delta \\ v_n & \text{on} \quad \partial A_\delta \end{cases}$$

in L_δ^+, and with θ equal to

$$\theta_n^- := \begin{cases} u & \text{on} \quad \partial A \\ \theta_0 & \text{on} \quad \Sigma \cap L_\delta \\ v_n & \text{on} \quad \partial A_\delta \end{cases}$$

in L_δ^-. We find, then, two double indexed sequences $(w_{n,k}^+)$ and $(w_{n,k}^-)$ in L_δ^+ and L_δ^-, respectively, yielding a sequence $(w_{n,k})$ such that $w_{n,k} \in W^{1,1}(L_\delta \setminus \Sigma; \mathbb{R}^d)$ and

$$w_{n,k} = v_n \text{ on } \partial A_\delta\,, \; w_{n,k} = u \text{ on } \partial A\,, \; w_{n,k}^+ = \lambda_0\,, w_{n,k}^- = \theta_0 \text{ on } L_\delta \cap \Sigma\,, \tag{2.2}$$

$$\lim_{k \to +\infty} \|w_{n,k} - u\|_{L^1(L_\delta; \mathbb{R}^d)} = 0\,, \tag{2.3}$$

and

$$\begin{aligned} \limsup_{k \to +\infty} |Dw_{n,k}|(L_\delta) \leq & |Du|(L_\delta \setminus \Sigma) + \int_{L_\delta \cap \Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-| \right) \, d\mathcal{H}^{N-1} \\ & + \int_{\partial A_\delta} |\text{tr } v_n - \text{tr } u| \, d\mathcal{H}^{N-1}. \end{aligned} \tag{2.4}$$

In view of (2.1) and (2.4), using a diagonalisation argument we may construct a new sequence $(\tilde{w}_n) := (w_{n,k_n})$, satisfying the same boundary and convergence conditions as in (2.2), (2.3), and such that, passing to the limit in n,

$$\limsup_{n\to+\infty} |D\tilde{w}_n|(L_\delta) \leq |Du|(L_\delta \setminus \Sigma) + \int_{L_\delta\cap\Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-|\right)\, d\mathcal{H}^{N-1}. \tag{2.5}$$

Set

$$w_n := \begin{cases} \tilde{w}_n & \text{in } L_\delta \\ v_n & \text{in } \overline{A}_\delta \\ u & \text{in } \Omega \setminus A. \end{cases}$$

Clearly (w_n) converges to u in $L^1(\Omega;\mathbb{R}^d)$, satisfies $w_n = u$ on $\partial A \cap \Omega$ and $w_n \in W^{1,1}(\Omega \setminus \Sigma;\mathbb{R}^d)$. In addition, in view of (H2) and (H4) and since $\mathcal{H}^{N-1}(\partial L_\delta \cap \Sigma) = 0$,

$$\begin{aligned} F(w_n;A) &= F(v_n;A_\delta) + F(\tilde{w}_n;L_\delta) \\ &\leq F(v_n;A) + \int_{L_\delta\cap\Omega} (a(x) + C|\nabla \tilde{w}_n|)\, dx \\ &\quad + \int_{L_\delta\cap\Sigma} \beta_0(x,\lambda_0,\theta_0)\, d\mathcal{H}^{N-1}. \end{aligned} \tag{2.6}$$

By (2.5), and passing to the limit in n, we obtain

$$\begin{aligned} \limsup_{n\to+\infty} F(w_n;A) \leq \lim_{n\to+\infty} F(v_n;A) &+ \int_{L_\delta\cap\Omega} a(x)\, dx + C|Du|(L_\delta\setminus\Sigma) \\ &+ C\int_{L_\delta\cap\Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-|\right)\, d\mathcal{H}^{N-1} \\ &+ \int_{L_\delta\cap\Sigma} \beta_0(x,\lambda_0,\theta_0)\, d\mathcal{H}^{N-1}. \end{aligned}$$

Since $\lim\limits_{\delta\to 0} \mathcal{H}^{N-1}(L_\delta\cap\Sigma) = 0$ and $\beta_0(\cdot,\lambda_0,\theta_0) \in L^1(\Sigma)$, letting δ go to 0 and using again a diagonal argument, we conclude the proof.

The case where $L_\delta \cap \Sigma = \emptyset$ can be proved in a similar way, where now $\theta_n^+ = \theta_n^- = \begin{cases} u & \text{on } \partial A, \\ v_n & \text{on } \partial A_\delta. \end{cases}$ □

Proposition 2.5. *Under hypotheses (H1), (H2), (H4) and (H5)(i), the functional $\mathcal{F}$ satisfies the following conditions :*

$$\mathcal{F}(u;\cdot) \text{ is the restriction to } \mathcal{A}(\mathbb{R}^N) \text{ of a Radon measure;} \tag{2.7}$$

there exists $C > 0$ such that, for all $A \in \mathcal{A}(\mathbb{R}^N)$

$$\frac{1}{C}|Du|(A\cap\Omega) - \int_{A\cap\Omega} \overline{a}(x)\, dx - C\mathcal{H}^{N-1}(A\cap\Sigma) \leq \mathcal{F}(u;A) \tag{2.8}$$

and

$$\mathcal{F}(u;A) \le \int_{A\cap\Omega} a(x)\ dx + C\Big(|Du|(A\cap\Omega) + \int_{A\cap\Sigma} (\alpha(x)+|u^+|+|u^-|) d\mathcal{H}^{N-1}\Big), \quad (2.9)$$

where $\alpha(x) := \beta_0(x,\lambda_0(x),\theta_0(x)) + |\lambda_0(x)| + |\theta_0(x)|$.

Proof. Estimate (2.8) is a straightforward consequence of hypotheses (H2) and (H4) and of the definition of $\mathcal{F}$.

We prove (2.9). In view of (2.0) (see Remark 2.3), it is enough to prove (2.9) for $\mathcal{F}_0$ instead of $\mathcal{F}$ and for all $A \in \mathcal{T}(\mathbb{R}^N)$ and $u \in BV(\Omega;\mathbb{R}^d)$. The case where $A\cap\Sigma = \emptyset$ is a staightforward consequence of (H2). We consider now A transversal to Σ. By Definition 2.2 we may split A into two sets A^+ and A^- with common boundary $\tilde{\Sigma}$ containing $\Sigma\cap A$; we fix A^+ to be the one whose outward normal coincides with ν_Σ, on Σ. Extend u to a fixed neigbourhood V of $\overline{\Omega}$ so that $|Du|(\partial\Omega) = 0$ and define

$$\theta^+ := \begin{cases} u & \text{on} \quad \partial A^+ \cup (\tilde{\Sigma}\setminus\Sigma) \\ \lambda_0 & \text{on} \quad \Sigma\cap A, \end{cases}$$

$$\theta^- := \begin{cases} u & \text{on} \quad \partial A^- \cup (\tilde{\Sigma}\setminus\Sigma) \\ \theta_0 & \text{on} \quad \Sigma\cap A. \end{cases}$$

Applying Lemma 2.1 to $w = u$ and $\theta = \theta^+$ in A^+, and to $w = u$ and $\theta = \theta^-$ in A^-, we contruct a sequence (v_n) in $W^{1,1}(A\setminus\Sigma;\mathbb{R}^d)$ such that

$$\|v_n - u\|_{L^1(A;\mathbb{R}^d)} \to 0,\ v_n^+ = \lambda_0 \text{ and } v_n^- = \theta_0 \text{ on } \Sigma\cap A,$$

and

$$\limsup_{n\to+\infty} \int_A |\nabla v_n| dx \le |Du|(A\setminus\Sigma) + \int_{A\cap\Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-|\right)\ d\mathcal{H}^{N-1}.$$

Since $|Du|(\partial\Omega) = 0$ and $|Du|(A\setminus\overline{\Omega}) \le \liminf_{n\to+\infty} \int_{A\setminus\overline{\Omega}} |\nabla v_n| dx$, the previous inequality yields

$$\begin{aligned}\limsup_{n\to+\infty} \int_{A\cap\Omega} |\nabla v_n| dx &\le \limsup_{n\to+\infty} \int_A |\nabla v_n| dx - |Du|(A\setminus\overline{\Omega}) \\ &\le |Du|(A\cap\Omega\setminus\Sigma) + \int_{A\cap\Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-|\right)\ d\mathcal{H}^{N-1}.\end{aligned}$$

Also, by virtue of hypotheses (H2) and (H4), we have

$$F(v_n;A) \le \int_{A\cap\Omega} a(x)\ dx + C\int_{A\cap\Omega} |\nabla v_n| + \int_{A\cap\Sigma} \beta_0(x,\lambda_0,\theta_0) d\mathcal{H}^{N-1}.$$

Consequently

$$\begin{aligned}\mathcal{F}_0(u;A) &\le \liminf_{n\to+\infty} F(v_n;A)\\ &\le \int_{A\cap\Omega} a(x)\,dx + \int_{A\cap\Sigma} \beta_0(x,\lambda_0,\theta_0)d\mathcal{H}^{N-1}\\ &\quad + C\left(|Du|((A\cap\Omega)\setminus\Sigma) + \int_{A\cap\Sigma} \left(|\lambda_0 - u^+| + |\theta_0 - u^-|\right)\,d\mathcal{H}^{N-1}\right),\end{aligned}$$

which yields (2.9).

