21-292 Midterm I – Fall 2016 - solutions

Name:______________________

Problem 1 (34 points):

Consider the following problem:

Maximize $Z = 2x_1 + 4x_2 - x_3$,

subject to

$$3x_2 - x_3 \leq 30 \quad \text{(resource 1)}$$
$$2x_1 - x_2 + x_3 \leq 10 \quad \text{(resource 2)}$$
$$4x_1 + 2x_2 - 2x_3 \leq 40 \quad \text{(resource 3)}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Work through the simplex method step by step to solve the problem.

Solution for Problem 4-4:

(a)

Iteration 0: After introducing slack variables, the following simplex tableau is obtained.

Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	-2	-4	1	0	0	0	0
x_4	(1)	0	0	3	-1	1	0	0	30
x_5	(2)	0	2	-1	1	0	1	0	10
x_6	(3)	0	4	2	-2	0	0	1	40

The negative coefficients of x_1 and x_2 in Eq. (0) indicate that the initial BF solution (0, 0, 0, 30, 10, 40) is not optimal.

Iteration 1: As the boxes in the above tableau indicate, the entering basic variable is x_2 and the leaving basic variable is x_4. Restoring proper form from Gaussian elimination then leads to the following tableau.

Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	-2	0	-0.333	1.333	0	0	40
x_2	(1)	0	0	1	-0.333	0.333	0	0	10
x_5	(2)	0	2	0	0.667	0.333	1	0	20
x_6	(3)	0	4	0	-1.333	-0.667	0	1	20

The negative coefficients of x_1 and x_3 in Eq. (0) indicate that the current BF solution (0, 10, 0, 0, 20, 20) is not optimal.

Iteration 2: As the boxes in the above tableau indicate, the entering basic variable is x_1 and the leaving basic variable is x_6. Restoring proper form from Gaussian elimination then leads to the following tableau.

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	0	-1	1	0	0.5	50
x_2	(1)	0	0	1	-0.333	0.333	0	0	10
x_5	(2)	0	0	0	1.333	0.667	1	-0.5	10
x_1	(3)	0	1	0	-0.333	-0.167	0	0.25	5

The negative coefficient of x_3 in Eq. (0) indicates that the current BF solution (5, 10, 0, 0, 10, 0) is not optimal.

Iteration 3: As the boxes in the above tableau indicate, the entering basic variable is x_3 and the leaving basic variable is x_5. Restoring proper form from Gaussian elimination then leads to the following tableau.

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	0	0	1.5	0.75	0.125	57.5
x_2	(1)	0	0	1	0	0.5	0.25	-0.125	12.5
x_3	(2)	0	0	0	1	0.5	0.75	-0.375	7.5
x_1	(3)	0	1	0	0	0	0.25	0.125	7.5

The fact that none of the coefficients in Eq. (0) are negative indicates that the current BF solution (7.5, 12.5, 7.5, 0, 0, 0) is optimal. Thus, when considering just the decision variables for the problem, the optimal solution is $(x_1^*, x_2^*, x_3^*) = (7.5, 12.5, 7.5)$ with
$Z^* = 57.5$.

Problem 3 (points):

Slim-Down Manufacturing makes a line of nutritionally complete, weight-reduction beverages. One of their products is a strawberry shake which is designed to be a complete meal. The strawberry shake consists of several ingredients. Some information about each of these ingredients is given below.

Ingredient	Calories from fat (per tbsp)	Total Calories (per tbsp)	Vitamin Content (mg/tbsp)	Thickeners (mg/tbsp)	Cost (¢/tbsp)
Strawberry flavoring	1	50	20	3	10
Cream	75	100	0	8	8
Vitamin supplement	0	0	50	1	25
Artificial sweetener	0	120	0	2	15
Thickening agent	30	80	2	25	6

The nutritional requirements are as follows. The beverage must total between 380 and 420 calories (inclusive). No more than 20% of the total calories should come from fat. There must be at least 50 milligrams (mg) of vitamin content. For taste reasons, there must be at least two tablespoons (tbsp) of strawberry flavoring for each tbsp of artificial sweetener. Finally, to maintain proper thickness, there must be exactly 15 mg of thickeners in the beverage.

Management would like to select the quantity of each ingredient for the beverage which would minimize cost while meeting the above requirements.

Formulate a linear programming model for this problem.

Solution:

The decision variables can be denoted and defined as follows:

S = Tablespoons of strawberry flavoring,
CR = Tablespoons of cream,
V = Tablespoons of vitamin supplement,
A = Tablespoons of artificial sweetener,
T = Tablespoons of thickening agent.

(Alternative notation for the decision variables is x_S, x_C, x_V, x_A, and x_T, respectively.) Also letting C (or Z) denote cost, the linear programming model for this problem is

Minimize $C = 10\,S + 8\,CR + 25\,V + 15\,A + 6\,T$,

subject to

$$50\,S + 100\,CR + 120\,A + 80\,T \geq 380$$
$$50\,S + 100\,CR + 120\,A + 80\,T \leq 420$$
$$S + 75\,CR + 30\,T \leq 0.2\,(50\,S + 100\,CR + 120\,A + 80\,T)$$
$$20\,S + 50\,V + 2\,T \geq 50$$
$$S \geq 2A$$
$$3\,S + 8\,CR + V + 2\,A + 25\,T = 15$$

and

$$S \geq 0,\ CR \geq 0,\ V \geq 0,\ A \geq 0,\ T \geq 0.$$