Name: ______________________________

Instructions: You have 50 minutes to complete this exam. Show your work and justify all of your responses. No calculators, notes, or other external aids are allowed. You may use the following theorems:

Theorem 1 (Markov's Inequality)**.** *If X is a nonnegative random variable and $\lambda > 0$ is a real number, then*

$$\mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}(X)}{\lambda}.$$

Theorem 2 (Chebyshev's Inequality)**.** *Let X be a random variable with finite variance and $\lambda > 0$ a real number. Then*

$$\mathbb{P}(|X - \mathbb{E}(X)| > \lambda) \leq \frac{\mathrm{Var}(X)}{\lambda^2}.$$

Theorem 3 (Chernoff Bound)**.** *Let $X_1, \cdots, X_n$ be independent random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$. Let $S = X_1 + \cdots + X_n$. Then for any $0 \leq \epsilon \leq 1$,*

$$\mathbb{P}\left(S \leq (1-\epsilon)pn\right) \leq e^{-\epsilon^2 pn/2}$$

$$\mathbb{P}\left(S \geq (1+\epsilon)pn\right) \leq e^{-\epsilon^2 pn/3}$$

Theorem 4 (Lovász Local Lemma)**.** *Let $A_1, A_2, \cdots, A_n$ be events in a probability space and let D be a dependency graph for them. If d has maximum degree d, $\mathbb{P}(A_i) \leq p$ for all i, and*

$$ep(d+1) \leq 1,$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) > 0.$$

1. (10 points) Let A be a subset of integers in $[n]$. A is called a *Sidon set* if for $x_1, x_2, x_3, x_4 \in A$, if
$$x_1 + x_2 = x_3 + x_4,$$
then it implies that $\{x_1, x_2\} = \{x_3, x_4\}$. Show that there is a Sidon subset $A \subset [n]$ with $|A| = \Omega\left(n^{1/3}\right)$ (Hint: you may want to show that the total number of solutions to $x_1 + x_2 = x_3 + x_4$ with $x_1, x_2, x_3, x_4 \in [n]$ is $O(n^3)$).

Solution: For any fixed $x_1, x_2, x_3 \in [n]$, there is at most one $x_4 \in [n]$ such that $x_4 = x_1 + x_2 - x_3$. Therefore there are at most n^3 solutions to the equation $x_1 + x_2 = x_3 + x_4$.

Choose $S \subset [n]$ randomly, putting each integer in S independently with probability p. Let $X = |S|$ and Y count the number of solutions to the equation $x_1 + x_2 = x_3 + x_4$ with $x_1, x_2, x_3, x_4 \in S$. Given our set S, we may make it a Sidon set by removing at most one element of S for each solution to the equation. Therefore, there is a Sidon set of size at least $X - Y$ for every outcome of this random process. In particular, there is a Sidon set of size at least $\mathbb{E}(X - Y) = pn - p^4 \cdot (\text{the number of solutions}) \geq pn - p^4 n^3$. Choosing $p = \frac{1}{2} n^{-2/3}$ yields the result.

An alternative solution using the Local Lemma:

Solution: Let $m = \epsilon n^{1/3}$ where $\epsilon > 0$ will be chosen later. For i from 1 to m, choose x_i uniformly independently at random from $[n]$. For $1 \leq i < j \leq m$, let A_{ij} be the event that $x_i = x_j$. For $\{i, j\} \neq \{r, s\}$, let $A_{ij,rs}$ be the event that $x_i + x_j = x_r + x_s$. Then if none of the events A_{ij} and $A_{ij,rs}$ occur, we have found a Sidon set of size m. Note that

$$\mathbb{P}(A_{ij}) = \frac{1}{n}$$

and

$$\mathbb{P}(A_{ij,rs}) = \frac{\text{number of solutions to Sidon equation}}{n^4} \leq \frac{n^3}{n^4} = \frac{1}{n}.$$

An event A_{ij} is independent of events A_{rs} and $A_{ab,cd}$ if $\{i, j\} \cap \{r, s\} = \emptyset$ or $\{i, j\} \cap \{a, b, c, d\} = \emptyset$. Therefore in the dependency graph A_{ij} has degree at most $2m + 8m^3$. Similarly, in the dependency graph the event $A_{ij,rs}$ has degree at most $8m + 16m^3$. If ϵ is a small enough constant, then

$$e \cdot \frac{1}{n} O(m^3) < 1$$

and therefore we may apply the LLL.

2. (10 points) Let G be a d-regular graph. Show that if

$$e \cdot \frac{k}{k^d} \cdot (d^2 + 1) < 1,$$

then there is a coloring of $V(G)$ with k colors such that each vertex sees at least 2 colors (ie no vertex has a monochromatic neighborhood).

Solution: Color the vertices of G independently and uniformly at random, each color with probability $\frac{1}{k}$. Order the vertices of G arbitrarily and let A_i be the event the the i'th vertex sees a monochromatic neighborhood. Then

$$\mathbb{P}(A_i) = \frac{k}{k^d}.$$

If vertex i and vertex j do not share any neighbors, then the events A_i and A_j are independent. Since G is d-regular, the number of vertices that may share a neighbor with a fixed vertex is at most d^2. Therefore the dependency graph has degree at most d^2. So, if

$$e \cdot \frac{k}{k^d} \cdot (d^2 + 1) < 1,$$

we may apply the LLL. Noting that if none of the A_i occur then we have colored our graph so that no vertex sees a monochromatic neighborhood yields the result.

3. (10 points) Take a random walk on the number line starting at 0 and lasting n steps. At each step you walk either right or left by 1 step independently with probability 1/2 each. Show that with probability tending to 1 (as $n \to \infty$), at the end of n steps you are not more than $\sqrt{n}\log n$ steps away from the origin.

Solution: Make random variables X_i where $X_i = 1$ if the i'th step was to the right and $X_i = 0$ if the i'th step was to the left. Let $X = X_1 + \cdots + X_n$ and note that the random walk is more than $\sqrt{n}\log n$ steps away from the origin if and only if X is more than $\sqrt{n}\log n/2$ away from $\frac{n}{2}$. Noting that $\mathbb{E}(X) = \frac{n}{2}$, we may apply the Chernoff Bound and say

$$\begin{aligned}\mathbb{P}\left(|X - \frac{n}{2}| > \sqrt{n}\log n/2\right) &= \mathbb{P}\left(|X - \mathbb{E}(X)| > \frac{\log n}{\sqrt{n}}\frac{n}{2}\right)\\ &\leq 2e^{-\left(\frac{\log n}{\sqrt{n}}\right)^2 \mathbb{E}(X)/3} = 2e^{-\log^2 n/6} \to 0.\end{aligned}$$