Next we prove (2.7). We claim that for every $u \in BV(\Omega;\mathbb{R}^d)$ and for every A, B, U in $\mathcal{A}(\mathbb{R}^N)$, the following implication holds :

$$U \subset\subset B \subset\subset A \;\Rightarrow\; \mathcal{F}(u;A) \le \mathcal{F}(u;B) + \mathcal{F}(u;A\setminus\overline{U}). \tag{2.10}$$

In view of (2.0), by considering subsets A', B' and U' such that $A', B', A'\setminus\overline{U'} \in \mathcal{T}(\mathbb{R}^N)$ and $U \subset\subset U' \subset\subset B' \subset\subset B \subset\subset A' \subset\subset A$, we may substitute $\mathcal{F}$ with $\mathcal{F}_0$ in (2.10).

Let (u_n) and (v_n) be two sequences converging to u in $L^1(\Omega;\mathbb{R}^d)$ and such that

$$\lim_{n\to+\infty} F(u_n;B) = \mathcal{F}_0(u;B) \quad\text{and}\quad \lim_{n\to+\infty} F(v_n;A\setminus\overline{U}) = \mathcal{F}_0(u;A\setminus\overline{U}).$$

We may assume, as in the proof of Lemma 2.4, that the sequences (u_n) and (v_n), as well as u, are defined in a fixed neigbourhood V of $\overline{\Omega}\cup A$, in such a way that $|Du|(\partial\Omega) = 0$, and (u_n) and (v_n) still converge to u in $L^1(V;\mathbb{R}^d)$. Since the measures $F(u_n;\cdot)$ and $F(v_n;\cdot)$ have their support on $\Omega\cup\Sigma$, the choice of these extensions will not affect the energies under consideration.

Choose an open Lipschitz domain $B_0 \in \mathcal{T}(\mathbb{R}^N)$ such that $U \subset\subset B_0 \subset\subset B$. Using Lemma 2.4 we find two other sequences (u'_n) and (v'_n), both converging to u in $L^1(V;\mathbb{R}^d)$, $u'_n = v'_n = u$ on ∂B_0, and satisfying

$$\begin{aligned}&\limsup_{n\to+\infty} F(v'_n;A\setminus\overline{B}_0) \le \liminf_{n\to+\infty} F(v_n;A\setminus\overline{B}_0),\\ &\limsup_{n\to+\infty} F(u'_n;B_0) \le \liminf_{n\to+\infty} F(u_n;B_0).\end{aligned}$$

Defining $w_n = v'_n$ in $V\setminus\overline{B}_0$ and $w_n = u'_n$ in B_0, we get $w_n \to u$ in $L^1(V;\mathbb{R}^d)$ and $w_n \in W^{1,1}(V\setminus\Sigma;\mathbb{R}^d)$. Since $B_0 \subset B$, $A\setminus\overline{B}_0 \subset A\setminus\overline{U}$, and $\mathcal{H}^{N-1}(\partial B_0\cap\Sigma) = 0$, we also obtain

$$\begin{aligned}\mathcal{F}_0(u;A) &\le \liminf_{n\to+\infty} F(w_n;A) = \liminf_{n\to+\infty}\left[F(u'_n;B_0) + F(v'_n;A\setminus\overline{B}_0)\right]\\ &\le \liminf_{n\to+\infty} F(u'_n;B_0) + \limsup_{n\to+\infty} F(v'_n;A\setminus\overline{B}_0)\\ &\le \lim_{n\to+\infty} F(u_n;B) + \lim_{n\to+\infty} F(v_n;A\setminus\overline{U})\\ &= \mathcal{F}_0(u;B) + \mathcal{F}_0(u;A\setminus\overline{U}),\end{aligned}$$

which proves (2.10).

Now we fix a bounded open subset V such that $V \supset \overline{\Omega}$, and we consider a sequence (u_n) such that

$$u_n \to u \quad \text{in } L^1(\Omega;\mathbb{R}^d), \qquad \lim_{n\to+\infty} F(u_n;V) = \mathcal{F}_0(u;V) \; . \tag{2.11}$$

Let τ be a Radon measure in $\mathbb{R}^N$ defined, up to a subsequence, as the weak-$*$ limit of $\tau_n := F(u_n;\cdot)$. Clearly τ is supported on $\overline{\Omega}$ and, in view of (1.1), noticing that the equality in (2.11) holds if we replace V by any open V' such that $\Omega \subset\subset V' \subset V$, we have

$$\mathcal{F}(u;V) \;\geq\; \tau(\overline{\Omega}) \;=\; \tau(\mathbb{R}^N) \; . \tag{2.12}$$

On the other hand, by definition of $\mathcal{F}$, for all $\varepsilon > 0$, $A \in \mathcal{A}(\mathbb{R}^N)$, there exists $A' \in \mathcal{A}(\mathbb{R}^N)$ such that $A' \subset\subset A$ and

$$\mathcal{F}(u;A) \;\leq\; \mathcal{F}_0(u;A') + \varepsilon \;\leq \liminf_{n\to+\infty} F(u_n;A') \;\leq\; \tau(\overline{A'}) + \varepsilon \;\leq\; \tau(A) + \varepsilon \; .$$

Letting $\varepsilon \to 0^+$, we get for all $A \in \mathcal{A}(\mathbb{R}^N)$

$$\mathcal{F}(u;A) \;\leq\; \tau(A). \tag{2.13}$$

We complete the proof of (2.7) by showing that

$$\tau(A) \;\leq\; \mathcal{F}(u;A). \tag{2.14}$$

holds for all $A \in \mathcal{A}(\mathbb{R}^N)$, which, together with (2.13), yields the equality $\tau \equiv \mathcal{F}(u;\cdot)$.

Let $A \in \mathcal{A}(\mathbb{R}^N)$. It is not restrictive to assume that $A \subset\subset V$. Let $\varepsilon > 0$ and find $U \in \mathcal{A}(\mathbb{R}^N)$ such that

$$U \subset\subset A \; , \quad \tau(A \setminus U) < \varepsilon \; . \tag{2.15}$$

By (2.10), (2.12), (2.13) and (2.15), we have

$$\begin{aligned} \tau(A) \leq \tau(U) + \varepsilon &\leq \tau(V) - \tau(V \setminus \overline{U}) + \varepsilon \\ &\leq \mathcal{F}(u;V) - \mathcal{F}(u;V \setminus \overline{U}) + \varepsilon \\ &\leq \mathcal{F}(u;A) + \varepsilon. \end{aligned}$$

Hence, (2.14) follows by letting $\varepsilon \to 0^+$.

□

To conclude this section we prove the following result

Proposition 2.6. *Under hypotheses (H1), (H2) and (H3),*

$$Q(f_0^\infty) = (Qf_0)^\infty,$$

where Qf stands for the quasiconvex envelope of f.

Proof. Since $Qf \le f$ we obtain $(Qf)^\infty \le f^\infty$ and $Q((Qf)^\infty) \le Q(f^\infty)$. Using (H2), (H3) and Fatou's Lemma, we get that $(Qf)^\infty$ is quasiconvex, which yields $(Qf)^\infty \le Q(f^\infty)$.

