Math 301: Homework 8

Due Wednesday November 8 at noon on Canvas

1. In this problem, we will "smash together" the two partite sets of the incidence graph of a projective plane and give an asymptotic formula for $\text{ex}(n, C_4)$. Let V be a 3-dimensional vector space over a finite field $\mathbb{F}_q$. We define a graph G_π^o where $V(G_\pi^o)$ is the set of one-dimensional subspaces of V. There are $q^2 + q + 1$ of these (to see this, think of a vector in V having 3 coordinates, and then for each subspace it is defined by a vector which you can normalize so that the first non-zero coordinate is 1). Two vertices are adjacent if and only if the subspaces are orthogonal to each other.

   (a) Show that each vertex has degree $q + 1$ (Hint: V is a 3-dimensional vector space. Given a fixed 1-dimensional subspace, the set of vectors orthogonal to it is 2-dimensional. How many 1-dimensional subspaces are in a 2-dimensional vector space over $\mathbb{F}_q$?)

   Solution: Let U be a fixed 1-dimensional subspace of $\mathbf{F}_q^3$. Then the number of neighbors of U is the number of 1-dimensional subspaces that are perpendicular to it. Since we are in a 3-dimensional vector space, this is equivalent to counting the number of 1-dimensional subspaces in a 2-dimensional vector space over $\mathbf{F}_q$ (the subspace which is perpendicular to U is isomorphic to $\mathbf{F}_q^2$). We may write subspaces in a 2-dimensional vector space over $\mathbf{F}_q$ as a vector of length 2 with coordinates from $\mathbf{F}_q$. Without loss of generality, we may normalize so that the leading nonzero coordinate is 1. Therefore the subspaces may be of the form $(1, x)$ with $x \in \mathbf{F}_q$ or $(0, 1)$ and so there are $q + 1$ of these.

   (b) Show that every pair of vertices has exactly one path of length 2 between them (Hint: this is *much* easier to do geometrically than algebraically).

   Solution: Let U and V be two distinct 1-dimensional subspaces in $\mathbf{F}_q^3$. Then since the cross product of two distinct vectors in a three dimensional space points in a unique direction (unique up to scalar multiplication), this direction is the unique subspace which is orthogonal to both U and V. Therefore U and V have exactly one common neighbor.

   (c) Show that there are loops in the graph (you may allow q to be of any form that is convenient for you).

   Solution: If for example $p \equiv 1 \pmod 4$, then -1 is a quadratic residue mod p, ie there is a y such that $y^2 \equiv -1 \pmod p$. In this case, the subspace coordinatized by $(1, y, 0)$ will be orthogonal to itself, and hence will have a loop.

(d) It is known that there are $q+1$ loops in this graph. Let G_π be the graph with the loops removed. Then G_π is a graph on q^2+q+1 vertices with q^2 vertices of degree $q+1$ and $q+1$ vertices of degree q.

(e) Use part (b) and (d) to count the number of triangles in G_π.

Solution: Part (b) and (d) together show that for any edge xy in G_π^o where both endpoints are not looped, there is a unique z such that xyz forms a triangle. Part (b) shows that two loops cannot be adjacent, and part (d) shows that for xy an edge with one loop on an endpoint, there is no triangle through xy. Therefore, the number of triangles is

$$\frac{1}{3}\text{(the number of edges with no loops on either end)} = \frac{1}{6}q^2(q+1).$$

(f) It is known that for any $\epsilon > 0$, there is an M such that for $m \geq M$, there is a prime number in the interval $[m,(1+\epsilon)m]$. Use this to show that $\mathrm{ex}(n, C_4) \sim \frac{1}{2}n^{3/2}$.

Solution: We already know from KST theorem that $\mathrm{ex}(n, C_4) \lesssim \frac{1}{2}n^{3/2}$. So we must show that for any $\delta > 0$, there exists an N such that for $n \geq N$

$$\mathrm{ex}(n, C_4) \geq \left(\frac{1}{2} - \delta\right) n^{3/2}.$$

Fix $\delta > 0$. Note that

$$\lim_{q\to\infty} \frac{\sqrt{q^2+q+1}}{\sqrt{q^2}} = 1,$$

so there exists an N such that for any $n \geq N$, there is a prime q with $q^2+q+1 < n$ and $q > (1-\delta)^{1/3}\sqrt{n}$. Now using this prime q, we may construct a graph on $q^2+q+1 < n$ vertices that has

$$\frac{1}{2}q(q+1)^2 > \frac{1}{2}q^3 \geq (1-\delta)n^{3/2}$$

edges.

2. The *multicolor Ramsey number* of a graph H, denoted $r_k(H)$ is the minimum n such that any k-coloring of the edges of K_n contains a monochromatic copy of H. We think of k as going to infinity. Assume that $\mathrm{ex}(n, H) = \Theta(n^\alpha)$ for some $1 < \alpha \leq 2$.

(a) Use the pigeonhole principle to show that $r_k(H) = O(k^{1/(2-\alpha)})$.

