Math 21-301 Exam 1 September 22, 2017

Name: Solutions

Instructions: You have 50 minutes to complete this exam. Show your work and justify all of your responses. No calculators, notes, or other external aids are allowed.

1. (10 points) Let $a_0 = 2$, $a_1 = 3/2$ and for $n \geq 2$, $2a_n = 3a_{n-1} - a_{n-2}$. Use generating functions to find an explicit formula for a_n as a function of n. (*Hint:* it may be useful to know that $\frac{4-3x}{x^2-3x+2} = \frac{1}{1-x} + \frac{2}{2-x}$).

Let $A(x) = \sum_{n \geq 0} a_n x^n$. Then

$$\sum_{n \geq 2} 2a_n x^n = \sum_{n \geq 2} 3a_{n-1} x^n - \sum_{n \geq 2} a_{n-2} x^n \implies$$

$$2A(x) - 2a_1 x - 2a_0 = 3xA(x) - 3xa_0 - x^2 A(x)$$

$$\implies A(x)(x^2 - 3x + 2) = 4 - 3x$$

$$\implies A(x) = \frac{4-3x}{x^2-3x+2} = \frac{1}{1-x} + \frac{2}{2-x}$$

$$a_n = [x^n]\left(\frac{1}{1-x} + \frac{2}{2-x}\right)$$

$$= [x^n]\left(\frac{1}{1-x} + \frac{1}{1-\frac{x}{2}}\right)$$

$$= \boxed{1 + \left(\frac{1}{2}\right)^n}$$

2. (10 points) Determine the number of functions $f : [n] \to [m]$ that miss exactly one point of $[m]$. That is, count the functions from $[n]$ to $[m]$ for which the size of the range of f is $m - 1$. Your answer may include summation symbols.

Let ~~g(k)~~ $g(k)$ = the number of surjections from $[n]$ to $[k]$.

For $1 \le i \le k$, let $A_i = \{f : [n] \to [k] \text{ s.t. } f(x) \ne i \ \forall x\}$

Then given $S \subseteq [k]$, $\left|\bigcap_{i \in S} A_i\right| = (k - |S|)^n$

By inclusion-exclusion,

$$g(k) = \sum_{S \subseteq [k]} (-1)^{|S|}(k - |S|)^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

To count functions which miss exactly one point, there are m choices for which point to miss, and then the function is a surjection on the remaining $m-1$ points. So the number of such functions is

$$\boxed{m \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^n}$$

3. (10 points) Count the number of integer solutions to

$$x_1 + x_2 + x_3 = 100$$

satisfying constraints

$$x_1, x_2, x_3 \geq 0$$
$$x_1 \text{ odd}$$
$$x_2 \text{ even}$$
$$x_3 \leq 40$$

Your answer may include binomial coefficients but should not include a summation symbol. (*Hint:* it may be useful to know that $\frac{x}{(1-x)^3(1+x)^2} = \frac{-1}{16(x+1)} - \frac{1}{8(x+1)^2} + \frac{1}{16(x+1)} - \frac{1}{4(x-1)^3}$.)

Let S be the set of sequences of length 3, $\{x_1, x_2, x_3\}$ where $x_1 \geq 0$ odd, $x_2 \geq 0$ even $0 \leq x_3 \leq 40$.

and define ~~[illegible]~~ weight $w : S \to \mathbb{N}$ by $w(\{x_1, x_2, x_3\}) = x_1 + x_2 + x_3$.

Let $S_1 = \{x : x \geq 0 \text{ odd}\}$ with $w(x) = x$
$S_2 = \{x : x \geq 0 \text{ even}\}$
$S_3 = \{x : 0 \leq x_3 \leq 40\}$

By the product lemma, $\Phi_S = \Phi_{S_1} \cdot \Phi_{S_2} \cdot \Phi_{S_3}$

$$= \frac{x}{1-x^2} \cdot \frac{1}{1-x^2} \cdot \frac{1-x^{41}}{1-x}$$

So the number of solutions is

$$[x^{100}]\,\Phi_S = [x^{100}]\left(\frac{x}{(1-x)^3(1+x)^2}\right) - [x^{60}]\left(\frac{x}{(1-x)^3(1+x)^2}\right)$$

Note: The partial fraction decomposition in the hint has a sign error. The third term should be $\frac{1}{16(x-1)}$ instead of $\frac{1}{16(x+1)}$. IF the hint were correct, we would proceed as:

$$[x^{100}]\left(\frac{x}{(1-x)^3(1+x)^3}\right) - [x^{60}]\left(\frac{x}{(1-x)^3(1+x)^2}\right)$$

$$\boxed{\begin{aligned} = &\left[-\frac{1}{16}\binom{-1}{100} - \frac{1}{8}\binom{-2}{100} + \frac{1}{16}\binom{-1}{100} + \frac{1}{4}\binom{-3}{100}(-1)^{100}\right] \\ - &\left[-\frac{1}{16}\binom{-1}{60} + -\frac{1}{8}\binom{-2}{60} + \frac{1}{16}\binom{-1}{60} + \frac{1}{4}(-1)^{60}\binom{-3}{60}\right] \end{aligned}}$$