Conversely,

$$(Qf)^\infty(\xi) = \limsup_{t\to+\infty} \frac{Qf(t\,\xi)}{t} = \limsup_{t\to+\infty} \int_Q \frac{f(t\,\xi + \nabla\varphi_t)}{t}, \tag{2.16}$$

for some $\varphi_t \in W^{1,\infty}(Q;\mathbb{R}^d)$. Defining $h_t(x) := |\xi + \nabla\varphi_t/t|$, hypothesis (H2) yields that $\|h_t\|_{L^1(Q;\mathbb{R}^d)} \le C'$, for a constant C' independent of t. By (H2), (H3), and in light of (2.16), we get

$$\begin{aligned}
(Qf)^\infty(\xi) \ge& \limsup_{t\to+\infty} \int_{Q\cap\{x|th_t(x)>L\}} \left(f^\infty(\xi + \nabla\varphi_t/t) - \frac{(h_t)^{1-m}}{t^m} \right) dx \\
\ge& \limsup_{t\to+\infty} \int_Q f^\infty(\xi + \nabla\varphi_t/t) - \limsup_{t\to+\infty} C \int_{Q\cap\{x|th_t(x)\le L\}} h_t(x)\, dx \\
&- \limsup_{t\to+\infty} \frac{1}{t^m} \int_Q (h_t(x))^{1-m}\, dx \\
\ge& Q(f^\infty) - \limsup_{t\to+\infty} \frac{1}{t^m} \left(\int_Q h_t(x)\, dx \right)^{1-m} \\
\ge& Q(f^\infty)(\xi) - \limsup_{t\to+\infty} \frac{(C')^{1-m}}{t^m} \\
\ge& Q(f^\infty)(\xi).
\end{aligned}$$

□

3. The global method of relaxation. The main theorem. Our goal here is to find an integral representation for the relaxed energy $\mathcal{F}$ introduced in (1.1), (1.2) and (1.3). Following [BFM], we introduce for $u \in BV(\Omega;\mathbb{R}^d)$ the following set functions on $\mathcal{A}_\infty(\mathbb{R}^N)$:

$$m_0(u;A) := \inf\left\{ F(v;A) |\ v = u \text{ on } \partial A \cap \Omega,\ v \in W^{1,1}(\Omega\setminus\Sigma;\mathbb{R}^d) \right\}, \tag{3.1}$$

$$m(u;A) := \inf\left\{ \mathcal{F}(v;A) |\ v = u \text{ on } \partial A \cap \Omega,\ v \in BV(\Omega;\mathbb{R}^d) \right\}. \tag{3.2}$$

Thanks to the upper bound (2.9) and Lemma 2.1, we can prove exactly as in [BFM]

Lemma 3.1. *There exists a constant $C > 0$ such that for all $u_1, u_2 \in BV(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}_\infty(\mathbb{R}^N)$,*

$$|m(u_1;A) - m(u_2;A)| \le C \int_{\partial A\cap\Omega} |\text{tr }(u_1 - u_2)|\, d\mathcal{H}^{N-1}.$$

In the remaining of this section, we fix an element $u \in BV(\Omega;\mathbb{R}^d)$ and consider the Radon measure μ, with support on $\Omega \cup \Sigma$,

$$\mu := \mathcal{L}^N \lfloor \Omega + |D_s u| \lfloor (\Omega \setminus \Sigma) + \mathcal{H}^{N-1} \lfloor \Sigma\, . \tag{3.3}$$

Propositions 3.2 and 3.3 below enable us to apply the representation formulæ of the global relaxation method introduced in [BFM]. In particular, we will be able to decouple and characterize separately the density of the relaxed energy $\mathcal{F}(u;\cdot)$ with respect to the surface density on Σ and the density with respect to $\mathcal{L}^N + |D_s u|$ on $\Omega \setminus \Sigma$ (the latter was already treated in [BFM]).

Proposition 3.2. *Let $u \in BV(\Omega;\mathbb{R}^d)$ and let μ be defined by (3.3). Then, under hypotheses (H1), (H2), (H4) and (H5)(i), we have*

$$m_0(u;A) = m(u;A) \tag{3.4}$$

for all set $A \in \mathcal{A}_\infty(\mathbb{R}^N)$.

Proof. In view of definitions (3.1), (3.2), and since $\mathcal{F}(u;A) \le F(u;A)$, it suffices to prove the inequality

$$m_0(u;A) \le m(u;A)\, .$$

For each $\delta > 0$ choose $v \in BV(\Omega;\mathbb{R}^d)$ such that $v = u$ on $\partial A \cap \Omega$ and

$$m(u;A) > \mathcal{F}(v;A) - \delta \ge \mathcal{F}_0(v;A_\varepsilon) - \delta, \tag{3.5}$$

where $A_\varepsilon := \{x \in A |\ \text{dist }(x,\partial A) > \varepsilon\ \}$, and $\varepsilon > 0$ is such that $A_\varepsilon \in \mathcal{T}(\mathbb{R}^N)$ and $|Dv|(\partial A_\varepsilon) = 0$.

Let (v_n^ε) be a sequence in $W^{1,1}(\Omega \setminus \Sigma;\mathbb{R}^d)$ converging to v in $L^1(\Omega;\mathbb{R}^d)$ and satisfying

$$\lim_{n\to+\infty} F(v_n^\varepsilon;A_\varepsilon) = \mathcal{F}_0(v;A_\varepsilon),\ v_n^\varepsilon = v \text{ on } \partial A_\varepsilon \cap \Omega, \tag{3.6}$$

where we have used Lemma 2.4. By Lemma 2.1, let $w_n^\varepsilon \in W^{1,1}(\Omega \setminus \Sigma;\mathbb{R}^d)$ be such that

$$w_n^\varepsilon := \begin{cases} v_n^\varepsilon & \text{in } A_\varepsilon \cap \Omega, \\ v & \text{in } \Omega \setminus A, \\ (v_n^\varepsilon)^+ = \lambda_0 & \text{on } \Sigma \cap (A \setminus A_\varepsilon), \\ (v_n^\varepsilon)^- = \theta_0 & \text{on } \Sigma \cap (A \setminus A_\varepsilon), \end{cases}$$

and

$$
\begin{aligned}
&F(w_n^\varepsilon; A \setminus \overline{A_\varepsilon}) \leq \\
&\leq C\left(\int_{(A\setminus\overline{A_\varepsilon})\cap\Omega} a(x)dx - |Dv|((A\setminus\overline{A_\varepsilon})\cap\Omega) - \int_{(A\setminus\overline{A_\varepsilon})\cap\Sigma} \beta_0(x,\lambda_0,\theta_0)d\mathcal{H}^{N-1}\right) \\
&= \mathcal{O}(\varepsilon).
\end{aligned} \tag{3.7}
$$

From (3.5), (3.6), (3.7), and since $w_n^\varepsilon = v$ on $\partial A \cap \Omega$, we conclude that

$$
m(u;A) \geq \limsup_{n\to+\infty} F(w_n^\varepsilon;A) - \delta + \mathcal{O}(\varepsilon) \geq m_0(u;A) - \delta + \mathcal{O}(\varepsilon).
$$

Letting $\varepsilon \to 0^+$ and then $\delta \to 0^+$, the result follows. □

By replacing the measure μ and the growth conditions on $\mathcal{F}$ used in [BFM] by the ones introduced in (3.3), (2.8), (2.9), arguments similar to those exploited in [BFM] yield the following two results :

Proposition 3.3. *Under hypotheses (H1), (H2), (H4) and (H5)(i),*

$$
\frac{d\mathcal{F}(u;\cdot)}{d\mu}(x_0) = \lim_{\varepsilon\to 0} \frac{m(u;Q_\nu(x_0,\varepsilon))}{\mu(u;Q_\nu(x_0,\varepsilon))} \quad \mu \text{ a.e. } x_0 \in \Omega, \text{ for all } \nu \in S^{N-1}. \tag{3.8}
$$

We now state our representation result. For the sake of simplicity in the presentation of the formulæ, as well as for simplification of technical details in the proof, we will consider only the case where the initial bulk energy f_0 is independent of (x,u). However, Theorem 3.4 may be extended to more general cases (see [BFM], Section 4), provided continuity conditions are imposed on f_0.

