$\frac{10}{10}$

Problem #1 5/5

First, for arbitrary θ_1, θ_2

$Z^*(\theta_1) = (c_1+\alpha_1\theta_1, c_2+\alpha_2\theta_1, \cdots, c_n+\alpha_n\theta_1)[X_1^*] = M_1 x_1^*$ ✓

$Z^*(\theta_2) = (c_1+\alpha_1\theta_2, c_2+\alpha_2\theta_2, \cdots, c_n+\alpha_n\theta_2)[X_2^*] = M_2 X_2^*$ ✓

for θ_1, X_1^* maximizes $(c_1+\alpha_1\theta_1, \cdots, c_n+\alpha_n\theta_1)X = M_1 X$ for X satisfying $\begin{cases} Ax \le b \\ x \ge 0 \end{cases}$ ✓

for θ_2, X_2^* maximizes $(c_1+\alpha_1\theta_2, \cdots, c_n+\alpha_n\theta_2)X = M_2 X$

then

$Z^*(\lambda\theta_1+(1-\lambda)\theta_2) = (c_1+\alpha_1(\lambda\theta_1+(1-\lambda)\theta_2), \cdots, c_n+\alpha_n(\lambda\theta_1+(1-\lambda)\theta_2)) = M_3 X_3^*$ ✓

X_1^*, X_2^*, X_3^* satisfies $\begin{cases} Ax \le b \\ x \ge 0 \end{cases}$

Then

$Z^*(\theta_1) = M_1 X_1^* \ge M_1 X_3^*$ ✓

$Z^*(\theta_2) = M_2 X_2^* \ge M_2 X_3^*$ ✓

let $\lambda \in [0,1]$

thus

$\lambda Z^*(\theta_1) + (1-\lambda) Z^*(\theta_2) = \lambda M_1 X_1^* + (1-\lambda) X_2^* \ge (\lambda M_1 + (1-\lambda) M_2) X_3^*$

$\lambda M_1 + (1-\lambda) M_2 = (c_1+\alpha_1(\lambda\theta_1+(1-\lambda)\theta_2), \cdots, c_n+\alpha_n(\lambda\theta_1+(1-\lambda)\theta_2))$ ✓

$= M_3$

thus

$\lambda Z^*(\theta_1) + (1-\lambda) Z^*(\theta_2) \ge (\lambda M_1 + (1-\lambda) M_2) X_3^* = M_3 X_3^* = Z^*(\lambda\theta_1 + (1-\lambda)\theta_2)$

we have showed that

$Z^*(\theta)$ is convex. ✓

Problem #2 5/5

Convert it to dual problem

$\min \sum_{i=1}^{m} (b_i + \beta_i \theta) y_i = y B_\theta \quad B_\theta = \begin{bmatrix} b_1 + \beta_1 \theta \\ \vdots \\ b_m + \beta_m \theta \end{bmatrix}$ ✓

st $yA \geq \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = C$ here $A = a_{ij} (1 \leq i \leq m, 1 \leq j \leq n)$

$y \geq 0$

then by the relationship between primal and dual, where both $\hat{x}, \hat{y}$ are feasible in their own problem.

$\hat{z}(\theta) = C\hat{x} = \hat{y}B$

for all feasible x and y. ✓

$C\hat{x} \geq Cx$, $\hat{y}B_\theta \leq yB_\theta$

Then fix θ_1, θ_2

$\hat{z}(\theta_1) = \hat{y}_1 B_{\theta_1} = C\hat{x}_1 \leq \hat{y}_3 B_{\theta_1}$

$\hat{z}(\theta_2) = \hat{y}_2 B_{\theta_2} = C\hat{x}_2 \leq \hat{y}_3 B_{\theta_2}$

$\hat{z}(\lambda\theta_1 + (1-\lambda)\theta_2) = \hat{z}(\theta_3) = \hat{y}_3 B_{\theta_3} = C\hat{x}_3$

Note: $\hat{y}_1, \hat{y}_2, \hat{y}_3$ are feasible since they are under the same constraint.

then

$\lambda \hat{z}(\theta_1) + (1-\lambda)\hat{z}(\theta_2)$

$= \lambda \hat{y}_1 B_{\theta_1} + (1-\lambda)\hat{y}_2 B_{\theta_2}$

$\leq \lambda \hat{y}_3 B_{\theta_1} + (1-\lambda)\hat{y}_3 B_{\theta_2}$

$= \hat{y}_3 (\lambda B_{\theta_1} + (1-\lambda) B_{\theta_2})$

$= \hat{y}_3 \begin{bmatrix} b_1 + (\lambda\theta_1 + (1-\lambda)\theta_2)\beta_1 \\ \vdots \\ b_n + (\lambda\theta_1 + (1-\lambda)\theta_2)\beta_n \end{bmatrix}$

$= \hat{y}_3 B_{\theta_3}$

$= \hat{z}(\lambda\theta_1 + (1-\lambda)\theta_2)$ for $\lambda \in$

Thus $\hat{z}(\theta)$ is concave ✓