Solution: By assumption $\mathrm{ex}(n, H) \leq Cn^\alpha$ for some constant C. Let $n = C_2k^{1/(2-\alpha)}$. We must show that for a large enough constant C_2, any k coloring of the edges of K_n must contain a monochromatic copy of H. By the pigeonhole principle, any k-coloring of $E(K_n)$ contains a color with at least

$$\frac{\binom{n}{2}}{k}$$

edges. If this many edges is more than Cn^α, then by the definition of the Turán number, there must be a copy of H in this color. So if

$$\frac{\binom{n}{2}}{k} \geq Cn^\alpha$$

then we are done. This happens if C_2 is a large enough constant relative to C.

(b) Use the probabilistic method to show that $r_k(H) = \Omega\left(k^{1/(2-\alpha)}/\text{polylog}(k)\right)$.

Solution: Let $n = k^{1/(2-\alpha)}/\text{polylog}(k)$. We must show that we may choose polylog(k) large enough that there is a k-coloring of $E(K_n)$ that has no monochromatic copy of H. This is equivalent to showing that there are subgraphs $G_1, \cdots, G_k$ each of which is H free such that

$$\bigcup E(G_i) = E(K_n).$$

To see this, given a k-coloring with no monochrome H, let G_i be the graph of edges with color i. Given a covering of $E(K_n)$ with $G_1, \cdots, G_k$ each of which is H free, let an edge xy be color i if $xy \in G_i$. If there are multiple choices for the color of an edge, choose one arbitrarily (note that the color classes will still be H free).

So we must show that for this choice of n, we can cover $E(K_n)$ with k subgraphs each of which are H-free. By assumption, we know that for some $\epsilon > 0$ there is a graph F on n vertices with ϵn^α edges which is H free. We put down copies of F "randomly" on $E(K_n)$. That is, choose $\pi_1, \cdots, \pi_k \in S_n$ uniformly and independently, and let G_i be a graph isomorphic to F with its vertices ordered by π_i. We are done if we can show that with positive probability, every edge in K_n is covered by at least one of the G_i. Fix an edge $xy \in E(K_n)$. Then

$$\mathbb{P}(xy \in E(G_i)) = \frac{\epsilon n^\alpha}{\binom{n}{2}}.$$

Therefore,

$$\mathbb{P}(xy \text{ not covered by any } G_i) = \left(1 - \frac{\epsilon n^\alpha}{\binom{n}{2}}\right)^k \leq e^{-\epsilon k n^\alpha / \binom{n}{2}}.$$

If this is less than $\binom{n}{2}^{-1}$ then by the union bound, the probability that there exists an edge which is not covered is strictly less than 1. This occurs when polylog(k) is a large enough constant (depending on α and ϵ) times $\log k$.

3. (**) Let G be a graph. A hypergraph H is said to be *Berge-G*, if there is a bijection $\phi : E(G) \to E(H)$ such that for each edge $e \in E(G)$, $e \subset \phi(e)$. We say that a hypergraph is Berge-G free if it does not contain any subhypergraph which is Berge-G. We denote by $\text{ex}_r(n, \text{Berge}-G)$ the maximum number of edges in an n-vertex r-uniform hypergraph which is Berge-G free. It is known that $\text{ex}_r(n, \text{Berge} - C_4) = O(n^{3/2})$.

(a) (***) Show that $\text{ex}_3(n, \text{Berge} - C_4) = \Omega(n^{3/2})$. What can you say for general r?

Solution: We use the graph G_π from problem 1 to construct a 3-uniform hypergraph H. Let $V(H) = V(G_\pi)$, so H has $n = q^2 + q + 1$ edges. We create a hyperedge in H xyz if and only if xyz form a triangle in G. By Problem 1, there are $\frac{1}{6}q^2(q+1) = \Omega(n^{3/2})$ edges in H.

To see that H is Berge-C_4-free, consider 4 vertices $xyzw$. If these vertices were to form a Berge-C_4 in H, this would mean that in G_π there were 4 triangles containing the edges xy yz zw and wx respectively. This would mean that $xyzw$ forms a C_4 in G_π, a contradiction.

It is not known how to construct Berge C_4 free hypergraphs for general r. You can do some tricks with projective planes if $r = 4$ and maybe if $r = 5$, please see me if you would like to know more details. To my knowledge, no construction is known with r at least say 7.

(b) (****) For a family of hypergraphs $\mathcal{F}$, define the multicolor Ramsey number $r_k^{(3)}(\mathcal{F})$ to be the minimum n such that for any k coloring of the edge set of the complete 3-uniform hypergraph, there is a monochromatic copy of some graph in $\mathcal{F}$. Show that $r_k^{(3)}(\text{Berge} - C_4) = \Theta(k^{2/3})$. (The upper bound is the same as in problem 2, using the Turán number and the pigeonhole principle. For the lower bound, it is equivalent to showing that the complete 3-uniform hypergraph on n vertices can be edge-partitioned into $O(n^{3/2})$ subgraphs, each of which has no Berge-C_4).

I am still offering a 5/30 exam point bounty for this one.