Theorem 3.4. *Under hypotheses (H1)-(H5) the functional $\mathcal{F}$ introduced in (1.0), (1.1) and (1.2), admits the following integral representation*

$$
\begin{aligned}
\mathcal{F}(u;A) = \int_{A\cap\Omega} Qf_0(\nabla u)\,dx &+ \int_{(A\setminus\Sigma)\cap S(u)} (Qf_0)^\infty([u]\otimes\nu_u)\,d\mathcal{H}^{N-1} \\
&+ \int_{A\cap\Sigma} \beta(x,u^+,u^-)\,d\mathcal{H}^{N-1} \\
&+ \int_{A\cap\Omega} (Qf_0)^\infty\left(\frac{dC(u)}{d|C(u)|}\right) d|C(u)|,
\end{aligned} \tag{3.9}
$$

for all $u \in BV(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\mathbb{R}^N)$, where Qf_0 is the quasiconvexification of f_0, and $\beta : \Sigma \times (\mathbb{R}^d)^2 \to [0,+\infty)$ is defined as follows

$$
\begin{aligned}
\beta(x_0,\lambda,\theta) := \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0\setminus\Sigma_0;\mathbb{R}^d) \\ v=u_{\lambda,\theta,\nu_0} \text{ on } \partial Q_{\nu_0}\cap\Pi_0}} \Big\{ &\int_{Q_{\nu_0}\cap\Pi_0} (Qf_0)^\infty(\nabla v)\,dy \\
&+ \int_{Q_{\nu_0}\cap\Sigma_0} \beta_0(x_0,v^+,v^-)\,d\mathcal{H}^{N-1} \Big\},
\end{aligned} \tag{3.10}
$$

with $\nu_0 := \nu_\Sigma(x_0)$, $\Sigma_0 := \{y \cdot \nu_0 = 0\}$, $u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \theta & \text{otherwise,} \end{cases}$ *and* $\Pi_0 := \mathbb{R}^N$ *if* $x_0 \in \Omega \cap \Sigma$, *and* $\Pi_0 := \{y \in \mathbb{R}^N \mid y \cdot \nu_0 < 0\}$ *if* $x_0 \in \partial\Omega \cap \Sigma$.

Proof. By Proposition 2.4, for any given $u \in BV(\Omega;\mathbb{R}^d)$, $\mathcal{F}(u;\cdot)$ is the trace in $\mathcal{A}(\mathbb{R}^N)$ of a Radon measure and its restriction to $BV(\Omega\setminus\Sigma;\mathbb{R}^d)\times\mathcal{A}(\Omega\setminus\Sigma)$ falls within the framework of the global method for relaxation introduced in [BFM] (Section 4.1, Theorem 4.1.3 and Remark 4.1.4) (see also [FM2] or [ADM] when f_0 is assumed to be quasiconvex) ; hence, for all $A \in \mathcal{A}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\mathcal{F}(u;A)=&\mathcal{F}(u;A\cap\Sigma)+\mathcal{F}(u;A\setminus\Sigma)\\ =&\mathcal{F}(u;A\cap\Sigma)+\int_{A\cap\Omega} Qf_0(\nabla u)\,dx+\int_{(A\setminus\Sigma)\cap S(u)}(Qf_0)^\infty([u]\otimes\nu_u)\,d\mathcal{H}^{N-1}\\ &+\int_{A\cap\Omega}(Qf_0)^\infty\left(\frac{dC(u)}{d|C(u)|}\right)d|C(u)|,\end{aligned}\tag{3.11}$$

where we have used the fact that $|C(u)|(\Sigma)=0$.

It remains to to identify the trace of the measure $\mathcal{F}(u;\cdot)$ on Σ which, by (2.9), is absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor\Sigma$. Hence the proof reduces to show that the density $\dfrac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}$ agrees with $\beta(x,u^+,u^-)$, where β is defined in (3.10). This identity will be etablished in the following two steps.

Step 1. Here we assume that in hypothesis (H5) the funtion u_0 is identically zero.

By Propositions 3.2 and 3.3, and using a blow-up argument combined with Lemma 3.1, we derive in the same way as in [BFM, proof of Thm. 3.7] that, for $\mathcal{H}^{N-1}$ a.e. $x_0 \in \Sigma$, there holds

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0)&=\frac{d\mathcal{F}(u;\cdot)}{d\mu}(x_0)\,\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0)\\ &=\lim_{\varepsilon\to0}\frac{m(u;Q_{\nu_0}(x_0,\varepsilon))}{\mu(Q_{\nu_0}(x_0,\varepsilon))}\lim_{\varepsilon\to0}\frac{\mu(Q_{\nu_0}(x_0,\varepsilon))}{\mathcal{H}^{N-1}(\Sigma\cap Q_{\nu_0}(x_0,\varepsilon))}\\ &=\lim_{\varepsilon\to0}\frac{m(u;Q_{\nu_0}(x_0,\varepsilon))}{\varepsilon^{N-1}},\\ &=\lim_{\varepsilon\to0}\frac{m_0(u_{x_0}(\cdot-x_0);Q_{\nu_0}(x_0,\varepsilon))}{\varepsilon^{N-1}}\end{aligned}\tag{3.12}$$

and $\displaystyle\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{N-1}}\int_{Q_{\nu_0}}a(x)\,dx=\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{N-1}}\int_{Q_{\nu_0}}\overline{a}(x)\,dx=0$, where ν_0 stands for the unit normal vector ν_Σ to Σ at x_0,

$$u_{x_0} := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu_0>0,\\ u^-(x_0) & \text{if } y\cdot\nu_0<0\,.\end{cases}$$

Using the change of variables $x = x_0 + \varepsilon y$ and defining $\Sigma_\varepsilon := (\Sigma - x_0)/\varepsilon$ and $\Omega_\varepsilon := (\Omega - x_0)/\varepsilon$, (3.12) reduces to

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega\varepsilon} \varepsilon\, f_0\Big(\frac{\nabla v}{\varepsilon}\Big)dy \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0(x_0+\varepsilon y, v^+, v^-)\, d\mathcal{H}^{N-1}\Big\} \\ &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega\varepsilon} f_0^\infty(\nabla v)dy \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0(x_0+\varepsilon y, v^+, v^-)\, d\mathcal{H}^{N-1}\Big\} \\ &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega_\varepsilon} Q(f_0^\infty)(\nabla v)\, dy + \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0(x_0+\varepsilon y, v^+, v^-)\, d\mathcal{H}^{N-1}\Big\}.\end{aligned} \tag{3.13}$$

where, for the second equality, we have used (H2) and (H3) to replace $\varepsilon\, f_0\Big(\dfrac{\nabla v}{\varepsilon}\Big)$ by $f_0^\infty\Big(\nabla v\Big)$, and, in the last equality, we have substituded f_0^∞ by its convexification $Q(f_0^\infty)$. This latter fact is justified since, on one hand, we have $Qf_0 \le f_0$ and, on the other hand, every competitor v in (3.13) can be approximated (see [FM1]) by a sequence of functions $v_n \in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)$ which agree with v on ∂Q_{ν_0} and on each side of $Q_{\nu_0}\cap\Sigma_\varepsilon$ and such that

$$\int_{Q_{\nu_0}\cap\Omega_\varepsilon} Q(f_0^\infty)(\nabla v)\, dx = \lim_{n\to+\infty}\int_{Q_{\nu_0}\cap\Omega_\varepsilon} f_0^\infty(\nabla v_n)\, dx.$$

We claim that (3.13) still holds if we replace $f_0(x)$ by $f_{u_0}(x,\xi) := f_0(\nabla u_0(x)+\xi)$, where u_0 is given by hypothesis (H5). This fact will be usefull to justify Step 2. In fact, since u_0 given by hypothesis (H5) satisfies, $\mathcal{H}^{N-1}$ a.e. $x_0\in\Sigma$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-1}}\int_{Q_{\nu_0}} |\nabla u_0(x)|\, dx = 0,$$

we deduce, as before,

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega\varepsilon} \varepsilon\, f_0\Big(\nabla u_0(x_0+\varepsilon y)+\frac{\nabla v}{\varepsilon}\Big)dy \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0\big(x_0+\varepsilon y, v^+, v^-\big)\, d\mathcal{H}^{N-1}\Big\} \\ &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega\varepsilon} f_0^\infty\Big(\varepsilon\nabla u_0(x_0+\varepsilon y)+\nabla v\Big)dy \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0\big(x_0+\varepsilon y, v^+, v^-\big)\, d\mathcal{H}^{N-1}\Big\} \\ &= \lim_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega_\varepsilon} Q(f_0^\infty)\Big(\varepsilon\nabla u_0(x_0+\varepsilon y)+\nabla v\Big)dy \\ &\quad + \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0\big(x_0+\varepsilon y, v^+, v^-\big)\, d\mathcal{H}^{N-1}\Big\}.\end{aligned} \tag{3.13'}$$

