

Math 301: Homework 5

Due Wednesday Wednesday October 4 at noon

1. A *dominating set* in a graph G is a subset $S \subset V(G)$ such that every vertex in G is either in S or has a neighbor in S . Let G be a graph of minimum degree δ . Show G has a dominating set of size at most

$$\frac{n}{\delta + 1}(\log(\delta + 1) + 1).$$

2. The goal of this problem is to prove the famous Szemerédi-Trotter Theorem. Given a set of points and lines in the Euclidean plane, an *incidence* is a point-line pair such that the point is on the line.

Theorem 1. *Given n points and m lines in the plane, the number of point-line incidences is*

$$O(n^{2/3}m^{2/3} + n + m).$$

First we need to prove a lemma. Given a graph G , the *crossing number* of G is the minimum number of edge-crossings possible among all drawings of the graph in the plane with the edges as straight line segments. The crossing number of a graph G is denoted by $\text{cr}(G)$.

Lemma 1. *Let G be a graph with e edges and n vertices. Then*

$$\text{cr}(G) \geq \frac{e^3}{64n^2} - n.$$

- a Show that the lemma is trivially true if $e < 4n$, so we may assume $e \geq 4n$.
- b Assume that G is drawn in the plane so that it has $\text{cr}(G)$ crossings.
- c Select a subset of vertices $S \subset V(G)$ independently with probability p , and let H be the subgraph induced by the selected vertices. Define random variables $X = |S|$, $Y = |E(H)|$. Define c_S to be the number of crossings that are left in the drawing after S is selected. Note that c_S and $\text{cr}(H)$ are random variables.
- d We need an easy bound on the crossing number of any graph. Let F be a graph and let F' be a planar subgraph of F with the maximum number of edges. Euler's formula says that $|E(F')| \leq 3|V(F)| - 6$. Since F' is maximal, adding any additional edge will create at least one crossing. Deduce that $\text{cr}(F) \geq |E(F)| - 3|V(F)|$ for any graph F .

e Deduce that

$$Y - 3X \leq \text{cr}(H) \leq c_S.$$

f From this, calculate the expected value of X , Y , and c_S and deduce that

$$p^2 e - 3pn \leq p^4 \text{cr}(G).$$

g Choose $p = \frac{4n}{e}$ (why is this a legitimate probability?) to finish the proof of the lemma.

Now we will prove the Szemerédi-Trotter Theorem.

a Let P and L be a set of n points and m lines. Construct a graph G with $V(G) = P$. Define adjacency in G by letting two points be adjacent if and only if they are consecutive on some line in L .

b Prove that $\text{cr}(G) < m^2$.

c Let x be the number of incidences between P and L . Prove that the number of edges in G is at least $(x - m)$.

d Deduce that

$$m^2 > \frac{(x - m)^3}{64n^2} - n$$

and show that this implies the theorem.

Give a construction that shows that the Szemerédi-Trotter Theorem is best possible up to the implied constant.

3. Let G be a random graph on n vertices where each edge is selected independently with probability p . Let $\omega(n)$ be a function that tends to infinity with n arbitrarily slowly.

a Use Markov's inequality to show that if $p \leq \frac{1}{\omega(n)n^{2/3}}$ then G does not contain a K_4 with probability tending to 1.

b Use Chebyshev's inequality to show that if $p \geq \frac{\omega(n)}{n^{2/3}}$ then G contains a K_4 with probability tending to 1.