The Sizes of Infinity

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Western PA ARML

September, 11 2016

A Story

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Then, Zero walks into Hilbert's Hotel...

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- B is **bigger than** A (denoted $|A| < |B|$) if $|A| \leq |B|$ and $|A| \neq |B|$.

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- This gives us a definition of 'infinite set'!
  That is, a set B is **infinite** if $\exists A \subsetneq B$ such that $|A| = |B|$.

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  - Proof: $0, -1, 1, -2, 2, -3, 3, \ldots$
  - In general, if A, B are countable, then $A \cup B$ is countable (Here: $A = \{0, 1, 2, \ldots\}$ and $B = \{-1, -2, -3, \ldots\}$).

The Rational Numbers

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$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

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- $g\left(\frac{p}{q}\right) = \text{sgn}(p) 2^{|p|}(2q-1)$
    - Almost, but not quite a bijection $\mathbb{Q} \to \mathbb{Z}$.

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- By (2), $|S_0 \cup S_1 \cup S_2 \cup S_3 \cup \ldots| = |\mathbb{N}|$.

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Proof: By contradiction. Assume that we can list the real numbers, say

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There are infinitely many sizes of infinity! We call the first few $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \dots$

The Continuum Hypothesis (CH)

We know that $\mathbb{N}$ is the smallest infinite set, i.e. $|\mathbb{N}| = \aleph_0$, and that $|\mathbb{R}| > |\mathbb{N}|$. But are there any sizes of infinity *between* the size of $\mathbb{N}$ and the size of $\mathbb{R}$? In other words, is $|\mathbb{R}| = \aleph_1$

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- In summary: the answer is whatever you want!

The Continuum Hypothesis (CH)

We know that $\mathbb{N}$ is the smallest infinite set, i.e. $|\mathbb{N}| = \aleph_0$, and that $|\mathbb{R}| > |\mathbb{N}|$. But are there any sizes of infinity *between* the size of $\mathbb{N}$ and the size of $\mathbb{R}$? In other words, is $|\mathbb{R}| = \aleph_1$

The Continuum Hypothesis: No intermediate sizes, $|\mathbb{R}| = \aleph_1$.

- Georg Cantor asked whether CH was true in 1878.
- Godel 1940: ZFC cannot disprove CH
- Cohen 1963: ZFC cannot prove CH
- In summary: the answer is whatever you want!

More specifically, the answer is **independent** of ZFC, meaning that you can add to ZFC exactly one of the following:

- CH is true
- CH is false

Either one will not lead you to a contradiction*

Proof Sketch

Proof Sketch

Just kidding.

Further Topics of Interest

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  - $|\mathbb{R}|$ could be any of $\aleph_1, \aleph_2, \aleph_3, \ldots$ and more!
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Thanks for listening!