Since $Q(f_0^\infty)$ is globally Lipschitz (see [Ma]), we obtain, for a convenient $C>0$,

$$\begin{aligned}&\int_{Q_{\nu_0}\cap\Omega_\varepsilon} |Q(f_0^\infty)\Big(\varepsilon\nabla u_0(x_0+\varepsilon y)+\nabla v\Big) - Q(f_0^\infty)(\nabla v)|\, dy \\ &\qquad \le C\int_{Q_{\nu_0}\cap\Omega_\varepsilon} |\varepsilon\nabla u_0(x_0+\varepsilon y)|\, dy = \frac{C}{\varepsilon^{N-1}}\int_{Q_{\nu_0}\cap\Omega_\varepsilon} |\nabla u_0(x)|\, dx \underset{\varepsilon}{\to} 0,\end{aligned}$$

which allow us to replace $Q(f_0^\infty)\Big(\varepsilon\nabla u_0(x_0+\varepsilon y)+\nabla v\Big)$ by $Q(f_0^\infty)(\nabla v)$ in the last equality of (3.13′).

By Proposition 2.6, we obtain from (3.13) (or (3.13′)) that, $\mathcal{H}^{N-1}$ a.e. $x_0\in\Sigma$,

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) = \lim_{\varepsilon\to 0} &\inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Omega_\varepsilon\setminus\Sigma_\varepsilon;\mathbb{R}^d)\\ v=u_{x_0} \text{ on } \partial Q_{\nu_0}\cap\Omega_\varepsilon}} \Big\{\int_{Q_{\nu_0}\cap\Omega_\varepsilon} (Qf_0)^\infty(\nabla v)\, dy \\ &+ \int_{Q_{\nu_0}\cap\Sigma_\varepsilon} \beta_0\big(x_0+\varepsilon y, v^+, v^-\big)\, d\mathcal{H}^{N-1}\Big\}.\end{aligned} \tag{3.14}$$

Without loss of generality, we assume in (3.14) that $\nu_0 = e_N$ and we set $Q := Q_{e_N}$, $Q' := (-\frac{1}{2},\frac{1}{2})^{N-1}$. For each $\varepsilon>0$ small enough, let $\varphi_\varepsilon : Q' \to \mathbb{R}$ be a C^1 function whose graph coincides with $Q\cap\Sigma_\varepsilon$. Consider the C^1 diffeomorfism $\Phi_\varepsilon : Q\to\mathbb{R}^N$ such that $\Phi_\varepsilon(y', y_N) := (y', y_N - \varphi_\varepsilon(y'))$, and let $\tilde{Q}_\varepsilon := \Phi_\varepsilon(Q)$. Note that $\nabla\varphi_\varepsilon(0) = 0$ and $\lim_{\varepsilon\to 0}\|\varphi_\varepsilon\|_{W^{1,\infty}(Q')} = 0$.

Representing by Σ_0 the hyperplane $\{y=(y',y_N)|y_N=0\}$, and defining

$$\overline{u}_\varepsilon(y) := u_{x_0}(\Phi_\varepsilon^{-1}(y)),$$

(3.14) becomes

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0)=\lim_{\varepsilon\to 0}\inf_{\substack{v\in W^{1,1}(\tilde{Q}_\varepsilon\cap\Pi_0\setminus\Sigma_0;\mathbb{R}^d)\\ v=\overline{u}_\varepsilon \text{ on } \partial\tilde{Q}_\varepsilon\cap\Pi_0}}\Big\{\int_{\tilde{Q}_\varepsilon\cap\Pi_0}(Qf_0)^\infty(\nabla v\,\nabla\Phi_\varepsilon(\Phi_\varepsilon^{-1}(y)))dy$$
$$+\int_{\tilde{Q}_\varepsilon\cap\Sigma_0}\beta_0\big(x_0+\varepsilon\Phi_\varepsilon^{-1}(y),v^+,v^-\big)\left(\sqrt{1+|\nabla\Phi_\varepsilon(\Phi_\varepsilon^{-1}(y))|^2}\right)^{-1}d\mathcal{H}^{N-1}(y)\Big\}. \tag{3.15}$$

We claim that (3.15) can be simplified as follows :

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0)=\lim_{\varepsilon\to 0}\inf_{\substack{v\in W^{1,1}(\tilde{Q}_\varepsilon\cap\Pi_0\setminus\Sigma_0;\mathbb{R}^d)\\ v=\overline{u}_\varepsilon \text{ on } \partial\tilde{Q}_\varepsilon\cap\Pi_0}}\Big\{\int_{\tilde{Q}_\varepsilon\cap\Pi_0}(Qf_0)^\infty(\nabla v)dy+$$
$$+\int_{\tilde{Q}_\varepsilon\cap\Sigma_0}\beta_0\big(x_0,v^+,v^-\big)\,d\mathcal{H}^{N-1}\Big\}. \tag{3.16}$$

Indeed, fix $\delta>0$ and for each $\varepsilon>0$ sufficiently small, consider a minimizing sequence (v_n) realizing the infimum in (3.15). Clearly the traces of v_n on $\partial\tilde{Q}_\varepsilon\cap\Pi_0$ are uniformly bounded and, due to coercivity hypotheses (H2), we have that $(|\nabla v_n|)$ is bounded in $L^1(\tilde{Q}_\varepsilon\cap\Pi_0)$ independently of n and ε. Therefore, using hypothesis (H5) (ii) (with $u_0\equiv 0$) and a version of Poincaré inequality in $W^{1,1}$, we have

$$\begin{aligned}\int_{\tilde{Q}_\varepsilon\cap\Sigma_0}&|\beta_0(x_0+\varepsilon\Phi_\varepsilon^{-1},v_n^+,v_n^-)-\beta_0(x_0,v_n^+,v_n^-)|d\mathcal{H}^{N-1}\\ &\le\int_{\tilde{Q}_\varepsilon\cap\Sigma_0}\delta\left(1+|v_n^+|+|v_n^-|\right)d\mathcal{H}^{N-1}\\ &\le C\delta\left(1+\int_{\tilde{Q}_\varepsilon\cap\Pi_0}|\nabla v_n|\,dx\right)\\ &\le C'\delta\,,\end{aligned}$$

where C' is a constant independent of ε, n and δ. From the above estimates using the fact that, due to (H2), $(Qf_0)^\infty$ is Lipschitz continuous and that the convergence of $\nabla\Phi_\varepsilon$ to the identity is uniform, we conclude (3.16) by letting ε, then δ tend to zero.

Now we remark that (3.16) would reduce to (3.10) if we could replace $\overline{u}_\varepsilon$ by u_{x_0} and $\tilde{Q}_\varepsilon$ by Q. In order to do so, noticing that $\tilde{Q}_\varepsilon\subset 2Q$ for small ε, we set $\Omega_0:=2Q\cap\Pi_0$, $\Sigma_0':=\Sigma_0\cap\overline{\Omega}_0$, and define

$$\overline{F}(w;A):=$$
$$\begin{cases}\displaystyle\int_{A\cap\Omega_0}(Qf_0)^\infty(\nabla w)\,dx+\int_{A\cap\Sigma_0'}\beta_0(x_0,w^+,w^-)d\mathcal{H}^{N-1} & \text{if } w\in W^{1,1}(\Omega_0\setminus\Sigma_0';\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$\overline{m}_0(w;A) := \inf\{\overline{F}(v;A) |\ v \in W^{1,1}(\Omega_0 \setminus \Sigma_0'; \mathbb{R}^d) \quad \text{and} \quad v = w \quad \text{on} \quad \partial A \cap \Omega_0\}.$$

