Math 301: Homework 6

Due Wednesday October 18 at noon on Canvas

For this homework you may use this version of the Chernoff Bound:

Theorem 1. *Let $X_1, \cdots, X_n$ be independent random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$. Let $S = X_1 + \cdots + X_n$. Then for any $0 \le \epsilon \le 1$,*

$$\mathbb{P}(S \le (1-\epsilon)pn) \le e^{-\epsilon^2 pn/2}$$

$$\mathbb{P}(S \ge (1+\epsilon)pn) \le e^{-\epsilon^2 pn/3}$$

1. Prove the Lopsided Lovász Local Lemma (if you promise to write neatly, you may handwrite this and scan it into your pdf).

   Theorem 2 (LLLL)**.** *Let $A_1, A_2, \cdots, A_n$ be events in a probability space and let D be a dependency graph for them. Suppose that there exist real numbers $x_1, x_2, \cdots, x_n \in [0, 1)$ such that for all i,*

   $$\mathbb{P}(A_i) \le x_i \prod_{(i,j)\in E(D)} (1 - x_j).$$

   Then

   $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) \ge \prod_{i=1}^{n} (1 - x_i) > 0.$$

2. Let G be a random graph on n vertices with edge probability $1/2$. Let $\epsilon > 0$ be arbitrary and let $k = (2 + \epsilon) \ln n$.

   (a) Use the Chernoff Bound to give an upper bound on the probability that any fixed set of k vertices forms an independent set.

   (b) Use part (a) to show that $\alpha(G) \le k$ with probability tending to 1.

3. The purpose of this problem is to show that any regular graph can be partitioned into parts such that between parts the graph is almost biregular. The constants $1/4$, $1/4$ and $1/2$ may obviously be changed depending on the situation. For a vertex v we denote its neighbors by $\Gamma(v)$. Show that for any $\epsilon > 0$ there exists a D_0 such that for any $d > D_0$, any d regular graph has a vertex partition into three parts A, B, C so that for any vertex v

   $$\left(\frac{1}{4} - \epsilon\right) d \le |\Gamma(v) \cap A| \le \left(\frac{1}{4} + \epsilon\right) d$$

$$\left(\frac{1}{4}-\epsilon\right)d \leq |\Gamma(v)\cap B| \leq \left(\frac{1}{4}+\epsilon\right)d$$
$$\left(\frac{1}{2}-\epsilon\right)d \leq |\Gamma(v)\cap C| \leq \left(\frac{1}{2}+\epsilon\right)d$$

(a) For each vertex, independently put it in A with probability $1/4$, into B with probability $1/4$ and into C with probability $1/2$. For each v, denote by A_v the event that either $|\Gamma(v)\cap A| < (1/4-\epsilon)d$ or $|\Gamma(v)\cap A| > (1/4+\epsilon)d$. Define events B_v and C_v similarly.

(b) Show that the probability of each of the events A_v, B_v, C_v is exponentially small as a function of d.

(c) Let D be a dependency graph for the events A_v, B_v, C_v. Show that the maximum degree of D is $O(d^2)$.

(d) Use the Lovász Local Lemma to prove the theorem.