Since $(Qf_0)^\infty$, $\beta_0(x_0, \cdot, \cdot)$ satisfy hypotheses (H1), (H2), (H4) and (H5) (i), in view of all the previous results, Lemma 3.1 and Proposition 3.2 still apply with Ω, Σ being replaced by Ω_0, Σ_0', respectively, and we have

$$\limsup_{\varepsilon\to 0} |\overline{m}_0(\overline{u}_\varepsilon; \tilde{Q}_\varepsilon) - \overline{m}_0(u_{x_0}; \tilde{Q}_\varepsilon)| \leq C \limsup_{\varepsilon\to 0} \int_{\partial\tilde{Q}_\varepsilon \cap \Omega_0} |\overline{u}_\varepsilon - u_{x_0}| d\mathcal{H}^{N-1} = 0.$$

We conclude then, from (3.16), that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) = \lim_{\varepsilon\to 0} \overline{m}_0(\overline{u}_\varepsilon; \tilde{Q}_\varepsilon) = \lim_{\varepsilon\to 0} \overline{m}_0(u_{x_0}; \tilde{Q}_\varepsilon). \tag{3.17}$$

The last step consists in replacing $\tilde{Q}_\varepsilon$ by Q in (3.17). To that aim, we define, for $k \in \mathbb{N}$, $Q_k' :=]-1/2\ ,1/2[^{N-1}\times\ (1+1/k)\ ]-1/2\ ,1/2[$ and $Q_k'' :=]-1/2\ ,1/2[^{N-1}\times\ (1-1/k)\ ]-1/2\ ,1/2[$. Since $\varphi_\varepsilon \to 0$ uniformly, let $\varepsilon(k)$ be such that, for all $\varepsilon < \varepsilon(k)$, $Q_k'' \subset\subset \tilde{Q}_\varepsilon \subset\subset Q_k'$. Using piecewise constant extensions of the admissible functions, one obtains

$$\overline{m}_0(u_{x_0}; Q_k') - \mathcal{O}(1/k) \leq \overline{m}_0(u_{x_0}; \tilde{Q}_\varepsilon) \leq \overline{m}_0(u_{x_0}; Q_k'') + \mathcal{O}(1/k). \tag{3.18}$$

Now, by considering a linear change of variables, in the direction e_N, mapping Q_k' onto Q (respectively Q_k'' onto Q) and since $(Qf_0)^\infty$ is Lipschitz, we obtain that,

$$\lim_{k\to+\infty} \overline{m}_0(u_{x_0}; Q_k') = \lim_{k\to+\infty} \overline{m}_0(u_{x_0}; Q_k'') = \overline{m}_0(u_{x_0}; Q).$$

Thus, letting $\varepsilon \to 0$ in (3.18) and then $k \to +\infty$,

$$\lim_{\varepsilon\to 0} \overline{m}_0(u_{x_0}; \tilde{Q}_\varepsilon) = \overline{m}_0(u_{x_0}; Q). \tag{3.19}$$

Finally, from (3.17) and (3.19), we conclude that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) = \overline{m}_0(u_{x_0}; Q) = \beta(x_0, u^+(x_0), u^-(x_0)),$$

where β is defined by (3.10).

Step 2. Consider in hypothesis (H5) a general $u_0 \in W^{1,1}(\Omega \setminus \Sigma; \mathbb{R}^d)$. Set

$$F_{u_0}(u;A) := F(u_0 + u; A)$$

We have

$$F_{u_0}(u;A) := \begin{cases} \displaystyle\int_{A\cap\Omega} f_{u_0}(x, \nabla u)\, dx & \\ \displaystyle + \int_{A\cap\Sigma} \beta_{u_0}(x, u^+, u^-)\, d\mathcal{H}^{N-1} & \text{if} \quad u \in W^{1,1}(\Omega \setminus \Sigma; \mathbb{R}^d), \\ +\infty & \text{otherwise}, \end{cases}$$

where

$$f_{u_0}(x,\xi) := f_0(\nabla u_0(x)+\xi) \quad \text{and} \quad \beta_{u_0}(x,r,s) := \beta_0(x,\lambda_0(x)+r,\theta_0(x)+s).$$

Representing by $\mathcal{F}_{u_0}(u;A)$ the corresponding relaxed functional, it follows that, for each $A \in \mathcal{A}(\mathbb{R}^N)$,

$$\mathcal{F}_{u_0}(u;A) = \mathcal{F}(u_0+u;A). \tag{3.20}$$

Since f_{u_0} and β_{u_0} satisfy hypotheses (H1), (H2), (H4) and (H5) (i) (note that $\beta_{u_0}(x,0,0) = \beta(x,\lambda_0,\theta_0)$), by Proposition 2.4 $\mathcal{F}_{u_0}(u-u_0;\cdot)$ is the trace of a Radon measure on $\mathcal{A}(\mathbb{R}^N)$, and by (3.20)

$$\begin{aligned} \frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) &= \frac{d\mathcal{F}_{u_0}(u-u_0;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) \\ &= \tilde{\beta}(x_0,(u-u_0)^+(x_0),(u-u_0)^-(x_0)) \\ &= \tilde{\beta}(x_0,u^+(x_0)-\lambda_0(x_0),u^-(x_0)-\theta_0(x_0)), \end{aligned} \tag{3.21}$$

where we have used Step 1 on $\mathcal{F}_{u_0}(u-u_0;\cdot)$, and where, according to (3.10),

$$\begin{aligned} &\tilde{\beta}(x_0,u^+(x_0)-\lambda_0(x_0),u^-(x_0)-\theta_0(x_0)) := \\ &= \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0\setminus\Sigma_0;\mathbb{R}^d) \\ v=u_{u^+(x_0)-\lambda_0(x_0),u^-(x_0)-\theta_0(x_0),\nu_0} \text{ on } \partial Q_{\nu_0}\cap\Pi_0}} \Big\{ \int_{Q_{\nu_0}\cap\Pi_0} (Qf_0)^\infty(\nabla v)\, dy \\ &\qquad + \int_{Q_{\nu_0}\cap\Sigma_0} \beta_{u_0}(x_0,v^+,v^-)\, d\mathcal{H}^{N-1} \Big\}. \end{aligned}$$

It is clear that if w is admissible for $\beta(x_0,u^+(x_0),u^-(x_0))$, then

$$v := \begin{cases} w-\lambda_0 & \text{if } y\cdot\nu_0 > 0, \\ w-\theta_0 & \text{otherwise}, \end{cases}$$

is admissible for $\tilde{\beta}(x_0,u^+(x_0)-\lambda_0(x_0),u^-(x_0)-\theta_0(x_0))$. In light of (3.21) we conclude that

$$\frac{d\mathcal{F}_{u_0}(u-u_0;\cdot)}{d\mathcal{H}^{N-1}\lfloor\Sigma}(x_0) = \beta_0(x_0,u^+(x_0),u^-(x_0)).$$

□

4. Examples.

4.1. Trace condition.

For fixed $\Phi_0 \in W^{1,1}(\Omega;\mathbb{R}^d)$ consider the energy functional

$$F(u;A) := \begin{cases} \int_{A\cap\Omega} f_0(\nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega\cap A;\mathbb{R}^d) \text{ and } u=\Phi_0 \text{ on } \Sigma\cap A, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.1.1}$$

where Σ is a closed and connected C^1 hypersurface contained in $\overline{\Omega}$. In order to establish a parallel with the analysis of the previous sections, we remark that F may be re-written as in (1.0), with

$$\beta_0(x,\lambda,\theta) := \begin{cases} 0 & \text{if} \lambda = \theta = \Phi_0(x), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1.2}$$

Suppose that f_0 satisfies hypotheses (H1)-(H3). It is clear that (H4) and (H5) hold with $u_0 := \Phi_0$.

We claim that for every $A \in \mathcal{A}(\mathbb{R}^N)$ and for every $u \in BV(\Omega;\mathbb{R}^d)$, the relaxed functional $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F}(u;A) = &\int_{A\cap\Omega} Qf_0(\nabla u)\, dx + \int_{(A\setminus\Sigma)\cap S(u)} (Qf_0)^\infty([u]\otimes\nu_u)\, d\mathcal{H}^{N-1} \\ &+ \int_{A\cap\Omega} (Qf_0)^\infty\left(\frac{dC(u)}{d|C(u)|}\right) d|C(u)| \\ &+ \int_{A\cap\Sigma} \beta(x,u^+,u^-) d\mathcal{H}^{N-1}, \end{aligned} \tag{4.1.3}$$

where

$$\beta(x_0,\lambda,\theta) := \begin{cases} (Qf_0)^\infty\Big((\lambda - \Phi_0(x_0))\otimes\nu_0\Big) + (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \Omega\cap\Sigma, \\ (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \partial\Omega\cap\Sigma. \end{cases} \tag{4.1.4}$$

By Theorem 3.4 it suffices to prove that (3.10) and (4.1.4) are one and the same surface energy density. According to (3.10) we have

$$\beta(x_0,\lambda,\theta) = \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0;\mathbb{R}^d) \\ v=u_{\lambda,\theta,\nu_0} \text{ on } \partial Q_{\nu_0}\cap\Pi_0 \\ v=\Phi_0(x_0) \text{ on } \Sigma_0}} \int_{Q_{\nu_0}\cap\Pi_0} (Qf_0)^\infty(\nabla v)\, dy, \tag{4.1.5}$$

where

$$u_{\lambda,\theta,\nu_0} := \begin{cases} \lambda & \text{if } \; y\cdot\nu_0 \geq 0, \\ \theta & \text{if } \; y\cdot\nu_0 < 0. \end{cases}$$

Recall that if $x_0 \in \Sigma$ then Π_0 is defined so that $Q^+_{\nu_0}\cap\Pi_0 = \emptyset$ if $x_0 \in \Sigma\cap\partial\Omega$, and that $Q^-_{\nu_0}\cap\Pi_0 = Q^-_{\nu_0}$ for all $x_0 \in \Sigma$.

Note that we may rewrite (4.1.5) as

$$\beta(x_0,\lambda,\theta) = \Lambda_0(\overline{u}^\lambda; Q^+_{\nu_0}\cap\Pi_0) + \Lambda_0(\overline{u}_\theta; Q^-_{\nu_0}), \tag{4.1.6}$$

where

$$\Lambda_0(u;A) := \inf_{\substack{v\in W^{1,1}(A;\mathbb{R}^d)\\ v=u \text{ on } \partial A}} \int_A (Qf_0)^\infty(\nabla v)\, dy,$$

$$Q^+_{\nu_0} := \{y \in Q_{\nu_0} |\ y\cdot\nu_0 \geq 0\},\ Q^-_{\nu_0} := \{y \in Q_{\nu_0} |\ y\cdot\nu_0 \leq 0\},$$

and $\overline{u}^\lambda \in W^{1,1}(Q^+_{\nu_0};\mathbb{R}^d)$ and $\overline{u}_\theta \in W^{1,1}(Q^-_{\nu_0};\mathbb{R}^d)$ are defined by

$$\overline{u}^\lambda := \begin{cases} \lambda & \text{on} \quad \partial Q_{\nu_0}\cap Q^+_{\nu_0},\\ \Phi_0(x_0) & \text{on} \quad \Sigma_0 \cap Q_{\nu_0}, \end{cases}$$

and

$$\overline{u}_\theta := \begin{cases} \Phi_0(x_0) & \text{on} \quad \Sigma_0 \cap Q_{\nu_0},\\ \theta & \text{on} \quad \partial Q_{\nu_0}\cap Q^-_{\nu_0}. \end{cases}$$

Since $(Qf_0)^\infty(0) = 0$, we have that

$$\Lambda_0(\overline{u}_\theta; Q^-_{\nu_0}) \geq \Lambda_0(u_{\Phi_0(x_0),\theta,\nu_0}; Q_{\nu_0}). \tag{4.1.7}$$

As $(Qf_0)^\infty$ is a quasiconvex function with linear growth then (see [FM2])

$$\Lambda_0(u_{\Phi_0(x_0),\theta,\nu_0}; Q_{\nu_0}) = (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big),$$

which, together with (4.1.7), yields

$$\Lambda_0(\overline{u}_\theta; Q^-_{\nu_0}) \geq (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big). \tag{4.1.8}$$

Similarly, if $Q^+_{\nu_0}\cap\Pi_0 \neq \emptyset$ then $\Lambda_0(\overline{u}^\lambda; Q^+_{\nu_0}\cap\Pi_0) \geq (Qf_0)^\infty\Big((\lambda-\Phi_0(x_0))\otimes\nu_0\Big)$, and from (4.1.6) and (4.1.8) we conclude that

$$\beta_0(x,\lambda,\theta) \geq \begin{cases} (Qf_0)^\infty\Big((\lambda-\Phi_0(x_0))\otimes\nu_0\Big) + (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \Omega\cap\Sigma, \\ (Qf_0)^\infty\Big((\Phi_0(x_0)-\theta)\otimes\nu_0\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \partial\Omega\cap\Sigma. \end{cases} \tag{4.1.9}$$

To establish the opposite inequality, for $\varepsilon > 0$ small enough consider ϕ^ε_θ in $W^{1,1}(Q^-_{\nu_0};\mathbb{R}^d)$ defined by

$$\phi^\varepsilon_\theta := \begin{cases} \theta & \text{if } y\cdot\nu_0 < -\varepsilon,\\ \Phi_0(x_0) + \varepsilon^{-1}(\Phi_0(x_0)-\theta)(y\cdot\nu_0), & \text{otherwise.} \end{cases}$$

Using Lemma 3.1 we have that

$$|\Lambda_0(\phi_\theta^\varepsilon; Q_{\nu_0}^-) - \Lambda_0(\overline{u}_\theta; Q_{\nu_0}^-)| \leq C \int_{\partial Q_{\nu_0}^-} |\phi_\theta^\varepsilon - \overline{u}_\theta| \, d\mathcal{H}^{N-1} = \mathcal{O}(\varepsilon), \tag{4.1.10}$$

and, since

$$\Lambda_0(\phi_\theta^\varepsilon; Q_{\nu_0}^-) \leq \int_{Q_{\nu_0}^-} (Qf_0)^\infty (\nabla \phi_\theta^\varepsilon) \, dy = (Qf_0)^\infty \Big((\Phi_0(x_0) - \theta) \otimes \nu_0 \Big),$$

from (4.1.11) we obtain, as ε goes to zero,

$$\Lambda_0(\overline{u}_\theta; Q_{\nu_0}^-) \leq (Qf_0)^\infty \Big((\Phi_0(x_0) - \theta) \otimes \nu_0 \Big).$$

Similarly, if $Q_{\nu_0}^+ \cap \Pi_0 \neq \emptyset$ then

$$\Lambda_0(\overline{u}^\lambda; Q_{\nu_0}^+) \leq (Qf_0)^\infty \Big((\lambda - \Phi_0(x_0)) \otimes \nu_0 \Big);$$

hence, by (4.1.6),

$$\beta_0(x, \lambda, \theta) \leq \begin{cases} (Qf_0)^\infty \Big((\lambda - \Phi_0(x_0)) \otimes \nu_0 \Big) + (Qf_0)^\infty \Big((\Phi_0(x_0) - \theta) \otimes \nu_0 \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \Omega \cap \Sigma, \\ (Qf_0)^\infty \Big((\Phi_0(x_0) - \theta) \otimes \nu_0 \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0 \in \partial\Omega \cap \Sigma. \end{cases} \tag{4.1.11}$$

Formula (4.1.4) now follows from (4.1.9) and (4.1.11).

4.2. Trace energy.

Consider now the functional

$$F(u; A) := \begin{cases} \displaystyle\int_{A \cap \Omega} f_0(\nabla u) dx + \int_{A \cap \Sigma} \beta_0(u) \, d\mathcal{H}^{N-1} & \text{if } u \in W^{1,1}(\Omega \cap A; \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2.1}$$

where, as before, Σ is a closed and connected C^1 hypersurface contained in $\overline{\Omega}$.

We assume that f_0 and β_0 satisfy hypotheses (H1) - (H5) and, in order to obtain more explicit formulæ, we suppose in addition that they are both convex. This last assumption falls into the case considered in [ADK], where $N = 2$, $f_0(\xi) := |\xi_1| + |\xi_2|$ and $\beta_0(x, s, t) := |s - t| + \left| \frac{s+t}{2} h_1(x) + h_2(x) \right|$, with h_1 and h_2 two suitable Borel functions.

We claim that the relaxed functional $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}(u;A) = &\int_{A\cap\Omega} f_0(\nabla u)\, dx + \int_{(A\setminus\Sigma)\cap S(u)} f_0^\infty([u]\otimes\nu_u)\, d\mathcal{H}^{N-1} \\ &+ \int_{A\cap\Omega} f_0^\infty\left(\frac{dC(u)}{d|C(u)|}\right) d|C(u)| + \int_{A\cap\Sigma} \beta(x,u^+,u^-)\, d\mathcal{H}^{N-1},\end{aligned} \tag{4.2.2}$$

where β can be described in terms of a minimization problem in $\mathbb{R}^d$, precisely,

$$\beta(x_0,\lambda,\theta) := \begin{cases} \inf_{w\in\mathbb{R}^d}\left\{ f_0^\infty\Big((\lambda-w)\otimes\nu_0\Big) + f_0^\infty\Big((w-\theta)\otimes\nu_0\Big) + \beta_0(w)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0\in\Omega\cap\Sigma, \\ \inf_{w\in\mathbb{R}^d}\left\{ f_0^\infty\Big((w-\theta)\otimes\nu_0\Big) + \beta_0(w)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_0\in\partial\Omega\cap\Sigma. \end{cases} \tag{4.2.3}$$

By Theorem 3.4, (4.2.2) holds if and only if β, as defined in (4.2.3), agrees with the formula provided by (3.10). Since f_0 is convex, (3.10) reduces to

$$\beta(x_0,\lambda,\theta) = \inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0;\mathbb{R}^d) \\ v=u_{\lambda,\theta,\nu_0} \text{ on } \partial Q_{\nu_0}\cap\Pi_0}} \left(\int_{Q_{\nu_0}\cap\Pi_0} f_0^\infty(\nabla v)\, dy + \int_{Q_{\nu_0}\cap\Sigma_0} \beta_0(v)\, d\mathcal{H}^{N-1}\right), \tag{4.2.4}$$

and so

$$\beta(x_0,\lambda,\theta) \geq \inf_{v\in\mathcal{P}(\lambda,\theta,\nu_0)} \left(\int_{Q_{\nu_0}\cap\Pi_0} f_0^\infty(\nabla v)\, dy + \int_{Q_{\nu_0}\cap\Sigma_0} \beta_0(v)\, d\mathcal{H}^{N-1}\right), \tag{4.2.5}$$

where

$$\mathcal{P}(\lambda,\theta,\nu_0) := \left\{ v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0;\mathbb{R}^d) \,|\, v(y) := \begin{cases}\lambda & \text{if } y\cdot\nu_0 = 1/2, \\ \theta & \text{if } y\cdot\nu_0 = -1/2,\end{cases} \right.$$
$$\left. \text{and } v \text{ periodic in the directions } R_{\nu_0}(e_i), i = 1,\cdots,N-1 \right\}.$$

In order to simplify the notations, set $\nu_0 = e_N, Q_{\nu_0} = Q, Q_{\nu_0}\cap\Sigma_0 = Q' = (-1/2,1/2)^{N-1}$. Let $v\in\mathcal{P}(\lambda,\theta,\nu_0)$ and define

$$\overline{v}(y_N) := \int_{Q'} v(y',y_N)\, dy'.$$

Clearly, $\overline{v}$ belongs to $\mathcal{P}(\lambda,\theta,e_N)$. Since f_0^∞ and β_0 are convex and f_0^∞ is positively homogeneous, by Jensen's inequality we have

$$\begin{aligned}\int_Q f_0^\infty(\nabla v)\, dy + \int_{Q'} \beta_0(v)\, dy' &\geq \int_{-1/2}^{1/2} f_0^\infty\Big(\int_{Q'} \nabla v\, dy'\Big)\, dy + \beta_0\Big(\int_{Q'} v\, dy'\Big) \\ &= \int_Q f_0^\infty(\nabla\overline{v})\, dy + \int_{Q'} \beta_0(\overline{v})\, dy',\end{aligned}$$

where we have used the fact that, due to the periodicity of $v(\cdot, y_N)$,

$$\int_{Q'} \nabla v \, dy' = \int_{Q'} \frac{\partial v}{\partial y_N} \, dy' = \nabla \overline{v}.$$

We conclude, therefore, that the infimum on the right hand side of (4.2.5) may be calculated whith functions depending only on $(y \cdot \nu_0)$ and, consequently, with constant trace along Σ_0. We have then, from (4.2.5),

$$\beta(x_0, \lambda, \theta) \geq \inf_{w \in \mathbb{R}^d} \left\{ \left(\inf_{\substack{v \in \mathcal{P}(\lambda,\theta,\nu_0) \\ v = w \text{ on } \Sigma_0}} \int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v) dy \right) + \beta_0(w) \right\}. \tag{4.2.6}$$

Since f_0^∞ is homogeneous of degree one, $f_0^\infty(0) = 0$ and we have, for $v \in \mathcal{P}(\lambda, \theta, \nu_0)$ and $v = w$ on Σ_0,

$$\int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v) dy = \int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v_1) dy + \int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v_2) dy, \tag{4.2.7}$$

where

$$v_1 := \begin{cases} v & \text{in } Q_{\nu_0}^+ \cap \Pi_0, \\ w, & \text{in } Q_{\nu_0}^-, \end{cases} \quad \text{and} \quad v_2 := \begin{cases} w & \text{in } Q_{\nu_0}^+ \cap \Pi_0, \\ v, & \text{in } Q_{\nu_0}^-. \end{cases}$$

Then, from (4.2.6) and (4.2.7), we get

$$\begin{aligned} &\beta(x_0, \lambda, \theta) \geq \\ &\geq \inf_{w \in \mathbb{R}^d} \left\{ \left(\inf_{v \in \mathcal{P}(\lambda, w, \nu_0)} \int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v) dy + \inf_{v \in \mathcal{P}(w,\theta,\nu_0)} \int_{Q_{\nu_0} \cap \Pi_0} f_0^\infty(\nabla v) dy \right) + \beta_0(w) \right\} \\ &= \begin{cases} \inf_{w \in \mathbb{R}^d} \left\{ f_0^\infty\Big((\lambda - w) \otimes \nu_0\Big) + f_0^\infty\Big((w - \theta) \otimes \nu_0\Big) + \beta_0(w) \right\} \\ \qquad\qquad\qquad\qquad \text{if } x_0 \in \Omega \cap \Sigma, \\ \inf_{w \in \mathbb{R}^d} \left\{ f_0^\infty\Big((w - \theta) \otimes \nu_0\Big) + \beta_0(w) \right\} \\ \qquad\qquad\qquad\qquad \text{if } x_0 \in \partial\Omega \cap \Sigma, \end{cases} \end{aligned}$$

where, to obtain the last equality, we have used the fact that (see [FM2])

$$\inf_{v \in \mathcal{P}(\lambda,\theta,\nu_0)} \int_{Q_{\nu_0}} f_0^\infty(\nabla v) \, dy = f_0^\infty\Big((\lambda - \theta) \otimes \nu_0\Big).$$

To prove the opposite inequality it is enough to remark that we have, from (4.2.4),

$$\beta(x_0,\lambda,\theta) \leq \inf_{w\in\mathbb{R}^d}\left\{\left(\inf_{\substack{v\in W^{1,1}(Q_{\nu_0}\cap\Pi_0;\mathbb{R}^d)\\ v=u_{\lambda,\theta,\nu_0}\text{ on }\partial Q_{\nu_0}\cap\Pi_0\\ v=w\text{ on }\Sigma_0}}\int_{Q_{\nu_0}\cap\Pi_0} f_0^\infty(\nabla v)\,dy\right)+\beta_0(w)\right\}$$

$$=\begin{cases}\inf_{w\in\mathbb{R}^d}\left[f_0^\infty\Big((\lambda-w)\otimes\nu_0\Big)+f_0^\infty\Big((w-\theta)\otimes\nu_0\Big)+\beta_0(w)\right] & \text{if } x_0\in\Omega\cap\Sigma,\\ \inf_{w\in\mathbb{R}^d}\left[f_0^\infty\Big((w-\theta)\otimes\nu_0\Big)+\beta_0(w)\right] & \text{if } x_0\in\partial\Omega\cap\Sigma.\end{cases}$$

where, to obtain the last equality, we have used the same arguments of example 4.1 to deduce (4.1.6) from (4.1.5).

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